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MATHEMATICS

q-Calculus as operational algebra

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Abstract. This second paper on operational calculus is a continuation of Ernst, T. q-Analogues of some operational formulas. *Algebras Groups Geom.*, 2006, **23**(4), 354–374. We find multiple q-analogues of formulas in Carlitz, L. A note on the Laguerre polynomials. *Michigan Math. J.*, 1960, **7**, 219–223, for the Cigler q-Laguerre polynomials (Ernst, T. A method for q-calculus. *J. Nonlinear Math. Phys.*, 2003, **10**(4), 487–525). The q-Jacobi polynomials (Jacobi, C. G. J. *Werke 6*. Berlin, 1891) are treated in the same way, we generalize further to q-analogues of Manocha, H. L. and Sharma, B. L. (Some formulae for Jacobi polynomials. *Proc. Cambridge Philos. Soc.*, 1966, **62**, 459–462) and Singh, R. P. (Operational formulae for Jacobi and other polynomials. *Rend. Sem. Mat. Univ. Padova*, 1965, **35**, 237–244). A field of fractions for Cigler's multiplication operator (Cigler, J. Operatormethoden für q-Identitäten II, q-Laguerre-Polynome. *Monatsh. Math.*, 1981, **91**, 105–117) is used in the computations. The formulas for q-Jacobi polynomials are mostly formal. We find q-orthogonality relations for q-Laguerre, q-Jacobi, and q-Legendre polynomials using q-integration by parts. This q-Legendre polynomial is given here for the first time, we also find its q-difference equations. An inequality for a q-exponential function is proved. The q-difference equation for ${}_p\phi_{p-1}(a_1,\ldots,a_p;b_1,\ldots,b_{p-1}|q,z)$ is given improving on Smith, E. R. *Zur Theorie der Heineschen Reihe und ihrer Verallgemeinerung*. Diss. Univ. München 1911, p. 11, by using e_k =elementary symmetric polynomial. Partial q-difference equations for the q-Appell and q-Lauricella functions are found, improving on Jackson, F. H. On basic double hypergeometric functions. *Quart. J. Math.*, Oxford Ser., 1942, **13**, 69–82, and Gasper, G. and Rahman, M. Basic hypergeometric series. Second edition. Cambridge, 2004, p. 299, where q-difference equations for q-Appell functions were given with different notation. The q-difference equation for Φ_1 can also be written in canonical form, a q-analogue of [p. 146] Mellin, H. J. Über den Zusammenhang zwischen den linearen Differential- und Differenzengleichunge, *Acta Math.*, 1901, **25**, 139–164.

Key words: q-difference equations, q-Laguerre, q-Jacobi polynomials, q-Legendre polynomials, q-orthogonality, formal equality, q-Appell function, q-Lauricella function, Rodriguez operator.

1. INTRODUCTION

The aim of this paper is to present q-calculus as a truly operational subject. Operational formulas were often used with great success in the theory of classical orthogonal polynomials and Bessel functions. The results obtained here are theoretically of certain interest, and also give important other formulas. The present paper is the second one in a series that tries to shed light on the mysteries of the so-called q-analogues of operational formulas; the first one was [25].

One example of operator is the Rodriguez operator operating on holomorphic functions, this is a generalization of the Rodriguez formula for Laguerre and Jacobi polynomials. This part of mathematics is not very well known, but it has been used extensively by a few experts on special functions, namely Cigler [15], Carlitz [12], Al-Salam [3], Chatterjea [13], and Jackson [39]. Leonard Carlitz (1907–1999) was the tutor of Waleed Al-Salam (1926–1996) at Duke University in 1958. Operational calculus went into the family as also N. A. Al-Salam, the wife of Waleed, published on the subject. Johann Cigler has tutored many students and built up a school in Vienna. His approach is centred around the Gauss q-binomial coefficients. We will show in this paper that this approach has many similarities with the method used by the author. Chatterjea represents the Indian school of q-calculus, which started after Wolfgang Hahn's (1911–1998) visit to India in 1959–1961. F. H. Jackson (1870–1960) was the first to use a so-called q-umbral calculus, which is treated in [23] and [24].

There are some similar approaches to this formal procedure in the literature. In [14] a parameter augmentation method for a reciprocal of a q-shifted factorial was used to obtain q-summation formulas. In [20] the equivalent approach (by the q-binomial theorem) to use the q-exponential function E_q to obtain formulas for q-Laguerre polynomials was used. As Fujiwara [30] showed, the most important property of the Jacobi, Laguerre, and Hermite polynomials is the generalized Rodriguez formula. That is why our treatment will use the Rodriguez formula for Cigler q-Laguerre polynomials (Chapter 2), q-Jacobi polynomials (Chapter 3), and orthogonality (Chapter 4). Here we will introduce a true operational form of q-Legendre polynomials. In [21] the author introduced q-functions of many variables. This treatment will be continued in Chapter 5, where partial q-difference equations will be found.

We will now describe the q-umbral method invented by the author [18–24], which also involves the Nalli–Ward-AlSalam (NWA) q-addition and the Jackson–Hahn–Cigler (JHC) q-addition. This method is a mixture of Heine [36] and Gasper and Rahman [31]. The advantages of this method have been summarized in [20, p. 495].

Definition 1. *The power function is defined by* $q^a \equiv e^{a\log(q)}$. *We always use the principal branch of the logarithm. The variables*

$$a,b,c,a_1,a_2,\dots,b_1,b_2,\dots \in \mathbb{C}$$

denote certain parameters. The variables i,j,k,l,m,n,p,r *will denote natural numbers except for certain cases where it will be clear from the context that i will denote the imaginary unit. The symbol* $\cong$ *will denote that an equality is purely formal. The q-analogues of a complex number a and of the factorial function are defined by*:

$$\{a\}_q \equiv \frac{1-q^a}{1-q},\ q \in \mathbb{C}\backslash\{1\}, \tag{1}$$

$$\{n\}_q! \equiv \prod_{k=1}^{n} \{k\}_q,\ \{0\}_q! \equiv 1,\ q \in \mathbb{C}. \tag{2}$$

Let the q-shifted factorial be defined by $(n = 0,1,2,\dots)$

$$\langle a;q\rangle_n \equiv \prod_{m=0}^{n-1}(1-q^{a+m});\ (a;q)_n \equiv \prod_{m=0}^{n-1}(1-aq^m). \tag{3}$$

The first formula will often be used, the second one is the Watson notation.

Since products of q-shifted factorials occur so often, to simplify them we shall frequently use the more compact notation

$$\langle a_1,\dots,a_m;q\rangle_n \equiv \prod_{j=1}^{m}\langle a_j;q\rangle_n. \tag{4}$$

Definition 2. *Assume that* $(m,l) = 1$, *i.e. m and l are relatively prime. The operator*

$$\widetilde{\tfrac{m}{l}} : \frac{\mathbb{C}}{\mathbb{Z}} \mapsto \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by

$$a \mapsto a + \frac{2\pi i m}{l\log q}. \tag{5}$$

Definition 3. *Generalizing Heine's* ${}_2\phi_1$ *series, we shall define a q-hypergeometric series by* (*compare* [31, p. 4]):

$$
{}_p\phi_r(\hat{a}_1,\dots,\hat{a}_p;\hat{b}_1,\dots,\hat{b}_r|q,z) \equiv {}_p\phi_r\left[\begin{array}{c}\hat{a}_1,\dots,\hat{a}_p\\ \hat{b}_1,\dots,\hat{b}_r\end{array}|q,z\right]
$$

$$
\equiv \sum_{n=0}^{\infty}\frac{\langle\hat{a}_1,\dots,\hat{a}_p;q\rangle_n}{\langle 1,\hat{b}_1,\dots,\hat{b}_r;q\rangle_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+r-p} z^n, \tag{6}
$$

where $q \neq 0$ *when* $p > r+1$, *and*

$$
\widehat{a} \equiv a \vee \widetilde{\tfrac{m}{l}\,a}. \tag{7}
$$

We will assume that all a in $\langle\hat{a};q\rangle_n$ *have this value in the rest of the paper.*

Let the Gauss q-binomial coefficient be defined by

$$
\binom{n}{k}_q \equiv \frac{\langle 1;q\rangle_n}{\langle 1;q\rangle_k\langle 1;q\rangle_{n-k}},\ k=0,1,\dots,n, \tag{8}
$$

and by

$$
\binom{\alpha}{\beta}_q \equiv \frac{\langle\beta+1,\alpha-\beta+1;q\rangle_\infty}{\langle 1,\alpha+1;q\rangle_\infty}, \tag{9}
$$

for complex α *and* β *when* $0<|q|<1$.

Let the Γ_q*-function be defined in the unit disk* $0<|q|<1$ *by*

$$
\Gamma_q(x) \equiv \frac{\langle 1;q\rangle_\infty}{\langle x;q\rangle_\infty}(1-q)^{1-x}. \tag{10}
$$

Here we deviate from the usual convention $q<1$, because we want to work with meromorphic functions of several variables. The reason is that the q-analogue of the Euler reflection formula involves the first Jacobi theta function, which by construction is a complex function, not only real [26]. The simple poles of Γ_q are located at $x=-n\pm\frac{2k\pi i}{\log q}$, $n,k\in\mathbb{N}$.

There is also a Γ_q-function for $q>1$.

We have $\lim_{q\to 1^-}\Gamma_q(x)=\Gamma(x)$ [48] and $\lim_{q\to 1^+}\Gamma_q(x)=\Gamma(x)$.

For $q<1$ the residue at $x=-n$ is [31, p. 17]

$$
\lim_{x\to -n}(x+n)\Gamma_q(x) = \frac{(1-q)^{n+1}}{\langle -n;q\rangle_n \log q^{-1}}. \tag{11}
$$

We find by L'Hospital's rule that $\lim_{q\to 1^-}$ of this is $\frac{(-1)^n}{n!}$.

Definition 4. *The following notations will sometimes be used*:

$$
\mathrm{QE}(x) \equiv q^x. \tag{12}
$$

$$
\varepsilon_i f(x_i) \equiv f(qx_i),\ i=1,2,\dots. \tag{13}
$$

Variants of the q-derivative were used by Euler and Heine, *but a real q-derivative was invented first by Jackson in 1908.*

$$
\mathrm{D}_q(x) \equiv \begin{cases} \dfrac{\varphi(x)-\varphi(qx)}{(1-q)x}, & \text{if } q\in\mathbb{C}\backslash\{1\},\ x\neq 0;\\ \dfrac{d\varphi}{dx}(x) & \text{if } q=1;\\ \dfrac{d\varphi}{dx}(0) & \text{if } x=0. \end{cases} \tag{14}
$$

Theorem 1.1.

$$\mathrm{D}_q^k \frac{x}{1-xq^{1+\alpha}} = \frac{q^{(1+\alpha)(k-1)}\{k\}_q!}{(xq^{1+\alpha};q)_{k+1}},\ k>0. \tag{15}$$

Definition 5. *Let a and b be any elements with commutative multiplication. Then the Nalli–Ward–AlSalam* (*NWA*) *q-addition*, *compare* [2, p. 240; 55, p. 345; 63, p. 256] *is given by*

$$(a \oplus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k},\ n=0,1,2,\ldots. \tag{16}$$

Furthermore, *we put*

$$(a \ominus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k (-b)^{n-k},\ n=0,1,2,\ldots. \tag{17}$$

There is a q-addition dual to the NWA. The following polynomial in 3 variables x,y,q originates from Gauss.

Definition 6. *The Jackson–Hahn–Cigler* (*JHC*) *q-addition*, *compare* [15, p. 91; 35, p. 362; 41, p. 78] *is the function*

$$(x \boxplus_q y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}}\, y^k x^{n-k} = P_{n,q}(x,y),\ n=0,1,2,\ldots. \tag{18}$$

$$(x \boxminus_q y)^n \equiv P_{n,q}(x,-y),\ n=0,1,2,\ldots. \tag{19}$$

The generalized (*noncommutative*) *NWA q-addition is the function*

$$(a \oplus_{q,t} b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^{n-k} b^k q^{t\left(nk-\binom{k}{2}\right)},\ n=0,1,2,\ldots. \tag{20}$$

Definition 7. *If* $|q|>1$, *or* $0<|q|<1$ *and* $|z|<|1-q|^{-1}$, *the q-exponential function* $\mathrm{E}_q(z)$ *was defined by Jackson* [39] *in 1904 and by Exton* [27]:

$$\mathrm{E}_q(z) \equiv \sum_{k=0}^{\infty} \frac{1}{\{k\}_q!} z^k. \tag{21}$$

We have now defined both Γ_q and $\mathrm{E}_q(z)$, so we can briefly state a result about the order and type of these functions:

Theorem 1.2. [64] *For* $0<q<1$ *and* $1 \le \mathrm{Re}(x) \le 2$, Γ_q *has order* 1 *and type* $<2\pi$. *Also* $\mathrm{E}_q(z)$ *has order* 1.

Definition 8. *In 1910 Jackson redefined the general q-integral* [31,40]

$$\int_0^a f(t,q)\, d_q(t) \equiv a(1-q) \sum_{n=0}^{\infty} f(aq^n,q) q^n,\ 0<|q|<1,\ a\in\mathbb{R}. \tag{22}$$

Following Jackson we will put

$$\int_0^\infty f(t,q)\, d_q(t) \equiv (1-q) \sum_{n=-\infty}^{\infty} f(q^n,q) q^n,\ 0<|q|<1, \tag{23}$$

provided the sum converges absolutely. Here we allow q to be complex, *but only real values correspond in the limit* $q \to 1$ *to ordinary integrals.*

In Chapter 4 we will need these two different kinds of q-integrals to treat orthogonality. In both cases the proof of orthogonality will be done by q-integration by parts. Since the proof of q-integration by parts for a finite interval is well known, we will only sketch the proof for a q-integral over $[0,\infty]$.

Theorem 1.3. *q-Integration by parts. If u and v are continuous on* $[0,\infty]$, $0<q<1$, *then*

$$\int_0^\infty u(t)\mathrm{D}_q v(t)\,d_q(t) = [u(t)v(t)]_0^\infty - \int_0^\infty v(qt)\mathrm{D}_q u(t)\,d_q(t), \tag{24}$$

provided that all expressions have a meaning.

Proof. This follows from a straightforward computation. We observe that $\lim_{n\to+\infty} q^n = 0$ and $\lim_{n\to-\infty} q^n = +\infty$. □

Definition 9. *Operators operate from left to right. Two operators are said to be equivalent if they give the same result when operating on* $\mathbb{C}[[x]]$. *Multiplication with x will be denoted by* $\mathbf{x}$.

If we work with operators, the definition of the Watson q-shifted factorial will be changed to

$$(a;q)_n \equiv \prod_{m=0}^{n-1}(I-q^m a), \tag{25}$$

where I denotes the identity operator, and a is an operator.

We will prove many operational formulas in this paper, so we need a designation for the functions to operate upon. This class of function will be called holomorphic. Convergence is not assumed, just as for formal power series.

Definition 10. *Let* H_q *denote holomorphic functions* $\mathbb{C}[[x]]$, *or more generally, functions of the form*

$$F(x) \equiv \sum_{k=0}^{\infty} \frac{\alpha_k x^{\beta_k}}{(x;q)_{\gamma_k}}. \tag{26}$$

2. CIGLER'S q-LAGUERRE POLYNOMIALS

In this paper we will be working with two different q-Laguerre polynomials. The polynomial $L_{n,q,c}^{(\alpha)}(x)$ was used by Cigler[16].

Definition 11.

$$L_{n,q,c}^{(\alpha)}(x) \equiv \sum_{k=0}^{n} \binom{n+\alpha}{n-k}_q \frac{\{n\}_q!}{\{k\}_q!} q^{k^2+\alpha k}(-1)^k x^k. \tag{27}$$

The Al-Salam q-Laguerre polynomial [1, p. 4] $L_{n,q}^{(\alpha)}(x)$ *is defined as follows*:

$$L_{n,q}^{(\alpha)}(x) = \frac{L_{n,q,c}^{(\alpha)}(x)}{\{n\}_q!}. \tag{28}$$

We will use the following operator operating on H_q as a basis for our calculations; the special case $\alpha=0,\ q=1$ was treated in [3, 1.1]. A related operator was used in [1, p. 4 (2.1)]. Compare [25].

$$\theta_{q,\alpha} \equiv \mathbf{x}(\{1+\alpha\}_q I + q^{1+\alpha}\mathbf{x}\mathrm{D}_{q,x}). \tag{29}$$

The q-Gould–Hopper [46, p. 77, 2.4; 3, 3.4] formula looks as follows:

$$\frac{1}{\mathbf{x}^n}\theta_{q,\alpha}^n = \prod_{k=1}^{n}(q^{k+\alpha}\mathbf{x}\mathrm{D}_q + \{\alpha+k\}_q I) = \sum_{k=0}^{n}\binom{n+\alpha}{n-k}_q \frac{\langle 1;q\rangle_n}{\langle 1;q\rangle_k} q^{k(k+\alpha)}\mathbf{x}^k\mathrm{D}_q^k(1-q)^{k-n}. \tag{30}$$

We will now give an alternative explanation of some of the previous operator formulas with the help of a paper by Viskov and Srivastava [62].

Theorem 2.1. *A q-analogue of* [62, p. 4]. *Let*

$$\triangle_{1,q} \equiv \mathbf{x}^{-\alpha}\mathrm{D}_q^n\mathbf{x}^{\alpha+n}\mathrm{D}_q^n, \tag{31}$$

$$\triangle_{2,q} \equiv (\mathbf{x}^{-\alpha}\mathrm{D}_q\mathbf{x}^{\alpha+1}\mathrm{D}_q)^n = (\mathbf{x}\mathrm{D}_q^2 + \mathrm{D}_q\varepsilon + q\{\alpha-1\}_q\mathrm{D}_q\varepsilon)^n, \tag{32}$$

$$\triangle_{3,q} \equiv \sum_{k=0}^{n}\binom{n+\alpha}{n-k}_q\binom{n}{k}_q\{n-k\}_q! q^{k(k+\alpha)}\mathbf{x}^k\mathrm{D}_q^{k+n}, \tag{33}$$

$$\triangle_{4,q} \equiv \prod_{k=1}^{n}(q^{k+\alpha}\mathbf{x}\mathrm{D}_q + \{\alpha+k\}_q I)\mathrm{D}_q^n. \tag{34}$$

Then these q-operators are all equivalent.

$$\triangle_{1,q} = \triangle_{2,q} = \triangle_{3,q} = \triangle_{4,q}. \tag{35}$$

Theorem 2.2. *A q-analogue of* [62, p. 6]. *Let*

$$\triangle_{5,q} \equiv \mathbf{x}^{-\alpha}\mathrm{D}_q^n\mathbf{x}^{\alpha+n}, \tag{36}$$

$$\triangle_{6,q} \equiv \mathbf{x}^{-n}(q\mathbf{x}^2\mathrm{D}_q + \varepsilon\{\alpha\}_q\mathbf{x} + \mathbf{x})^n = \mathbf{x}^{-n}(\mathbf{x}^{-\alpha+1}\mathrm{D}_q\mathbf{x}^{\alpha+1})^n, \tag{37}$$

$$\triangle_{7,q} \equiv \sum_{k=0}^{n}\binom{n+\alpha}{n-k}_q\binom{n}{k}_q\{n-k\}_q! q^{k(k+\alpha)}\mathbf{x}^k\mathrm{D}_q^{k}, \tag{38}$$

$$\triangle_{8,q} \equiv \prod_{k=1}^{n}(q^{k+\alpha}\mathbf{x}\mathrm{D}_q + \{\alpha+k\}_q I). \tag{39}$$

Then these four q-operators are equivalent.

$$\triangle_{5,q} = \triangle_{6,q} = \triangle_{7,q} = \triangle_{8,q}. \tag{40}$$

The q-Gould–Hopper formula follows from the equivalence of $\triangle_{7,q}$ and $\triangle_{8,q}$. All expressions in (40) are equal to $\frac{1}{\mathbf{x}^n}\theta_{q,\alpha}^n$.

As was pointed out in [1, p. 4], the operator $\theta_{q,\alpha}$ is particularly useful in dealing with q-Laguerre polynomials. We find that the q-Laguerre Rodriguez operator to be presented shortly is equal to $\mathrm{E}_{\frac{1}{q}}(x)\triangle_{8,q}\mathrm{E}_q(-x)$, and we obtain

Theorem 2.3. *The following equation is a q-analogue of the corr. version of* [3, 3.9]

$$\theta_{q,\alpha}^n\mathrm{E}_q(-x) = x^n\mathrm{E}_q(-x)L_{n,q,c}^{(\alpha)}(x). \tag{41}$$

Definition 12. *A q-analogue of Chatterjea* [13, p. 245]. *The q-Laguerre Rodriguez operator is given by*

$$\Omega_{n,q}^{(\alpha)} f(x) \equiv x^{-\alpha} \mathrm{E}_{\frac{1}{q}}(x) \mathrm{D}_q^n (x^{\alpha+n} \mathrm{E}_q(-x) f(x)),\ f(x) \in \mathrm{H}_q. \tag{42}$$

We immediately obtain a q-analogue of [13, p. 245].

Theorem 2.4.

$$\Omega_{n,q}^{(\alpha)} = \sum_{k=0}^{n} \binom{n+\alpha}{n-k}_q \frac{\{n\}_q!}{\{k\}_q!} q^{k^2+\alpha k} x^k (\mathrm{D}_q \ominus_q \varepsilon)^k. \tag{43}$$

Proof. Use the q-Leibniz theorem. □

We will now give an operator expression for $\Omega_{n,q}^{(\alpha)}$ and an extension of Khan's q-analogue of this paper [46, p. 79]. It turns out that we obtain an equivalence class of six objects for each element in Carlitz's paper [12].

Theorem 2.5. *A q-analogue of Carlitz* [12, p. 219]. *All the products begin with $k = n$ and end with $k = 1$.*

$$\begin{aligned}
\Omega_{n,q}^{(\alpha)} &= \prod_{k=1}^{n} (q^{k+\alpha}(I+(1-q)\mathbf{x})\mathbf{x}\mathrm{D}_q + \{\alpha+k\}_q I - q^{k+\alpha}\mathbf{x}) \\
&= \prod_{k=1}^{n} (q^{k+\alpha}\mathbf{x}\mathrm{D}_q + \{\alpha+k\}_q I - q^{k+\alpha}\mathbf{x}\varepsilon) \\
&= \prod_{k=1}^{n} (\mathbf{x}\mathrm{D}_q + \{\alpha+k\}_q \varepsilon - q^{k+\alpha}\mathbf{x}\varepsilon) \\
&= \prod_{k=1}^{n} (q^{k+\alpha}(I+(1-q)\mathbf{x})\mathbf{x}\mathrm{D}_q + \{\alpha+k\}_q (I+(1-q)\mathbf{x}) - \mathbf{x}) \\
&= \prod_{k=1}^{n} ((I+(1-q)\mathbf{x})\mathbf{x}\mathrm{D}_q + \{\alpha+k\}_q (I+(1-q)\mathbf{x}) - \mathbf{x}) \\
&= \prod_{k=1}^{n} (\mathbf{x}\mathrm{D}_q + \{\alpha+k\}_q (I+(1-q)\mathbf{x}) - \mathbf{x}\varepsilon).
\end{aligned} \tag{44}$$

Proof. We will use (42). We only prove the first identity. The five others are proved in a similar way by permutation of the three functions involved in the q-differentiation. We will use [15, (13), p. 91] in the computations.

$$\begin{aligned}
\Omega_{n+1,q}^{(\alpha)} f(x) =& x^{-\alpha} \mathrm{E}_{\frac{1}{q}}(x) \mathrm{D}_q^n \left[(1+(1-q)x) q^{n+1+\alpha} \mathrm{E}_q(-x) x^{n+1+\alpha} \mathrm{D}_q \right. \\
&\left. + \{\alpha+n+1\}_q x^{n+\alpha} \mathrm{E}_q(-x) - (xq)^{n+1+\alpha} \mathrm{E}_q(-x) \right] f(x) \\
=& \Omega_{n,q}^{(\alpha)} \left[\{\alpha+n+1\}_q - xq^{n+1+\alpha} + (1+(1-q)x) q^{n+1+\alpha} x \mathrm{D}_q \right] f(x).
\end{aligned} \tag{45}$$

□

Remark 1. This was the first occasion where multiple q-analogues occurred because of the q-Leibniz theorem. We had three functions and got $\binom{3}{2}$ q-analogues.

Theorem 2.6. *A first q-analogue of* [12, (4), p. 219; 13, p. 246].

$$\Omega_{n,q}^{(\alpha)} f(x) = \{n\}_q! \sum_{k=0}^{n} \frac{x^k}{\{k\}_q!} L_{n-k,q}^{(\alpha+k)}(x) \varepsilon^{n-k} \mathrm{D}_q^k f(x). \tag{46}$$

Proof.

$$\Omega_{n,q}^{(\alpha)} f(x) = x^{-\alpha} \mathrm{E}_{\frac{1}{q}}(x) \sum_{k=0}^{n} \binom{n}{k}_q \mathrm{D}_q^{n-k}(x^{\alpha+n}\mathrm{E}_q(-x))\varepsilon^{n-k}\mathrm{D}_q^k f(x)$$

$$= x^{-\alpha} \mathrm{E}_{\frac{1}{q}}(x) \sum_{k=0}^{n} \binom{n}{k}_q x^{\alpha+k}\mathrm{E}_q(-x) L_{n-k,q,c}^{(\alpha+k)}(x)\varepsilon^{n-k}\mathrm{D}_q^k f(x) = RHS. \quad (47)$$

□

The following special case of (46) is a q-analogue of [12, (6), p. 220], see also [46, p. 79].

$$L_{n,q,c}^{(\alpha)}(x) = \Omega_{n,q}^{(\alpha)} 1. \quad (48)$$

The following formula is the first q-analogue of [12, (7), p. 221], the proof is the same.

$$\binom{m+n}{m}_q L_{m+n,q}^{(\alpha)}(x) = \sum_{k=0}^{\min(m,n)} \frac{(-x)^k}{\{k\}_q!} L_{m-k,q}^{(\alpha+n+k)}(x) q^{\alpha k+k^2} L_{n-k,q}^{(\alpha+k)}(xq^m). \quad (49)$$

Theorem 2.7. *A second q-analogue of* [12, (4), p. 219; 13, p. 246].

$$\Omega_{n,q}^{(\alpha)} f(x) = \{n\}_q! \sum_{k=0}^{n} \frac{x^k}{\{k\}_q!} q^{k(\alpha+k)}(-(1-q)x;q)_k L_{n-k,q}^{(\alpha+k)}(xq^k)\mathrm{D}_q^k f(x). \quad (50)$$

Proof. We will use Cigler [15, (13), p. 91] in the computations.

$$\Omega_{n,q}^{(\alpha)} f(x) = x^{-\alpha} \mathrm{E}_{\frac{1}{q}}(x) \sum_{k=0}^{n} \binom{n}{k}_q \varepsilon^k \mathrm{D}_q^{n-k}(x^{\alpha+n}\mathrm{E}_q(-x))\mathrm{D}_q^k f(x)$$

$$= x^{-\alpha} \mathrm{E}_{\frac{1}{q}}(x) \sum_{k=0}^{n} \binom{n}{k}_q \varepsilon^k \left[x^{\alpha+k}\mathrm{E}_q(-x) L_{n-k,q,c}^{(\alpha+k)}(x)\right] \mathrm{D}_q^k f(x)$$

$$= x^{-\alpha} \mathrm{E}_{\frac{1}{q}}(x) \sum_{k=0}^{n} \binom{n}{k}_q (xq^k)^{\alpha+k}\mathrm{E}_q(-xq^k) L_{n-k,q,c}^{(\alpha+k)}(xq^k)\mathrm{D}_q^k f(x) = RHS. \quad (51)$$

□

The following formula is the second q-analogue of [12, (7), p. 221].

$$\binom{m+n}{m}_q L_{m+n,q}^{(\alpha)}(x) = \sum_{k=0}^{\min(m,n)} \frac{(-x)^k}{\{k\}_q!}(-(1-q)x;q)_k$$

$$\times L_{m-k,q}^{(\alpha+n+k)}(xq^k) q^{k(2\alpha+n+2k)} L_{n-k,q}^{(\alpha+k)}(xq^k), \; n > 1, \; m \geq n. \quad (52)$$

An interesting consequence of (49) is the following q-analogue of [12, (10), p. 222].

Theorem 2.8.

$$\sum_{n=0}^{\infty} \binom{m+n}{m}_q L_{m+n,q}^{(\alpha-n)}(x) t^n q^{\binom{n}{2}-n\alpha} = L_{m,q}^{(\alpha)}(x \oplus_{q,-1} xtq^{-\alpha}) \frac{\mathrm{E}_{\frac{1}{q}}(-xtq^m)}{(-t;q)_{-\alpha}}. \quad (53)$$

Proof.

$$LHS=\sum_{n=0}^{\infty}\sum_{k=0}^{\min(m,n)}\frac{(-x)^k}{\{k\}_q!}L_{m-k,q}^{(\alpha+k)}(x)\mathrm{QE}\left((\alpha-n)k+\binom{k+1}{2}+\binom{k}{2}\right)L_{n-k,q}^{(\alpha-n+k)}(xq^m)t^nq^{\binom{n}{2}-n\alpha}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{m}\frac{(-x)^k}{\{k\}_q!}L_{m-k,q}^{(\alpha+k)}(x)q^{\frac{k^2-k}{2}}L_{n,q}^{(\alpha-n)}(xq^m)t^{n+k}q^{\binom{n}{2}-n\alpha}=RHS, \tag{54}$$

where we have used [20, 5.29, p. 28], a q-Taylor formula, and (77) in the last step. □

3. q-JACOBI POLYNOMIALS

We now come to the definition of q-Jacobi polynomials. In the literature there is a very similar so-called little q-Jacobi polynomial. We will however use the original definition, because it leads to a nice Rodriguez formula with corresponding orthogonality.

Definition 13. *A q-analogue of* [43, p. 192; 17, p. 76; 7; 6, p. 162; 45, p. 467; 28, p. 242, (1)].

$$P_{n,q}^{(\alpha,\beta)}(x)\equiv\frac{\langle 1+\alpha;q\rangle_n}{\langle 1;q\rangle_n}{}_2\phi_1(-n,\beta+n;1+\alpha|q,xq^{\alpha+1-\beta})$$

$$\equiv\frac{\langle 1+\alpha;q\rangle_n}{\langle 1;q\rangle_n}\sum_{k=0}^{n}\binom{n}{k}_q\frac{\langle\beta+n;q\rangle_k}{\langle 1+\alpha;q\rangle_k}(-x)^kq^{\binom{k}{2}+(\alpha+1-\beta-n)k}. \tag{55}$$

Theorem 3.1.

$$\lim_{\beta\to-\infty}P_{n,q}^{(\alpha,\beta)}(-x(1-q))=L_{n,q}^{(\alpha)}(x). \tag{56}$$

The following Rodriguez formula is a q-analogue of [43, p. 192, (7); 28, p. 242; 60, p. 220, (1.1); 8, p. 99].

Theorem 3.2. *Let* $x\in(0,|q^{\beta-\alpha-1}|)$. *Then*

$$P_{n,q}^{(\alpha,\beta)}(x)=\frac{x^{-\alpha}}{\{n\}_q!(xq^{\alpha+1-\beta};q)_{\beta-\alpha-1}}\mathrm{D}_q^n\left(\frac{x^{\alpha+n}}{(x;q)_{\alpha+1-\beta-n}}\right). \tag{57}$$

Proof. The q-Leibniz formula gives

$$RHS=\frac{x^{-\alpha}}{\{n\}_q!(xq^{-\beta+\alpha+1};q)_{\beta-\alpha-1}}\sum_{k=0}^{n}\frac{\langle 1;q\rangle_n\{1+\alpha-\beta-n\}_{k,q}\{1+\alpha+k\}_{n-k,q}x^{\alpha+k}q^{k\alpha+k^2}}{\langle 1;q\rangle_k\langle 1;q\rangle_{n-k}(x;q)_{-\beta+\alpha+k+1-n}}$$

$$=\sum_{k=0}^{n}\frac{x^k\langle 1+\alpha-\beta-n;q\rangle_k\langle 1+\alpha;q\rangle_nq^{k\alpha+k^2}}{\langle 1+\alpha;q\rangle_k(xq^{-\beta+\alpha+1};q)_{k-n}\langle 1;q\rangle_k\langle 1;q\rangle_{n-k}}$$

$$=\sum_{k=0}^{n}\frac{x^k\langle 1+\alpha-\beta-n,-n;q\rangle_k\langle 1+\alpha;q\rangle_nq^{k\alpha+k^2-\binom{k}{2}+nk}(-1)^k}{\langle 1,1+\alpha;q\rangle_k(xq^{-\beta+\alpha+1};q)_{k-n}\langle 1;q\rangle_n}$$

$$=\frac{\langle 1+\alpha;q\rangle_n}{\langle 1;q\rangle_n(xq^{-\beta+\alpha+1};q)_{-n}}$$

$$\times{}_2\phi_2\left(-n,-n-\beta+\alpha+1;\alpha+1|q,xq^{n+\alpha+1}||-;xq^{-n-\beta+\alpha+1}\right)=LHS.$$

The interval for x is chosen to make certain infinite products converge, compare [56, p. 300]. □

Corollary 3.3.

$$P_{n,q}^{(\alpha,\beta)}(xq^{\gamma}) = \frac{x^{-\alpha}}{\{n\}_q!(xq^{\alpha+\gamma+1-\beta};q)_{\beta-\alpha-1}} \mathrm{D}_q^n \left(\frac{x^{\alpha+n}}{(xq^{\gamma};q)_{\alpha+1-\beta-n}} \right). \tag{58}$$

Proof. Same as above. □

Corollary 3.4. *A function $F(x) \in \mathrm{H}_q$*

$$F(x) = \sum_{k=0}^{\infty} \frac{\alpha_k x^{\beta_k}}{(x;q)_{\gamma_k}}$$

has nth q-difference given by

$$\mathrm{D}_q^n F(x) = \sum_{k=0}^{\infty} \alpha_k P_{n,q}^{(\beta_k-n,\beta_k+1-2n-\gamma_k)}(x)\{n\}_q! \frac{x^{\beta_k-n}}{(x;q)_{\gamma_k+n}}, \tag{59}$$

$x \in (0, |q^{-n-\gamma_k}|)$, $\forall k$.

Definition 14. *The q-Jacobi Rodriguez operator is a q-analogue of Singh* [58, p. 238].

$$\Omega_{n,q}^{(\alpha,\beta)} f(x) \equiv \frac{x^{-\alpha}}{\{n\}_q!(xq^{\alpha+1-\beta};q)_{\beta-\alpha-1}} \mathrm{D}_q^n \left(\frac{x^{\alpha+n}}{(x;q)_{\alpha+1-\beta-n}} f(x) \right), \tag{60}$$

$f(x) \in \mathrm{H}_q, x \in (0, |q^{\beta-\alpha-1}|)$.

$$\Omega_{0,q}^{(\alpha,\beta)} \equiv I. \tag{61}$$

Theorem 3.5. *Almost a q-analogue of Singh* [58, 2.2, p. 238]. $\Theta_{k,q}^{(\alpha,\beta)}$ *is a bilinear function of* D_q *and* ε *with coefficients in the field of fractions of* $\mathbb{C}[\mathbf{x}]$.

$$\Omega_{n,q}^{(\alpha,\beta-n+1)} f(x) \cong \prod_{k=2}^{n} \left(\frac{1-xq^{\alpha-\beta+1}}{\{k\}_q} \right) \Omega_{1,q}^{(\alpha,\beta)} \prod_{k=2}^{n} \Theta_{k,q}^{(\alpha,\beta)} f(x),\ n \geq 1, \tag{62}$$

where $\Theta_{k,q}^{(\alpha,\beta)}$ *is given by one of the following six equivalent expressions.*

$$\begin{aligned}
\Theta_{k,q}^{(\alpha,\beta)} &\cong \frac{q^{k+\alpha}(I-\mathbf{x})}{I-\mathbf{x}q^{2-k+\alpha-\beta}} \mathbf{x}\mathrm{D}_q + \{\alpha+k\}_q I + \frac{q^{k+\alpha}\{2-k+\alpha-\beta\}_q}{I-\mathbf{x}q^{2-k+\alpha-\beta}} \mathbf{x} \\
&\cong \{\alpha+k\}_q I + \frac{q^{k+\alpha}\{2-k+\alpha-\beta\}_q}{I-\mathbf{x}q^{2-k+\alpha-\beta}} \mathbf{x}\varepsilon + q^{k+\alpha}\mathbf{x}\mathrm{D}_q \\
&\cong \mathbf{x}\mathrm{D}_q + \{\alpha+k\}_q \varepsilon + \frac{q^{k+\alpha}\{2-k+\alpha-\beta\}_q}{I-\mathbf{x}q^{2-k+\alpha-\beta}} \mathbf{x}\varepsilon \\
&\cong \frac{\{2-k+\alpha-\beta\}_q}{I-\mathbf{x}q^{2-k+\alpha-\beta}} \mathbf{x} + \frac{q^{k+\alpha}(I-\mathbf{x})}{I-\mathbf{x}q^{2-k+\alpha-\beta}} \mathbf{x}\mathrm{D}_q + \frac{\{\alpha+k\}_q(I-\mathbf{x})}{I-\mathbf{x}q^{2-k+\alpha-\beta}} \\
&\cong \frac{\{2-k+\alpha-\beta\}_q}{I-\mathbf{x}q^{2-k+\alpha-\beta}} \mathbf{x} + \frac{(I-\mathbf{x})}{I-\mathbf{x}q^{2-k+\alpha-\beta}} \mathbf{x}\mathrm{D}_q + \frac{\{\alpha+k\}_q(I-\mathbf{x})}{I-\mathbf{x}q^{2-k+\alpha-\beta}} \varepsilon \\
&\cong \mathbf{x}\mathrm{D}_q + \frac{\{\alpha+k\}_q(I-\mathbf{x})}{I-\mathbf{x}q^{2-k+\alpha-\beta}} \varepsilon + \frac{\{1+\alpha-\beta\}_q}{I-\mathbf{x}q^{1+\alpha-\beta}} \mathbf{x}\varepsilon.
\end{aligned} \tag{63}$$

Proof. We only prove the first identity for $\Theta_{k,q}^{(\alpha,\beta)}$. The five others are proved in a similar way by permutation of the three functions involved in the q-differentiation.

$$\begin{aligned}\Omega_{n+1,q}^{(\alpha,\beta-1)} f(x) \cong & \frac{x^{-\alpha}}{\{n+1\}_q!(xq^{\alpha+2-\beta};q)_{\beta-\alpha-2}} \mathrm{D}_q^n \\ & \times \left[\left[\frac{\{\alpha+n+1\}_q x^{\alpha+n}}{(x;q)_{1+\alpha-\beta-n}} + \frac{(xq)^{\alpha+n+1}\mathrm{D}_q}{(xq;q)_{1+\alpha-\beta-n}} + \frac{(xq)^{\alpha+n+1}\{\alpha+1-\beta-n\}_q}{(x;q)_{2+\alpha-\beta-n}} \right] f(x) \right] \\ \cong & \frac{1-xq^{\alpha+1-\beta}}{\{n+1\}_q} \Omega_{n,q}^{(\alpha,\beta)} \\ & \times \left[\left[\frac{(1-x)q^{n+1+\alpha}}{1-xq^{1+\alpha-\beta-n}} x\mathrm{D}_q + \{\alpha+n+1\}_q + \frac{xq^{n+1+\alpha}\{1+\alpha-\beta-n\}_q}{1-xq^{1+\alpha-\beta-n}} \right] f(x) \right]. \end{aligned} \tag{64}$$

The assertion now follows by induction. □

The following generalization of (46) is a first q-analogue of Singh [58, 2.3, p. 239], with the difference that in the present paper Jacobi's original polynomial definition is used.

Theorem 3.6.

$$\Omega_{n,q}^{(\alpha,\beta)} f(x) \cong \sum_{k=0}^{n} \frac{x^k}{\{k\}_q!} (xq^{\alpha+1-k-\beta};q)_k P_{n-k,q}^{(\alpha+k,\beta+2k)}(x) \varepsilon^{n-k} \mathrm{D}_q^k f(x). \tag{65}$$

Proof.

$$\begin{aligned}\Omega_{n,q}^{(\alpha,\beta)} f(x) \cong & \frac{x^{-\alpha}}{\{n\}_q!(xq^{\alpha+1-\beta};q)_{\beta-\alpha-1}} \sum_{k=0}^{n} \binom{n}{k}_q \mathrm{D}_q^{n-k} \left[\frac{x^{\alpha+n}}{(x;q)_{-\beta+\alpha+1-n}} \right] \varepsilon^{n-k} \mathrm{D}_q^k f(x) \\ \cong & \frac{1}{\{n\}_q!(xq^{\alpha+1-\beta};q)_{\beta-\alpha-1}} \sum_{k=0}^{n} \binom{n}{k}_q x^k P_{n-k,q}^{(\alpha+k,\beta+2k)}(x) \\ & \times \{n-k\}_q!(xq^{\alpha-k+1-\beta};q)_{\beta-\alpha+k-1} \varepsilon^{n-k} \mathrm{D}_q^k f(x) \cong RHS. \end{aligned} \tag{66}$$

□

Lemma 3.7.

$$\Omega_{n,q}^{(\alpha,\beta)} \frac{x}{1-xq^{1+\alpha-\beta-n}} \cong P_{n,q}^{(\alpha,\beta)}(x) \frac{xq^n}{1-xq^{1+\alpha-\beta}} + \sum_{k=1}^{n} x^k P_{n-k,q}^{(\alpha+k,\beta+2k)}(x) \frac{q^{(\alpha+1-\beta-n)(k-1)}}{1-xq^{1+\alpha-\beta}}. \tag{67}$$

Proof. Use (65) and (15). □

Theorem 3.8.

$$\begin{aligned} P_{n+1,q}^{(\alpha,\beta-1)}(x) \cong & \frac{1-xq^{\alpha+1-\beta}}{\{n+1\}_q} \{\alpha+n+1\}_q P_{n,q}^{(\alpha,\beta)}(x) \\ & + \frac{q^{\alpha+n+1}\{\alpha+1-\beta-n\}_q}{\{n+1\}_q} \left[\sum_{k=1}^{n} x^k P_{n-k,q}^{(\alpha+k,\beta+2k)}(x) q^{(\alpha+1-\beta-n)(k-1)} + P_{n,q}^{(\alpha,\beta)}(x) xq^n \right]. \end{aligned} \tag{68}$$

Proof. Apply (64) to 1 and use (67). □

Corollary 3.9.

$$L_{n+1,q}^{(\alpha)}(x) \cong \frac{1}{\{n+1\}_q} \times [\{\alpha+n+1\}_q L_{n,q}^{(\alpha)}(x) - q^{n+\alpha+1}[xL_{n-1,q}^{(\alpha+1)}(x) + q^n x L_{n,q}^{(\alpha)}(x)]]. \tag{69}$$

Proof.

$$\begin{aligned} LHS = \lim_{\beta\to-\infty} \frac{1+x(1-q)q^{\alpha+1-\beta}}{\{n+1\}_q} \Bigg[&\{\alpha+n+1\}_q P_{n,q}^{(\alpha,\beta)}(-x(1-q)) \\ &+ q^{\alpha+n+1}\{\alpha+1-\beta-n\}_q \Bigg[\sum_{k=1}^{n} (-x(1-q))^k P_{n-k,q}^{(\alpha+k,\beta+2k)}(-x(1-q)) \\ &\times \frac{q^{(\alpha+1-\beta-n)(k-1)}}{1+x(1-q)q^{1+\alpha-\beta}} + P_{n,q}^{(\alpha,\beta)}(-x(1-q)) \frac{-x(1-q)q^n}{1+x(1-q)q^{1+\alpha-\beta}} \Bigg] \Bigg] = RHS. \end{aligned} \tag{70}$$

□

The following generalization of (50) is the second q-analogue of Singh [58, 2.3, p. 239].

Theorem 3.10.

$$\Omega_{n,q}^{(\alpha,\beta)} f(x) \cong \sum_{k=0}^{n} \frac{x^k}{\{k\}_q!} q^{k(\alpha+k)} (x;q)_k P_{n-k,q}^{(\alpha+k,\beta+2k)}(xq^k) \mathrm{D}_q^k f(x). \tag{71}$$

Proof.

$$\Omega_{n,q}^{(\alpha,\beta)} f(x) \cong \frac{x^{-\alpha}}{\{n\}_q!(xq^{\alpha+1-\beta};q)_{\beta-\alpha-1}} \sum_{k=0}^{n} \binom{n}{k}_q \varepsilon^k \left[\mathrm{D}_q^{n-k} \left[\frac{x^{\alpha+n}}{(x;q)_{-\beta+\alpha+1-n}} \right] \right] \mathrm{D}_q^k f(x) \cong RHS. \tag{72}$$

□

Lemma 3.11.

$$\begin{aligned} \Omega_{n,q}^{(\alpha,\beta)} \frac{x}{1-xq^{1+\alpha-\beta-n}} \cong & P_{n,q}^{(\alpha,\beta)}(x) \frac{x}{1-xq^{1+\alpha-\beta-n}} \\ & + \sum_{k=1}^{n} x^k q^{k(\alpha+k)} P_{n-k,q}^{(\alpha+k,\beta+2k)}(xq^k) \frac{q^{(\alpha+1-\beta-n)(k-1)}(x;q)_k}{(xq^{\alpha+1-\beta-n});q)_{k+1}}. \end{aligned} \tag{73}$$

Proof. Use (71) and (15). □

Theorem 3.12.

$$\begin{aligned} \frac{\{n+1\}_q}{1-xq^{\alpha+1-\beta}} P_{n+1,q}^{(\alpha,\beta-1)}(x) \cong & \{\alpha+n+1\}_q P_{n,q}^{(\alpha,\beta)}(x) + q^{\alpha+n+1}\{\alpha+1-\beta-n\}_q \\ \times & \left(P_{n,q}^{(\alpha,\beta)}(x) \frac{x}{1-xq^{\alpha+1-\beta-n}} + \sum_{k=1}^{n} x^k q^{k(\alpha+k)} (x;q)_k P_{n-k,q}^{(\alpha+k,\beta+2k)}(xq^k) \frac{q^{(\alpha+1-\beta-n)(k-1)}}{(xq^{\alpha+1-\beta-n};q)_{k+1}} (x;q)_k \right). \end{aligned} \tag{74}$$

Proof. Apply (64) to 1 and use (73). □

Theorem 3.13. *A variation of the Rodriguez formula.*

$$P_{n,q}^{(\alpha,\beta)}(x) = \frac{x^{-\alpha-n-1}}{\{n\}_q!(xq^{\alpha+1-\beta};q)_{\beta-\alpha-1}}(x^2 D_q)^n \left(\frac{x^{\alpha+1}}{(x;q)_{\alpha+1-\beta-n}}\right). \tag{75}$$

Proof. This follows from a q-analogue of [60, p. 220]. □

The limit to q-Laguerre polynomials for the above equation leads to [20, 6.11, p. 31].

The second of the following equations shows that the operator $\mathrm{D}_q\varepsilon^{-1}$ keeps the same function argument, while D_q does not. This will be important in future applications.

$$\mathrm{D}_q^m P_{n,q}^{(\alpha,\beta)}(x) = P_{n-m,q}^{(\alpha+m,\beta+m)}(xq^m)\frac{(-1)^m\langle\beta+n;q\rangle_m}{(1-q)^m}\mathrm{QE}\left(\binom{m}{2}+m(\alpha+1-\beta-n)\right). \tag{76}$$

$$(\mathrm{D}_q\varepsilon^{-1})^m P_{n,q}^{(\alpha,\beta)}(x) = P_{n-m,q}^{(\alpha+m,\beta+m)}(x)\frac{(-1)^m\langle\beta+n;q\rangle_m}{(1-q)^m}\mathrm{QE}(m(\alpha-\beta-n)). \tag{77}$$

Theorem 3.14. *The first q-analogue of Manocha and Sharma* [52, (6), p. 459] *and Feldheim* [29, p. 134].

$$\begin{aligned}\binom{m+n}{m}_q (xq^{\alpha+1+n-\beta};q)_{\beta-n-\alpha-1}P_{m+n,q}^{(\alpha,\beta-n)}(x) \cong \sum_{k=0}^{\min(m,n)} \frac{P_{n-k,q}^{(\alpha+k,\beta+2k-n)}(x)x^k}{\{k\}_q!(x;q)_{\alpha+1+n-\beta-k}} \\ \times P_{m-k,q}^{(\alpha+n+k,\beta+n+k)}(xq^n)(-1)^k\frac{\langle\beta+n+m;q\rangle_k}{(1-q)^k}\mathrm{QE}\left(\binom{k}{2}+k(\alpha+1-\beta-m)\right).\end{aligned} \tag{78}$$

Proof.

$$\begin{aligned}P_{m+n,q}^{(\alpha,\beta-n)}(x) &\cong \frac{x^{-\alpha}}{\{m+n\}_q!(xq^{\alpha+1-\beta+n};q)_{\beta-\alpha-1-n}}\mathrm{D}_q^{m+n}\left(\frac{x^{\alpha+m+n}}{(x;q)_{\alpha+1-\beta-m}}\right) \\ &\cong \frac{x^{-\alpha}\{m\}_q!}{\{m+n\}_q!(xq^{\alpha+1-\beta+n};q)_{\beta-\alpha-1-n}}\mathrm{D}_q^n[x^{\alpha+n}P_{m,q}^{(\alpha+n,\beta+n)}(x)(xq^{\alpha+1-\beta};q)_{\beta-\alpha-1}] \\ &\cong \frac{\{m\}_q!}{\{m+n\}_q!(xq^{\alpha+1-\beta+n};q)_{\beta-\alpha-1-n}}\sum_{k=0}^{n}\binom{n}{k}_q \\ &\quad\times \frac{P_{n-k,q}^{(\alpha+k,\beta+2k-n)}(x)x^k\{n-k\}_q!}{(x;q)_{\alpha+1-\beta-k+n}}\varepsilon^{n-k}\mathrm{D}_q^k P_{m,q}^{(\alpha+n,\beta+n)}(x) \\ &\overset{\text{by}(76)}{\cong} \frac{\{m\}_q!\{n\}_q!}{\{m+n\}_q!(xq^{\alpha+1-\beta+n};q)_{\beta-\alpha-1-n}}\sum_{k=0}^{\min(m,n)}\frac{P_{n-k,q}^{(\alpha+k,\beta+2k-n)}(x)x^k}{\{k\}_q!(x;q)_{\alpha+1-\beta-k+n}} \\ &\quad\times P_{m-k,q}^{(\alpha+n+k,\beta+n+k)}(xq^n)(-1)^k\frac{\langle\beta+n+m;q\rangle_k}{(1-q)^k}\mathrm{QE}\left(\binom{k}{2}+k(\alpha+1-\beta-m)\right).\end{aligned} \tag{79}$$

□

Theorem 3.15. *The second q-analogue of Manocha and Sharma* [52, (6), p. 459] *and Feldheim* [29, p. 134].

$$\binom{m+n}{m}_q (xq^{\alpha+1+n-\beta};q)_{\beta-n-\alpha-1} P_{m+n,q}^{(\alpha,\beta-n)}(x)$$
$$\cong \sum_{k=0}^{\min(m,n)} \frac{P_{n-k,q}^{(\alpha+k,\beta+2k-n)}(xq^k)x^k}{\{k\}_q!(xq^k;q)_{\alpha+1+n-\beta-k}} P_{m-k,q}^{(\alpha+n+k,\beta+n+k)}(xq^k)(-1)^k$$
$$\times \frac{\langle \beta+n+m;q\rangle_k}{(1-q)^k} \mathrm{QE}\left(\binom{k}{2}+k(\alpha+1-\beta-m)+k+\alpha\right). \tag{80}$$

In the limit we obtain the following third q-analogue of [12, (7), p. 221].

$$\binom{m+n}{m}_q L_{m+n,q}^{(\alpha)}(x) = \sum_{k=0}^{\min(m,n)} \frac{(-x)^k}{\{k\}_q!} L_{m-k,q}^{(\alpha+n+k)}(xq^n) \mathrm{QE}\left(k(n+\alpha+1)+2\binom{k}{2}\right) L_{n-k,q}^{(\alpha+k)}(x). \tag{81}$$

The following two formulas are q-analogues of Manocha and Sharma [52, (9), p. 460].

Theorem 3.16.

$$P_{n,q}^{(\alpha+\gamma,\beta+\delta)}(x) \cong \sum_{k=0}^{n} q^{(n-k)(\gamma-k)} P_{n-k,q}^{(\alpha+k,\beta+k)}(x) P_{k,q}^{(\gamma-k,1+\delta-k)}(xq^{\alpha+1-\beta}). \tag{82}$$

Proof.

$$LHS \cong \frac{x^{-\alpha-\gamma}}{\{n\}_q!(xq^{\alpha+1+\gamma-\beta-\delta};q)_{\beta+\delta-\alpha-\gamma-1}} \mathrm{D}_q^n\left(\frac{x^{\alpha+\gamma+n}}{(x;q)_{\alpha+\gamma+1-\beta-\delta-n}}\right)$$
$$\cong \frac{x^{-\alpha-\gamma}}{\{n\}_q!(xq^{\alpha+1+\gamma-\beta-\delta};q)_{\beta+\delta-\alpha-\gamma-1}} \sum_{k=0}^{n} \binom{n}{k}_q \mathrm{D}_q^{n-k}\left(\frac{x^{\alpha+n}}{(x;q)_{\alpha+1-\beta-n+k}}\right)$$
$$\times \varepsilon^{n-k}\mathrm{D}_q^k\left(\frac{x^\gamma}{(xq^{\alpha+1-\beta-n+k};q)_{\gamma-\delta-k}}\right) \overset{\text{by}(58)}{\cong} \frac{x^{-\alpha-\gamma}}{(xq^{\alpha+1+\gamma-\beta-\delta};q)_{\beta+\delta-\alpha-\gamma-1}}$$
$$\times \sum_{k=0}^{n} x^{\alpha+k}(xq^{\alpha+1-\beta};q)_{\beta-\alpha-1} P_{n-k,q}^{(\alpha+k,\beta+k)}(x)$$
$$\times \varepsilon^{n-k}\left[x^{\gamma-k}(xq^{\gamma+\alpha-\beta-\delta+1-n+k};q)_{\delta-\gamma} P_{k,q}^{(\gamma-k,1+\delta-k)}(xq^{\alpha+1-\beta-n+k})\right] \cong RHS. \tag{83}$$

□

Theorem 3.17.

$$P_{n,q}^{(\alpha+\gamma,\beta+\delta)}(x) \cong \sum_{k=0}^{n} q^{k(\alpha+k)} P_{n-k,q}^{(\alpha+k,\beta+k)}(xq^k) P_{k,q}^{(\gamma-k,1+\delta-k)}(xq^{\alpha+1-\beta-n+k})$$
$$\times \frac{(x;q)_k}{(xq^{\alpha+1-\beta+\gamma-\delta};q)_{-n+k}(xq^{\alpha+1-\beta-n+k};q)_n}. \tag{84}$$

Proof. A slight modification of the previous proof. □

4. ORTHOGONALITY

In this chapter we consider orthogonality relations for q-Jacobi, q-Laguerre, and q-Legendre polynomials. The proofs will all use q-integration by parts, a method equivalent to the previously used recurrence technique. The orthogonality relations are all of discrete type, a well-known phenomenon.

Theorem 4.1.

$$\int_0^{q^{\beta-\alpha-1}} P_{n,q}^{(\alpha,\beta)}(x)P_{m,q}^{(\alpha,\beta)}(x)x^{\alpha}\{n\}_q!(xq^{-\beta+\alpha+1};q)_{\beta-\alpha-1}\,d_q(x)$$
$$=\delta(m,n)\frac{\langle\beta+n;q\rangle_n}{(1-q)^n}\mathrm{QE}((1+\alpha)(-\alpha+\beta+n))B_q(\beta-\alpha+n,\alpha+1+n). \quad (85)$$

Proof. q-Integration by parts gives

$$\int_0^{q^{\beta-\alpha-1}} P_{n,q}^{(\alpha,\beta)}(x)P_{m,q}^{(\alpha,\beta)}(x)x^{\alpha}\{n\}_q!(xq^{-\beta+\alpha+1};q)_{\beta-\alpha-1}\,d_q(x)$$
$$=\int_0^{q^{\beta-\alpha-1}} \mathrm{D}_q^n\left(\frac{x^{\alpha+n}}{(x;q)_{\alpha+1-\beta-n}}\right)P_{m,q}^{(\alpha,\beta)}(x)\,d_q(x)$$
$$=\left[\mathrm{D}_q^{n-1}\left(\frac{x^{\alpha+n}}{(x;q)_{\alpha+1-\beta-n}}\right)\varepsilon^{-1}P_{m,q}^{(\alpha,\beta)}(x)\right]_0^{q^{\beta-\alpha-1}}$$
$$-\int_0^{q^{\beta-\alpha-1}} \mathrm{D}_q^{n-1}\left(\frac{x^{\alpha+n}}{(x;q)_{\alpha+1-\beta-n}}\right)\mathrm{D}_q\varepsilon^{-1}P_{m,q}^{(\alpha,\beta)}(x)\,d_q(x)=\ldots$$
$$=\sum_{l=1}^{n}(-1)^{l+1}\left[\mathrm{D}_q^{n-l}\left(\frac{x^{\alpha+n}}{(x;q)_{\alpha+1-\beta-n}}\right)(\varepsilon^{-1}\mathrm{D}_q)^{l-1}\varepsilon^{-1}P_{m,q}^{(\alpha,\beta)}(x)\right]_0^{q^{\beta-\alpha-1}}$$
$$+(-1)^n\int_0^{q^{\beta-\alpha-1}} x^{\alpha+n}(xq^{-\beta+\alpha+1-n};q)_{\beta+n-\alpha-1}(\mathrm{D}_q\varepsilon^{-1})^n[P_{m,q}^{(\alpha,\beta)}(x)]\,d_q(x)$$
$$=\sum_{l=1}^{n}\left[(-1)^{l+1}\sum_{k=0}^{n-l}\left(\binom{n-l}{k}_q\frac{\{-\beta-n+\alpha+1\}_{k,q}\{\alpha+1+k+l\}_{n-l-k,q}x^{\alpha+k+l}}{(xq^{n-k-l};q)_{\alpha+1-\beta+k-n}}\right)\right.$$
$$\left.\times(\varepsilon^{-1}\mathrm{D}_q)^{l-1}\varepsilon^{-1}P_{m,q}^{(\alpha,\beta)}(x)\right]_0^{q^{\beta-\alpha-1}}$$
$$+(-1)^n\int_0^{q^{\beta-\alpha-1}} x^{\alpha+n}(xq^{-\beta-n+\alpha+1};q)_{\beta-\alpha-1}(\mathrm{D}_q\varepsilon^{-1})^n[P_{m,q}^{(\alpha,\beta)}(x)]\,d_q(x)=RHS.$$

The q-integral can be computed as follows.

$$\int_0^{q^{\beta-\alpha-1}} x^{\alpha+n}(xq^{-\beta+\alpha+1-n};q)_{\beta-\alpha-1+n}\,d_q(x)$$
$$= q^{\beta-\alpha-1}(1-q)\sum_{m=n+1}^{\infty}\langle m-n;q\rangle_{n-\alpha+\beta-1}q^{(n+\alpha)(-\alpha+\beta+m-1)+m}$$
$$= q^{\beta-\alpha-1}(1-q)\sum_{m=n+1}^{\infty}\frac{\langle m-n;q\rangle_\infty}{\langle m-\alpha+\beta-1;q\rangle_\infty}q^{(n+\alpha)(-\alpha+\beta+m-1)+m}$$
$$= q^{\beta-\alpha-1}(1-q)\sum_{l=0}^{\infty}\frac{\langle l+1;q\rangle_\infty}{\langle l+n-\alpha+\beta;q\rangle_\infty}q^{(n+\alpha)(-\alpha+\beta+l+n)+l+1+n}$$
$$= q^{\beta-\alpha-1}(1-q)\sum_{l=0}^{\infty}\frac{\langle 1;q\rangle_\infty\langle n-\alpha+\beta;q\rangle_l}{\langle n-\alpha+\beta;q\rangle_\infty\langle 1;q\rangle_l}q^{(n+\alpha)(-\alpha+\beta+l+n)+l+1+n}$$
$$= q^{(\beta-\alpha+n)(n+\alpha+1)}(1-q)\frac{\langle 1,2n+\beta+1;q\rangle_\infty}{\langle n-\alpha+\beta,n+\alpha+1;q\rangle_\infty}$$
$$= B_q(\beta-\alpha+n,\alpha+1+n)q^{(\beta-\alpha+n)(n+\alpha+1)}. \tag{86}$$

□

The orthogonality for q-Laguerre polynomials has a weight function which consists of x^α times a q-exponential function with negative function argument. This q-exponential function can also be written as an inverse q-shifted factorial $\frac{1}{(x(1-q);q)_\infty}$.

We can use the definition of $\mathrm{E}_q(-x)$ for $x < \frac{1}{1-q}$. For larger x we use the inverse q-shifted factorial formula. We now prove an inequality for a q-exponential function. In the recent past, several research papers were published presenting inequalities for various q-functions. In particular, there exist numerous inequalities for the q-gamma function, e.g. [4,5,9,33,34,37,38,47,50,51,54,57].

Theorem 4.2. *An inequality for* $\mathrm{E}_q(-x)$.

$$\mathrm{E}_q(-x) > e^{-x}, 0<q<1,\ \ x>0. \tag{87}$$

Proof. Denote

$$P_N \equiv \prod_{k=0}^{N}\frac{1}{1+x(1-q)q^k}. \tag{88}$$

Then

$$P_N > \exp(-\sum_{k=0}^{N}x(1-q)q^k) = \exp(-x(1-q^N)). \tag{89}$$

Now

$$\mathrm{E}_q(-x) = \lim_{N\to\infty}P_N > e^{-x}. \tag{90}$$

□

To do the complete proof of the following theorem, we need a formula for a certain q-integral.

Lemma 4.3. *Compare Jackson* [40, p. 200, (22)]. *The moments of order n for the q-Laguerre weight function are given by*

$$\int_0^\infty x^{\alpha+n}\mathrm{E}_q(-x)\,d_q(x)=\mathrm{QE}\left(-\binom{n+\alpha+1}{2}\right)\Gamma_q(n+\alpha+1). \tag{91}$$

Since the Stieltje moment problem for q-Laguerre polynomials is indeterminate, there are many orthogonality relations. One of these is the following.

Theorem 4.4. *A q-analogue of* [61, p. 214, (1.6)]. *Let* Re $\alpha>-1$. *Then*

$$\int_0^\infty L_{n,q}^{(\alpha)}(x)L_{m,q}^{(\alpha)}(x)x^\alpha\mathrm{E}_q(-x)\,d_q(x)=\delta(m,n)\frac{1}{\{n\}_q!}\mathrm{QE}\left(\binom{n}{2}+n\alpha-\binom{n+\alpha+1}{2}\right)\Gamma_q(n+\alpha+1). \tag{92}$$

Proof. Assume that $n\geq m$. q-Integration by parts gives

$$\begin{aligned}
&\int_0^\infty L_{n,q}^{(\alpha)}(x)L_{m,q}^{(\alpha)}(x)x^\alpha\{n\}_q!\mathrm{E}_q(-x)\,d_q(x)\\
&\quad=\int_0^\infty L_{m,q}^{(\alpha)}(x)\mathrm{D}_q^n(x^{\alpha+n}\mathrm{E}_q(-x))\,d_q(x)\\
&\quad=[\varepsilon^{-1}L_{m,q}^{(\alpha)}(x)\mathrm{D}_q^{n-1}(x^{\alpha+n}\mathrm{E}_q(-x))]_0^\infty-\int_0^\infty(\mathrm{D}_q\varepsilon^{-1})(L_{m,q}^{(\alpha}(x))\mathrm{D}_q^{n-1}(x^{\alpha+n}\mathrm{E}_q(-x))\,d_q(x)=\ldots\\
&\quad=\sum_{l=1}^n(-1)^{l+1}[(\varepsilon^{-1}\mathrm{D}_q)^{l-1}(\varepsilon^{-1}L_{m,q}^{(\alpha)}(x))\mathrm{D}_q^{n-l}(x^{\alpha+n}\mathrm{E}_q(-x))]_0^\infty\\
&\qquad+(-1)^n\int_0^\infty(\mathrm{D}_q\varepsilon^{-1})^n[L_{m,q}^{(\alpha)}(x)]x^{\alpha+n}\mathrm{E}_q(-x)\,d_q(x)\\
&\quad=\sum_{l=1}^n\Bigg[(-1)^{l+1}\sum_{k=0}^{n-l}\binom{n-l}{k}_q\{\alpha+1+k+l\}_{n-l-k,q}x^{\alpha+k+l}(-1)^k\\
&\qquad\times q^{k(\alpha+k+l)}\mathrm{E}_q(-x)(\varepsilon^{-1}\mathrm{D}_q)^{l-1}\varepsilon^{-1}L_{m,q}^{(\alpha)}(x)\Bigg]_0^\infty+(-1)^n\int_0^\infty(\mathrm{D}_q\varepsilon^{-1})^n[L_{m,q}^{(\alpha)}(x)]x^{\alpha+n}\mathrm{E}_q(-x)\,d_q(x)\\
&\quad=\delta(m,n)\mathrm{QE}\left(\binom{n}{2}+n\alpha\right)\int_0^\infty x^{\alpha+n}\mathrm{E}_q(-x)\,d_q(x).
\end{aligned}$$

Finally use the lemma to complete the proof. □

The following polynomial is defined by the Rodriguez formula to enable an easy orthogonality relation. q-Legendre polynomials have been given before, but these do not have the same orthogonality range in the limit $q\to1$ as in the classical case. To be able to treat orthogonality properly, we only consider the Rodriguez formula.

Definition 15. *The q-Legendre polynomial is defined by*

$$P_{n,q}(x)\equiv\frac{q^{-\binom{n}{2}}(-1)^n}{\{n\}_q!(1\boxplus_q q^{-n})^n}\mathrm{D}_q^n((1\boxminus_q x)^n(1\boxplus_q x)^n)\,. \tag{93}$$

This implies

Theorem 4.5. *An explicit combinatorial formula for q-Legendre polynomials*:

$$P_{n,q}(x) = \frac{q^{-\binom{n}{2}}(-1)^n}{\{n\}_q!(1 \boxplus_q q^{-n})^n} \sum_{k=0}^{n} \binom{n}{k}_q \{n-k+1\}_{k,q}\, q^{\binom{k}{2}}(-1)^k$$
$$\times (1 \boxminus_q q^n x)^{n-k} q^{\binom{n-k}{2}} (1 \boxplus_q q^{n-k} x)^k \{k+1\}_{n-k,q}. \tag{94}$$

Proof. We use the following lemma: □

Lemma 4.6.

$$\mathrm{D}_q^k (1 \boxplus_q x)^l = \{l-k+1\}_{k,q} q^{\binom{k}{2}} (1 \boxplus_q q^k x)^{l-k},\ l \geq k. \tag{95}$$

Theorem 4.7. *For simplicity we put*

$$\widetilde{P_{n,q}(x)} \equiv \mathrm{D}_q^n \left((1 \boxminus_q x)^n (1 \boxplus_q x)^n\right). \tag{96}$$

Orthogonality relation for q-Legendre polynomials:

$$\int_{-q^{1-m}}^{q^{1-m}} \widetilde{P_{m,q}(x)} \widetilde{P_{n,q}(x)}\, d_q(x)$$
$$= \delta(m,n)(-1)^n \int_{-q^{1-m}}^{q^{1-m}} (1 \boxminus_q x)^m (1 \boxplus_q x)^n (\mathrm{D}_q \varepsilon^{-1})^n \widetilde{P_{n,q}(x)}\, d_q(x),\ n \geq m. \tag{97}$$

Proof. q-Integration by parts gives

$$\int_{-q^{1-m}}^{q^{1-m}} \widetilde{P_{m,q}(x)} \widetilde{P_{n,q}(x)}\, d_q(x) = \int_{-q^{1-m}}^{q^{1-m}} \mathrm{D}_q^m \left((1 \boxminus_q x)^m (1 \boxplus_q x)^m\right) \widetilde{P_{n,q}(x)}\, d_q(x)$$
$$= \left[\mathrm{D}_q^{m-1} \left((1 \boxminus_q x)^m (1 \boxplus_q x)^m\right) \widetilde{P_{n,q}(xq^{-1})}\right]_{-q^{1-m}}^{q^{1-m}}$$
$$- \int_{-q^{1-m}}^{q^{1-m}} \mathrm{D}_q^{m-1} \left((1 \boxminus_q x)^m (1 \boxplus_q x)^m\right) \mathrm{D}_q \widetilde{P_{n,q}(xq^{-1})}\, d_q(x) = \ldots$$
$$= \sum_{k=1}^{n} (-1)^{k+1} \left[\sum_{l=0}^{m-k} \binom{m-k}{l}_q \prod_{j=0}^{l-1} \{m-j\}_q\, q^{\binom{l}{2}} (-1)^l (1 \boxminus_q q^{m-k} x)^{m-l} q^{\binom{m-l-k}{2}} \right.$$
$$\left. \times (1 \boxplus_q q^{m-l-k} x)^{k+l} \prod_{j=0}^{m-l-k-1} \{m-j\}_q (\varepsilon^{-1} \mathrm{D}_q)^{k-1} \varepsilon^{-1} \widetilde{P_{n,q}(x)} \right]_{-q^{1-m}}^{q^{1-m}}$$
$$+ \delta(m,n)(-1)^n \int_{-q^{1-m}}^{q^{1-m}} \sum_{k=0}^{m-n} \binom{m-n}{k}_q \prod_{l=0}^{k-1} \{m-l\}_q\, q^{\binom{k}{2}} (-1)^k (1 \boxminus_q q^{m-n} x)^{m-k}$$
$$\times q^{\binom{m-n-k}{2}} (1 \boxplus_q q^{m-n-k} x)^{k+n} \prod_{l=0}^{m-n-k-1} \{m-l\}_q (\mathrm{D}_q \varepsilon^{-1})^n \widetilde{P_{n,q}(x)}\, d_q(x).$$

All terms disappear when $n > m$. □

Theorem 4.8. *The q-Legendre polynomials $P_{n,q}(x)$ for small index are solutions of the following q-difference equations*:

$$(1-x^2)\mathrm{D}_q^2 f(x,q) - \{2\}_q x\mathrm{D}_q f(x,q) + \{2\}_q f(x,q) = 0 \tag{98}$$

has the solution $f(x,q) = P_{1,q}(x)$.

$$(1-x^2q^2)\mathrm{D}_q^2 f(x,q) - q^3\{2\}_q x\mathrm{D}_q f(x,q) + q^2\{3\}_q! f(x,q) = 0 \tag{99}$$

has the solution $f(x,q) = P_{2,q}(x)$.

$$(1-x^2q^6)\mathrm{D}_q^2 f(x,q) - q^3\{2\}_q x\mathrm{D}_q f(x,q) + q^3\{3\}_q(\{2\}_q)^2(q^2-q+1)f(x,q) = 0 \tag{100}$$

has the solution $f(x,q) = P_{3,q}(x)$.

Theorem 4.9. *The function $P_{n,q}(x)$ is the solution of the following linear second-order q-difference equation with the initial value $f(q^{-n}) = 1$:*

$$(x^2q^{2n+2}-1)\mathrm{D}_q^2 f(x,q) + q^n(\{2\}_q\{n+1\}_q - q\{2n\}_q)x\mathrm{D}_q f(x,q) - q^n\{n\}_q\{n+1\}_q f(x,q) = 0. \tag{101}$$

Proof. A q-analogue of [17, p. 73]. Let

$$u = (-1)^n(x^2;q^2)_n, \tag{102}$$

then

$$(x^2-1)\mathrm{D}_q u = \{2n\}_q xu. \tag{103}$$

Operate with D_q^{n+1} on this, and use the q-Leibniz theorem to obtain (101). □

5. SYSTEMS OF PARTIAL q-DIFFERENCE EQUATIONS FOR THE q-APPELL AND q-LAURICELLA FUNCTIONS

In 1880 Paul Emile Appell (1855–1930) [8] introduced some 2-variable hypergeometric series now called the Appell functions.

They have the following q-analogues [41,42]. The convergence area in the x_1 x_2 plane is slightly larger than for the corresponding Appell functions. The convergence areas given are those for $q = 1$.

$$\Phi_1(a;b,b';c|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}} x_1^{m_1}x_2^{m_2}, \quad \max(|x_1|,|x_2|) < 1. \tag{104}$$

$$\Phi_2(a;b,b';c,c'|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1}\langle c';q\rangle_{m_2}} x_1^{m_1}x_2^{m_2}, \quad |x_1|+|x_2| < 1. \tag{105}$$

$$\Phi_3(a,a';b,b';c|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1}\langle a';q\rangle_{m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}} x_1^{m_1}x_2^{m_2}, \quad \max(|x_1|,|x_2|) < 1. \tag{106}$$

$$\Phi_4(a;b;c,c'|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1+m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1}\langle c';q\rangle_{m_2}} x_1^{m_1}x_2^{m_2}, \quad |\sqrt{x_1}|+|\sqrt{x_2}| < 1. \tag{107}$$

Definition 16. *Partial q-derivatives are denoted* $\mathrm{D}^2_{q,i,j}$ *etc. Let* $\{\theta_i\}_q \equiv x_i\mathrm{D}_{q,i}$. *The following inverse pair of symbolic operators defined in* [21,41] *will be used in some of the computations.*

$$\triangledown_q(h) \equiv \Gamma_q\left[\begin{array}{c} h, h+\{\theta_1\}_q+\{\theta_2\}_q \\ h+\{\theta_1\}_q, h+\{\theta_2\}_q \end{array}\right],\ \triangle_q(h) \equiv \Gamma_q\left[\begin{array}{c} h+\{\theta_1\}_q, h+\{\theta_2\}_q \\ h+\{\theta_1\}_q+\{\theta_2\}_q, h \end{array}\right]. \tag{108}$$

In this chapter we are going to find q-difference equations for q-Appell and q-Lauricella functions. So as a preliminary lemma we need the q-difference equations for a ${}_2\phi_1$ q-hypergeometric series.

Lemma 5.1. *The series* ${}_2\phi_1(a,b;c|q,x)$ *satisfies the q-difference equation due to Heine*:

$$x(q^c - xq^{a+b+1})\mathrm{D}^2_q + \left[\{c\}_q - (\{a\}_q q^b + \{b\}_q q^a + q^{a+b})x\right]\mathrm{D}_q - \{a\}_q\{b\}_q I = 0. \tag{109}$$

Proof. The q-difference equation can be written

$$-x\{\theta+a\}_q\{\theta+b\}_q + \{\theta\}_q\{\theta+c-1\}_q = 0. \tag{110}$$

This can be restated as

$$\begin{aligned} & -x(q^a\{\theta\}_q+\{a\}_q)(q^b\{\theta\}_q+\{b\}_q)+\{\theta\}_q(q^c\{\theta-1\}_q+\{c\}_q) \\ = & -xq^{a+b}(qx^2\mathrm{D}^2_q+x\mathrm{D}_q)-x^2\mathrm{D}_q(\{a\}_q q^b+\{b\}_q q^a)-x\{a\}_q\{b\}_q+\{\theta\}_q(q^{c-1}\{\theta\}_q-q^{c-1}+\{c\}_q) \\ = & -x^3q^{a+b+1}\mathrm{D}^2_q-x^2q^{a+b}\mathrm{D}_q-x^2\mathrm{D}_q(\{a\}_q q^b+\{b\}_q q^a)-x\{a\}_q\{b\}_q+q^c x^2\mathrm{D}^2_q+x\mathrm{D}_q\{c\}_q=0. \end{aligned} \tag{111}$$

□

There is also a third form [59, p. 11], which is presented for the generalized series ${}_p\phi_{p-1}(a_1,\ldots,a_p;b_1,\ldots,b_{p-1}|q,z)$. We have put $b_p = 1$, and e_k = elementary symmetric polynomial.

$$\sum_{k=0}^{p}(-1)^k(e_k(q^{b_i})q^{-k}-e_k(q^{a_i})x)f(q^kx)=0. \tag{112}$$

Some of the following equations appeared in different form and different notation in [41, p. 79–80]. The partial q-difference equations for the q-Appell functions

$$\Phi_1(a;b,b';c|q;x_1,x_2), \Phi_2(a;b,b';c,c'|q;x_1,x_2),$$
$$\Phi_3(a,a';b,b';c|q;x_1,x_2), \Phi_4(a;b;c,c'|q;x_1,x_2)$$

are in a corrected form

$$\begin{aligned} & x_1(q^c-x_1q^{a+b+1})\varepsilon_2\mathrm{D}^2_{q,1,1}+x_2\left[q^c+x_1(q^a-q^{a+b}-q^{a+b+1})\right]\mathrm{D}^2_{q,1,2}-\{b\}_q q^a x_2\mathrm{D}_{q,2} \\ & +\left[\{c\}_q-(\{a\}_q q^b+\{b\}_q q^a+q^{a+b})x_1\right]\mathrm{D}_{q,1}-\{a\}_q\{b\}_q I=0. \end{aligned} \tag{113}$$

$$\begin{aligned} & x_1(q^c-x_1q^{a+b+1}\varepsilon_2)\mathrm{D}^2_{q,1,1}+x_1x_2(q^a-q^{a+b}-q^{a+b+1})\mathrm{D}^2_{q,1,2}-\{b\}_q q^a x_2\mathrm{D}_{q,2} \\ & +\left[\{c\}_q-(\{a\}_q q^b+\{b\}_q q^a+q^{a+b})x_1\right]\mathrm{D}_{q,1}-\{a\}_q\{b\}_q I=0. \end{aligned} \tag{114}$$

$$\begin{aligned} & x_1(q^c\varepsilon_2-x_1q^{a+b+1})\mathrm{D}^2_{q,1,1}+x_2q^c\mathrm{D}^2_{q,1,2} \\ & +\left[\{c\}_q-(\{a\}_q q^b+\{b\}_q q^a+q^{a+b})x_1\right]\mathrm{D}_{q,1}-\{a\}_q\{b\}_q I=0. \end{aligned} \tag{115}$$

$$x_1(q^c - x_1 q^{a+b+1}\varepsilon_2^2)\mathrm{D}^2_{q,1,1} - 2q^{a+b}\varepsilon_2 x_1 x_2 \mathrm{D}^2_{q,1,2} - [\{a\}_q q^b + \{b\}_q q^a + q^{a+b-1}\varepsilon_2]x_2\mathrm{D}_{q,2} + \left[\{c\}_q - \varepsilon_2(\{a\}_q q^b + \{b\}_q q^a + \varepsilon_2 q^{a+b})x_1\right]\mathrm{D}_{q,1} - q^{a+b}x_2^2 D^2_{q,2,2} - \{a\}_q\{b\}_q I = 0. \tag{116}$$

The proof of (113) goes as follows: Write the first q-Appell function in the form

$$\Phi_1(a;b,b';c|q;x_1,x_2) = \sum_{m_2=0}^{\infty} \frac{\langle a,b';q\rangle_{m_2}}{\langle 1,c;q\rangle_{m_2}} \sum_{m_1=0}^{\infty} \frac{\langle a+m_2,b;q\rangle_{m_1}}{\langle 1,c+m_2;q\rangle_{m_1}} x_1^{m_1} x_2^{m_2}. \tag{117}$$

Then the q-difference equation for the inner sum becomes:

$$\sum_{m_2=0}^{\infty} \frac{\langle a,b';q\rangle_{m_2}}{\langle 1,c;q\rangle_{m_2}} \Big[x_1(q^{c+m_2} - x_1 q^{a+b+1+m_2})\mathrm{D}^2_{q,1,1} + \left[\{c+m_2\}_q - (\{a+m_2\}_q q^b + \{b\}_q q^{a+m_2} + q^{a+b+m_2})x_1\right]\mathrm{D}_{q,1} - \{a+m_2\}_q\{b\}_q I\Big] \sum_{m_1=0}^{\infty} \frac{\langle a+m_2,b;q\rangle_{m_1}}{\langle 1,c+m_2;q\rangle_{m_1}} x_1^{m_1} x_2^{m_2} = 0. \tag{118}$$

We have

$$\{c+m_2\}_q = \begin{cases} \{c\}_q + q^c\{m_2\}_q \\ \{m_2\}_q + q^{m_2}\{c\}_q. \end{cases} \tag{119}$$

Therefore we get the terms

$$\begin{cases} \{c\}_q \mathrm{D}_{q,1} + q^c x_2 \mathrm{D}^2_{q,1,2} \\ x_2 \mathrm{D}^2_{q,1,2} + \varepsilon_2\{c\}_q \mathrm{D}_{q,1}. \end{cases} \tag{120}$$

In the same way we have

$$-(\{a+m_2\}_q q^b + \{b\}_q q^{a+m_2} + q^{a+b+m_2}) = \begin{cases} -(q^b\{a\}_q + q^a\{b\}_q + q^{a+b} + \{m_2\}_q(q^{a+b+1} + q^{a+b} - q^a)) \\ -(\{m_2\}_q q^{a+b+1} + q^b\{a+1\}_q + q^{a+m_2}\{b\}_q). \end{cases} \tag{121}$$

Therefore we get the terms

$$-(\{a\}_q q^b + \{b\}_q q^a + q^{a+b})x_1\mathrm{D}_{q,1} + x_1x_2(-q^a + q^{a+b} + q^{a+b+1})\mathrm{D}^2_{q,1,2} \tag{122}$$

or

$$-(x_1x_2 q^{a+b+1}\mathrm{D}^2_{q,1,2} + x_1[\{a+1\}_q q^b + q^a\varepsilon_2\{b\}_q]\mathrm{D}_{q,1}). \tag{123}$$

This gives us 8 equivalent q-difference equations for Φ_1, 4 equivalent q-difference equations for Φ_2, 2 equivalent q-difference equations for Φ_3 and 16 equivalent q-difference equations for Φ_4. These equations are stated in a different form in [32, p. 299]. The q-difference equation for Φ_1 can be written in the following canonical form, a q-analogue of [53, p. 146].

$$-x_1\{\theta_1 + b\}_q\{\theta_1 + \theta_2 + a\}_q + \{\theta_1\}_q\{\theta_1 + \theta_2 + c - 1\}_q = 0. \tag{124}$$

The q-difference equation for Φ_2 can be written in the canonical form

$$-x_1\{\theta_1 + a\}_q\{\theta_1 + \theta_2 + b\}_q + \{\theta_1\}_q\{\theta_1 + c - 1\}_q = 0. \tag{125}$$

The q-difference equation for Φ_3 can be written in the canonical form

$$-x_1\{\theta_1+a\}_q\{\theta_1+b\}_q+\{\theta_1\}_q\{\theta_1+\theta_2+c-1\}_q=0. \tag{126}$$

The q-difference equation for Φ_4 can be written in the canonical form

$$-x_1\{\theta_1+\theta_2+a\}_q\{\theta_1+\theta_2+\theta_2+b\}_q+\{\theta_1\}_q\{\theta_1+c-1\}_q=0. \tag{127}$$

The q-difference equation for Φ_1 can be rewritten in the operator form

$$\begin{aligned}(q^c-x_1q^{a+b+1})\frac{\varepsilon_2}{(1-q)^2}q^{-1}(\varepsilon_1^2-(1+q)\varepsilon_1+q)+[q^c+x_1(q^a-q^{a+b}-q^{a+b+1})]\frac{1}{(1-q)^2}[1-\varepsilon_1][1-\varepsilon_2]\\ -\{b\}_q\frac{x_1q^a}{1-q}[1-\varepsilon_2]-x_1\{a\}_q\{b\}_q+[\{c\}_q-(\{a\}_qq^b+\{b\}_qq^a+q^{a+b})x_1]\frac{1}{1-q}[1-\varepsilon_1]=0.\end{aligned} \tag{128}$$

Another q-difference equation satisfied by Φ_1 is (special thanks to Axel Riese for finding this equation using *Mathematica*)

$$x_2\{b'\}_qx_1\mathrm{D}_{q,x_1}f-x_1\{b\}_qx_2\mathrm{D}_{q,x_2}f+(-x_1q^b+x_2q^{b'})x_2\mathrm{D}_{q,x_2}x_1\mathrm{D}_{q,x_1}f=0. \tag{129}$$

Theorem 5.2. *Equation* (114) *is also satisfied by* (*compare* [59, p. 34, (65)] *where all the solutions of a homogeneous second order q-difference equation were found*)

$$x_1^{1-c}\Phi_2(a-c+1;b-c+1,b';2-c,c'|q;x_1,x_2).$$

Assume a solution to (114) of the form

$$\sum_{m_1,m_2=0}^{\infty}a_{m_1,m_2}x_1^{m_1+\mu_1}x_2^{m_2+\mu_2}.$$

Then the method of Frobenius gives the indicial equation for the term $a_{0,0}x_1^{\mu_1-1}x_2^{\mu_2}$.

$$\{\mu_1\}_q(\{c\}_q+q^c\{\mu_1-1\}_q)=\{\mu_1\}_q\{\mu_1+c-1\}_q. \tag{130}$$

The four q-Lauricella functions are defined by, compare [44, p. 15]

Definition 17.

$$\Phi_A^{(n)}(a,b_1,\ldots,b_n;c_1,\ldots,c_n|q;x_1,\ldots,x_n)\equiv\sum_{\mathbf{m}}\frac{\langle a;q\rangle_{m_1+\ldots+m_n}\langle b_1;q\rangle_{m_1}\ldots\langle b_n;q\rangle_{m_n}\prod_{j=1}^n x_j^{m_j}}{\langle c_1;q\rangle_{m_1}\ldots\langle c_n;q\rangle_{m_n}\prod_{j=1}^n\langle 1;q\rangle_{m_j}}, \tag{131}$$

$$\Phi_B^{(n)}(a_1,\ldots,a_n,b_1,\ldots,b_n;c|q;x_1,\ldots,x_n)\equiv\sum_{\mathbf{m}}\frac{\prod_{j=1}^n\langle a_j,b_j;q\rangle_{m_j}x_j^{m_j}}{\langle c;q\rangle_{m_1+\ldots+m_n}\prod_{j=1}^n\langle 1;q\rangle_{m_j}}, \tag{132}$$

$$\Phi_C^{(n)}(a,b;c_1,\ldots,c_n|q;x_1,\ldots,x_n)\equiv\sum_{\mathbf{m}}\frac{\langle a,b;q\rangle_{m_1+\ldots+m_n}\prod_{j=1}^n x_j^{m_j}}{\langle c_1;q\rangle_{m_1}\ldots\langle c_n;q\rangle_{m_n}\prod_{j=1}^n\langle 1;q\rangle_{m_j}}, \tag{133}$$

$$\Phi_D^{(n)}(a,b_1,\ldots,b_n;c|q;x_1,\ldots,x_n)\equiv\sum_{\mathbf{m}}\frac{\langle a;q\rangle_{m_1+\ldots+m_n}\langle b_1;q\rangle_{m_1}\ldots\langle b_n;q\rangle_{m_n}\prod_{j=1}^n x_j^{m_j}}{\langle c;q\rangle_{m_1+\ldots+m_n}\prod_{j=1}^n\langle 1;q\rangle_{m_j}}. \tag{134}$$

Theorem 5.3. *The partial q-difference equations for the three q-Lauricella functions* $\Phi_A^{(3)}, \Phi_B^{(3)}, \Phi_D^{(3)}$ *are:*

$$\begin{aligned}x_1(q^{c_1}-x_1q^{a+b_1+1}\varepsilon_2\varepsilon_3)\mathrm{D}^2_{q,1,1}+(q^a-q^{a+b_1}-q^{a+b_1+1})\varepsilon_3\{\theta_1\}_q\{\theta_2\}_q\\ -\{b_1\}_q q^a\varepsilon_3\{\theta_2\}_q-q^{a+b_1}\{\theta_1\}_q\{\theta_3\}_q-\{b_1\}_q q^a\{\theta_3\}_q\\ +\left[\{c_1\}_q-(\{a\}_q q^{b_1}+\varepsilon_3(\{b_1\}_q q^a+q^{a+b_1}))x_1\right]\mathrm{D}_{q,1}-\{a\}_q\{b_1\}_q I=0.\end{aligned} \tag{135}$$

$$\begin{aligned}x_1(q^c\varepsilon_2\varepsilon_3-x_1q^{a_1+b_1+1})\mathrm{D}^2_{q,1,1}+x_2q^c\mathrm{D}^2_{q,1,2}\\ +\left[\{c\}_q+q^c\theta_3\varepsilon_2-(\{a_1\}_q q^{b_1}+\{b_1\}_q q^{a_1}+q^{a_1+b_1})x_1\right]\mathrm{D}_{q,1}-\{a_1\}_q\{b_1\}_q I=0.\end{aligned} \tag{136}$$

$$\begin{aligned}0=&x_1(q^c-x_1q^{a+b_1+1})\varepsilon_2\varepsilon_3\mathrm{D}^2_{q,1,1}-\{\theta_3\}_q q^a(q^{b_1}\{\theta_1\}_q+\{b_1\}_q)\\ &+x_2\left[q^c+x_1(q^a-q^{a+b_1}-q^{a+b_1+1})\right]\varepsilon_3\mathrm{D}^2_{q,1,2}-\varepsilon_3\{b_1\}_q q^a\{\theta_2\}_q\\ &+\left[\{c\}_q+q^c\{\theta_3\}_q-\{a\}_q q^{b_1}x_1-\varepsilon_3(\{b_1\}_q q^a+q^{a+b_1})x_1\right]\mathrm{D}_{q,1}-\{a\}_q\{b_1\}_q I.\end{aligned} \tag{137}$$

Proof. Write the first q-Lauricella function in the form

$$\Phi_A^{(3)}(a,\vec{b};\vec{c}|q;\vec{x})=A\Phi_2(a+m_3;b_1,b_2;c_1,c_2|q;x_1,x_2)x_3^{m_3}, \tag{138}$$

where

$$A\equiv\sum_{m_3=0}^{\infty}\frac{\langle a,b_3;q\rangle_{m_3}}{\langle 1,c_3;q\rangle_{m_3}}. \tag{139}$$

Then the q-difference equation (114) for the inner sum is valid:

$$\begin{aligned}0=&A\Big(x_1(q^{c_1}-x_1q^{a+m_3+b_1+1}\varepsilon_2)\mathrm{D}^2_{q,1,1}\\ &+x_1x_2(q^{a+m_3}-q^{a+m_3+b_1}-q^{a+m_3+b_1+1})\mathrm{D}^2_{q,1,2}-\{b_1\}_q q^{a+m_3}x_2\mathrm{D}_{q,2}\\ &+\left[\{c_1\}_q-(\{a+m_3\}_q q^{b_1}+\{b_1\}_q q^{a+m_3}+q^{a+m_3+b_1})x_1\right]\mathrm{D}_{q,1}\\ &-\{a+m_3\}_q\{b_1\}_q I\Big)\Phi_2x_3^{m_3}.\end{aligned} \tag{140}$$

This can be simplified to

$$\begin{aligned}0=&A\Big(x_1(q^{c_1}-x_1q^{a+b_1+1}\varepsilon_2\varepsilon_3)\mathrm{D}^2_{q,1,1}+(q^a-q^{a+b_1}-q^{a+b_1+1})\varepsilon_3\{\theta_1\}_q\{\theta_2\}_q\\ &-\{b_1\}_q q^a\varepsilon_3\{\theta_2\}_q-q^{a+b_1}\{\theta_1\}_q\{\theta_3\}_q-\{b_1\}_q q^a\{\theta_3\}_q\\ &+\left[\{c_1\}_q-(\{a\}_q q^{b_1}+\varepsilon_3(\{b_1\}_q q^a+q^{a+b_1}))x_1\right]\mathrm{D}_{q,1}-\{a\}_q\{b_1\}_q I\Big)\Phi_2x_3^{m_3}.\end{aligned} \tag{141}$$

□

There are 16 equivalent q-difference equations for $\Phi_A^{(3)}$, 4 equivalent q-difference equations for $\Phi_B^{(3)}$, 16 equivalent q-difference equations for $\Phi_C^{(3)}$ and 64 equivalent q-difference equations for $\Phi_D^{(3)}$. The most complicated one, for $\Phi_C^{(3)}$, will be treated in a subsequent paper.

6. DISCUSSION

The first discussion of formal computations was Cardan's formula [11], which led to the complex numbers. Arbogast [10, p. 127] and Fourier regarded symbolic calculus as an elegant way of discovering, expressing, or verifying theorems, rather than as a valid method of proof [49, p. 172]. DeMorgan said that the symbolic algebra method gives a strong presumption of truth, not a method of proof [49, p. 234]. As we see, our method works better for q-Laguerre polynomials than for q-Jacobi polynomials. The reason is that the weight function for q-Laguerre polynomials can be made entire. The weight function for q-Jacobi polynomials is however defined only in a small interval.

REFERENCES

1. Al-Salam, N. A. On some q-operators with applications. *Nederl. Akad. Wetensch. Indag. Math.*, 1989, **51**(1), 1–13.
2. Al-Salam, W. A. q-Bernoulli numbers and polynomials. *Math. Nachr.*, 1959, **17**, 239–260.
3. Al-Salam, W. A. Operational representations for the Laguerre and other polynomials. *Duke Math. J.*, 1964, **31**, 127–142.
4. Alzer, H. Sharp bounds for the ratio of q-gamma functions. *Math. Nachr.*, 2001, **222**, 5–14.
5. Alzer, H. and Grinshpan, A. Z. Inequalities for the gamma and q-gamma functions. *J. Approx. Theory*, 2007, **144**(1), 67–83.
6. Angelesco, P. A. Sur les polynômes hypergeometriques. *Ann. Math.*, 1925, **3**, 161–177.
7. Appell, P. Sur la serie hypergeometrique et les polynômes de Jacobi. *C. R. Paris*, 1879, **89**, 31–33.
8. Appell, P. and Kampé de Fériet, J. *Fonctions hypergéométriques et hypersphériques*. Paris, 1926.
9. Askey, R. The q-gamma and q-beta functions. *Applicable Anal.*, 1978/79, **8**(2), 125–141.
10. Arbogast, L. *Du calcul des dérivation*. Strasbourg, 1800.
11. Cardano, G. *Artis magnae, sive de regulis algebraicis* (also known as *Ars magna*). Nürnberg, 1545.
12. Carlitz, L. A note on the Laguerre polynomials. *Michigan Math. J.*, 1960, **7**, 219–223.
13. Chatterjea, S. K. Operational formulae for certain classical polynomials. I. *Quart. J. Math. Oxford*, Ser. (2), 1963, **14**, 241–246.
14. Chen, W. Y. C. and Liu, Zhi-guo. *Parameter Augmentation for Basic Hypergeometric Series, I. Mathematical Essays in Honor of Gian-Carlo Rota*. Birkhäuser, 1998, 111–129.
15. Cigler, J. Operatormethoden für q-Identitäten. *Monatsh. Math.*, 1979, **88**, 87–105.
16. Cigler, J. Operatormethoden für q-Identitäten II, q-Laguerre-Polynome. *Monatsh. Math.*, 1981, **91**, 105–117.
17. Courant, R. and Hilbert, D. *Methoden der Mathematischen Physik*. I. 3rd edn. Heidelberger Taschenbücher, Vol. 30. Springer-Verlag, Berlin, 1968.
18. Ernst, T. *The History of q-Calculus and a New Method*. Uppsala, 2000.
19. Ernst, T. *A New Method for q-Calculus*. Uppsala dissertations, 2002.
20. Ernst, T. A method for q-calculus. *J. Nonlinear Math. Phys.*, 2003, **10**(4), 487–525.
21. Ernst, T. Some results for q-functions of many variables. *Rend. Padova*, 2004, **112**, 199–235.
22. Ernst, T. q-Generating functions for one and two variables. *Simon Stevin*, 2005, **12**(4), 589–605.
23. Ernst, T. A renaissance for a q-umbral calculus. In *Proceedings of the International Conference Munich, Germany 25–30 July 2005*. World Scientific, 2007.
24. Ernst, T. q-Bernoulli and q-Euler polynomials, an umbral approach. *Int. J. Difference Equations Dynamical Systems*, 2006, **1**(1), 31–80.
25. Ernst, T. q-Analogues of some operational formulas. *Algebras Groups Geom.*, 2006, **23**(4), 354–374.
26. Ernst, T. The different tongues of q-calculus. *Proc. Estonian Acad. Sci.*, 2008, **57**, 81–99.
27. Exton, H. *q-Hypergeometric Functions and Applications*. Ellis Horwood, 1983.
28. Feldheim, E. Équations intégrales pour les polynômes d'Hermite à une et plusieurs variables, pour les polynômes de Laguerre, et pour les fonctions hypergéométriques les plus générales. *Ann. Scuola Norm. Super. Pisa*, 1940, **9**(2), 225–252.
29. Feldheim, E. Relations entre les polynômes de Jacobi, Laguerre et Hermite. *Acta Math.*, 1943, **75**, 117–138.
30. Fujiwara, I. A unified presentation of classical orthogonal polynomials. *Math. Japon.*, 1966, **11**, 133–148.
31. Gasper, G. and Rahman, M. *Basic Hypergeometric Series*. Cambridge, 1990.
32. Gasper, G. and Rahman, M. *Basic Hypergeometric Series*. 2nd edn. Cambridge, 2004.
33. Grinshpan, A. Z. and Ismail, M. E. H. Completely monotonic functions involving the gamma and q-gamma functions. *Proc. Amer. Math. Soc.*, 2006, **134**(4), 1153–1160.
34. Gupta, A. On a q-extension of "incomplete" beta function. *J. Indian Math. Soc.* (N.S.), 1999, **66**(1–4), 193–201.
35. Hahn, W. Beiträge zur Theorie der Heineschen Reihen. *Math. Nachr.*, 1949, **2**, 340–379.
36. Heine, E. Über die Reihe $1+[(1-q^{\alpha})(1-q^{\beta})]/[(1-q)(1-q^{\gamma})]x+[(1-q^{\alpha})(1-q^{\alpha+1})(1-q^{\beta})(1-q^{\beta+1})]/[(1-q)(1-q^2)(1-q^{\gamma}(1-q^{\gamma+1})]x^2+\ldots$ *J. reine angew. Math.*, 1846, **32**, 210–212.
37. Ismail, M. E. H., Lorch, L., and Muldoon, M. E. Completely monotonic functions associated with the gamma function and its q-analogues. *J. Math. Anal. Appl.*, 1986, **116**(1), 1–9.

38. Ismail, M. E. H. and Muldoon, M. E. Inequalities and monotonicity properties for gamma and q-gamma functions. In *Approximation and Computation (West Lafayette, IN, 1993)*. Birkhäuser, 1994, 309–323.
39. Jackson, F. H. A basic-sine and cosine with symbolical solution of certain differential equations. *Proc. Edinburgh Math. Soc.*, 1904, **22**, 28–39.
40. Jackson, F. H. On q-definite integrals. *Quart. J. Pure Appl. Math.*, 1910, **41**, 193–203.
41. Jackson, F. H. On basic double hypergeometric functions. *Quart. J. Math.*, Oxford Ser., 1942, **13**, 69–82.
42. Jackson, F. H. Basic double hypergeometric functions. *Quart. J. Math.*, Oxford Ser., 1944, **15**, 49–61.
43. Jacobi, C. G. J. *Werke 6*. Berlin, 1891.
44. Jain, V. K. and Srivastava, H. M. Some general q-polynomial expansions for functions of several variables and their applications to certain q-orthogonal polynomials and q-Lauricella functions. *Bull. Soc. Roy. Sci. Liege*, 1989, **58**(1), 13–24.
45. Kamke, E. *Differentialgleichungen Lösungsmethoden und Lösungen*. Chelsea, 1959.
46. Khan, M. A. q-Analogues of certain operational formulae. *Houston J. Math.*, 1987, **13**(1), 75–82.
47. Kim, T. and Adiga, C. On the q-analogue of gamma functions and related inequalities. *JIPAM. J. Inequal. Pure Appl. Math.*, 2005, **6**(4), Article 118, 4 pp.
48. Koornwinder, T. H. Jacobi functions as limit cases of q-ultraspherical polynomials. *J. Math. Anal. Appl.*, 1990, **148**(1), 44–54.
49. Koppelman, E. The calculus of operations and the rise of abstract algebra. *[J] Arch. Hist. Exact Sci.*, 1971, **8**, 155–242.
50. Mansour, M. An asymptotic expansion of the q-gamma function $\Gamma_q(x)$. *J. Nonlinear Math. Phys.*, 2006, **13**(4), 479–483.
51. Mansour, T. Some inequalities for the q-gamma function. *JIPAM. J. Inequal. Pure Appl. Math.*, 2008, **9**(1), Article 18, 4 pp.
52. Manocha, H. L. and Sharma, B. L. Some formulae for Jacobi polynomials. *Proc. Cambridge Philos. Soc.*, 1966, **62**, 459–462.
53. Mellin, H. J. Über den Zusammenhang zwischen den linearen Differential- und Differenzengleichunge. *Acta Math.*, 1901, **25**, 139–164.
54. Moak, D. S. The q-gamma function for $q > 1$. *Aequat. Math.*, 1980, **20**(2–3), 278–285.
55. Nalli, P. Sopra un procedimento di calcolo analogo alla integrazione. *Palermo Rend.*, 1923, **47**, 337–374.
56. Rudin, W. *Real and Complex Analysis*. McGraw-Hill, New York, 1987.
57. Sellami, M., Brahim, K., and Bettaibi, N. New inequalities for some special and q-special functions. *JIPAM. J. Inequal. Pure Appl. Math.*, 2007, **8**(2), Article 47, 7 pp.
58. Singh, R. P. Operational formulae for Jacobi and other polynomials. *Rend. Sem. Mat. Univ. Padova*, 1965, **35**, 237–244.
59. Smith, E. R. *Zur Theorie der Heineschen Reihe und ihrer Verallgemeinerung*. Diss. Univ. München, 1911.
60. Toscano, L. Sui polinomi ipergeometrici a più variabili del tipo F_D di Lauricella. *Matematiche (Catania)*, 1972, **27**, 219–250.
61. Tricomi, F. G. Vorlesungen über Orthogonalreihen. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Vol. LXXVI. Springer-Verlag, Berlin, 1955.
62. Viskov, O. V. and Srivastava, H. M. New approaches to certain identities involving differential operators. *J. Math. Anal. Appl.*, 1994, **186**(1), 1–10.
63. Ward, M. A calculus of sequences. *Amer. J. Math.*, 1936, **58**, 255–266.
64. Zhang, C. Sur la fonction q-gamma de Jackson. *Aequat. Math.*, 2001, **62**(1–2), 60–78.

q-arvutus algebralistes teisendustes

Thomas Ernst

On jätkatud artiklis [23] alustatud tuntud arvutusvalemite üldistuste, nn q-analoogide analüüsi. Seejuures piirjuhul $q = 1$ on tegemist tavaliste üldkasutatavate valemitega. On vaadeldud Laguerre'i, Jacobi ja Legendre'i polünoomide vastavaid laiendusi. See nõuab mitmesuguste tuntud funktsioonide ja nendega seotud diferentsvõrrandite q-laiendite esitust ning omaduste uurimist.