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ON ZERO-OUTPUT CONSTRAINED DYNAMICS OF DISCRETE-TIME NONLINEAR SYSTEM

(Presented by Ü. Jaaksoo)

Abstract. The paper considers the output zeroing problem for discrete-time nonlinear system with measurable input disturbances. The objective is to identify, if possible, the set of all pairs consisting of an initial state and a control sequence, which produce an identically zero output for the given disturbance sequence. For this purpose the zero dynamics algorithm is generalized for discrete-time nonlinear system with measurable input disturbances. Under some mild regularity assumptions, the local solution around an equilibrium point of the system is derived on the basis of zero dynamics algorithm.

1. Introduction

The system zero dynamics represent the nonlinear analogue of the zeros of the transfer function of a linear system, and are defined as the internal dynamics arising in a system when control sequence and initial conditions are chosen so as to constrain the output to remain zero for some interval of time. The concept of zero dynamics has been introduced by Byrnes and Isidori [1] for continuous-time nonlinear systems, and generalized by Monaco and Normand-Cyrot [2] for discrete-time nonlinear systems. Nonlinear systems with zero dynamics asymptotically stable at a given equilibrium point are called minimum-phase systems [1-3].

The concept of zero dynamics has proven to be important through its applications to different control problems. One of the examples is the output reproducibility of a given trajectory. In [4], a simple condition for solvability of this problem and the associated control law is derived on the basis of the zero-dynamics algorithm. Moreover, if these conditions are not satisfied and/or some uncertainties are present, sliding control is shown to perform asymptotic tracking whenever the zero dynamics of an extended system related to the given system and to the reference trajectory satisfies certain stability properties. Another example is the output regulation problem, that is the problem of controlling a plant in order to have its output tracking (or rejecting) reference (or disturbance) signals produced by some external generator (the exosystem). The solvability conditions of this problem have been formulated [5] in terms of properties of the zero dynamics of a composite system whose output is defined as tracking error and which incorporates the plant dynamics as well as the exosystem dynamics. The third example is the system stabilization. It has been shown [6] that, if the system with the same number of inputs and outputs has a zero dynamics which is asymptotically stable at an equilibrium point, it is always possible to find a smooth state feedback that asymptotically stabilizes the system even in the case of systems whose linear approximation has uncontrollable modes associated with eigenvalues on the imaginary axis.

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The serval alternative algorithms to compute the zero dynamics for continuous-time nonlinear system have been presented in [3, 7]. For discrete-time nonlinear system, the zero-dynamics algorithm has been presented in [2].

The objective of this paper is to generalize the zero-dynamics algorithm for discrete-time nonlinear systems with input disturbances. Besides its immediate importance for solving the output zeroing problem for nonlinear systems with input disturbances, in terms of this algorithm the necessary and sufficient conditions for the solvability of the strong model matching problem [8], and the exact model-matching problem from the origin, can be given [9].

2. Locally maximal output zeroing submanifold

Consider the system

$$x(t+1)=f(x(t),\ u(t),\ w(t)),\qquad x(0)=x_0,$$
$$y(t)=h(x(t)),\tag{1}$$

where the states $x(\cdot)$ belong to an open subset X of R^n, the controls $u(\cdot)$ belong to an open subset U of R^m, the measurable disturbances $w(\cdot)$ belong to an open subset W of R^r and the outputs $y(\cdot)$ belong to an open subset Y of R^p. The f and h are supposed to be smooth (i.e. C^∞) mappings. We are assumed to work in a neighbourhood of an equilibrium point (x^0, u^0, w^0) of the system (1) (i.e. $f(x^0, u^0, w^0)=x^0$) for which the following equality holds: $h(x^0)=0$.

In this paper, we shall discuss the problem of how the output $y(t)$ of the nonlinear discrete-time system of the form (1) can be set to zero by means of a proper choice of the initial state x_0 and control input $u(t)$. If the initial state of (1) is equal to x^0, the control input $u(t)=u^0$ for all $t\geqslant 0$, then also the output $y(t)$ is zero for all $t\geqslant 0$. Our purpose is to identify, if possible, the set of *all pairs* consisting of an initial state x_0 and a control sequence $\{u(t),\ t\geqslant 0\}$, which produce an identically zero output for the given disturbance sequence $\{w(t),\ t\geqslant 0\}$. Let us first introduce some terminology.

Let M be a smooth connected submanifold of X which contains the point x^0. The manifold M is said to be *locally controlled invariant* at x^0 if there exist the neighbourhoods X^0 of x^0, W^0 of w^0, U^0 of u^0 and a smooth mapping $\alpha: X^0\times W^0\to U^0$, such that $f(x, \alpha(x, w), w)\in M$ for all $x\in M\cap X^0$ and for all $w\in W^0$.

An *output zeroing submanifold* of (1) is a smooth connected submanifold M of X which contains the point x^0 and satisfies

(i) for each $x\in M$, $h(x)=0$;

(ii) M is a locally controlled invariant at x^0.

In other words, an output zeroing submanifold is a submanifold M of the state space with the property that — for some choice of feedback control $\alpha(x, w)$ — the trajectories of the closed loop system

$$x(t+1)=f(x(t), \alpha(x(t), w(t)), w(t)),$$
$$y(t)=h(x(t))$$

which start in M stay in M for $t\leqslant t_F$ (for some finite t_F), and the corresponding output is identically zero for $t\leqslant t_F$.

If M and M' are two connected smooth submanifolds of X which both contain the point x^0, it is said that M *locally contains* M' (or, M coincides with M') if, for some neighbourhood X^0 of x^0, $M\cap X^0\supset M'\cap X^0$ (or $M\cap X^0=M'\cap X^0$). An output zeroing submanifold M is *locally* *maximal* if, for some neighbourhood X^0 of x^0, any other output zeroing submanifold M' satisfies $M \cap X^0 \supset M' \cap X^0$. The locally maximal output zeroing submanifold M is called the (local) *zero-dynamics submanifold*.

In general, it is not clear whether or not the local zero-dynamics submanifold might exist at all. However, under some mild regularity assumptions, in a neighbourhood of x^0 the zero-dynamics submanifold L^* — if it exists — can be found by the algorithm presented in the next section and called the zero-output constrained dynamics algorithm, or, in short, zero-dynamics algorithm. The algorithm we present is a generalization of the zero-dynamics algorithm for discrete-time nonlinear systems presented in [2] to the case of systems with measurable input disturbances. Note that this algorithm is quite similar to the structure (inversion) algorithm [10].

3. Zero-dynamics algorithm

Consider the system (1) and assume that (x^0, u^0, w^0) is an equilibrium point of (1) also satisfying $h(x^0)=0$.

Step 1. Assume that the function $\lambda_0=h$ has constant rank s_0 in a neighbourhood of x^0 in $h^{-1}(0)$. Define $\sigma_0:=s_0$. Then, locally, $L_0:=h^{-1}(0)$ is a $(n-\sigma_0)$-dimensional subset of X. Let $x=(x'_0, \tilde{x}_1)$ be an adapted coordinate system around x^0 such that locally $L_0=\{x \mid x'_0=0\}$, and let f be partitioned accordingly:

$$f(x, u, w)=\begin{bmatrix} f'_0(x'_0, \tilde{x}_1, u, w) \\ f_1(x'_0, \tilde{x}_1, u, w) \end{bmatrix}. \tag{2}$$

Let us denote the equilibrium point in this new coordinate system by $(x'^0_0, \tilde{x}^0_1, u^0, w^0)$.

The constraint $y(t)=0$ for all t implies $x'_0(t)=0$ for all t, and hence

$$x'_0(t+1)=f'_0(0, \tilde{x}_1(t), u(t), w(t))=0.$$

Define $x^*_0(t+1) = x'_0(t+1)$ and $F_0(0, \tilde{x}_1, u, w) = f'_0(0, \tilde{x}_1, u, w)$. In these notations the last equation takes the form

$$x^*_0(t+1)=F_0(0, \tilde{x}_1(t), u(t), w(t))=0. \tag{3}$$

Assume that the rank of the matrix

$$\frac{\partial}{\partial u} F_0(0, \tilde{x}_1, u, w)$$

is constant, say r_0, around $(\tilde{x}^0_1, u^0, w^0)$.

If the equation (3) can be solved for $u(t)$ (the sufficient condition for this is $r_0=s_0$)

$$u(t)=\varphi(\tilde{x}_1(t), w(t)), \tag{4}$$

then the algorithm stops because L_0 (around x^0) is clearly the set we are looking for. Really, the feedback control (4) is such as to keep in L_0 the trajectory starting from any point of L_0, and the system

$$\tilde{x}_1(t+1)=f_1(0, \tilde{x}_1(t), \varphi(\tilde{x}_1(t), w(t)), w(t))$$

characterizes the zero-output constrained dynamics. In this case, we may still set, formally, $L_1=L_0$.

If the Eq. (3) cannot be solved for $u(t)$, then permute temporarily, if necessary, the components of $F_0(\cdot)$ so that the first r_0 rows of $\partial F_0(\cdot)/\partial u$ are linearly independent. Decompose $x_0^*(t+1)$ and $F_0(\cdot)$ according to

$$x_0^*(t+1)=\begin{bmatrix} x_{01}^*(t+1) \\ x_{02}^*(t+1) \end{bmatrix}, \quad F_0(\cdot)=\begin{bmatrix} F_{01}(\cdot) \\ F_{02}(\cdot) \end{bmatrix},$$

where $x_{01}^*(t+1)$ and $F_{01}(\cdot)$ consist of the first r_0 elements. Since the last rows of $\partial F_0(\cdot)/\partial u$ are linearly dependent on the first r_0 rows, we can write

$$x_{01}^*(t+1)=F_{01}(0, \tilde{x}_1(t), u(t), w(t)),$$

$$x_{02}^*(t+1)=R_0(x_{01}^*(t+1), \tilde{x}_1(t), w(t)).$$

The first components of $x_0^*(t+1)$, that is $x_{01}^*(t+1)$, can be made zero by suitable choice of the control. The last s_0-r_0 components of $x_0^*(t+1)$, that is $x_{02}^*(t+1)$, do not depend on the control, and so (3) clearly implies that the following equality has to hold for every $w(t)\in W^0$

$$x_{02}^*(t+1)=R_0(0, \tilde{x}_1(t), w(t))=0. \tag{5}$$

If it is possible to find a solution with respect to $\tilde{x}_1$ of the equation

$$R_0(0, \tilde{x}_1, w)=0$$

which does not depend on w, then define $\lambda_1(\tilde{x}_1)$ as the left-hand side of this solution in the implicit form $\lambda_1(\tilde{x}_1)=0$.

If it is not possible to find such a solution, then the system (1) with disturbances does not have the zero dynamics, and the algorithm stops.

If the solution $\lambda_1(\tilde{x}_1)$ exists, it can be found in the following way. For the r-dimensional vector w, let $w^{[l]}$ denote the following C^l_{r+l-1}-dimensional vector

$$w^{[l]}=[w_1^l,\ w_1^{l-1}w_2,\ \ldots,\ w_1^{l-1}w_r,\ w_1^{l-2}w_2^2,\ w_1^{l-2}w_2w_3\ \ldots$$
$$\ldots,\ w_1^{l-2}w_2w_r,\ \ldots,\ w_r^l]^{\mathrm{T}}.$$

In a neighbourhood W^0 of w^0, the components R_0^s $(s=1, \ldots, s_0-r_0)$ of the nonlinear vector-valued function R_0 can be represented in terms of the Taylor series expansion

$$R_0^s(0, \tilde{x}_1, w)=R_{00}^s(\tilde{x}_1)+\sum_{l\geqslant 1} R_{0l}^s(\tilde{x}_1)(w-w^0)^{[l]}.$$

Define the vector-valued function $\lambda_1(\tilde{x}_1)$ as the one formed from the independent (over the field of analytic functions of $\tilde{x}_1$) components of R_{00}^s, R_{0l}^s, $l\geqslant 1$, $s=1, \ldots, s_0-r_0$.

Note that $\lambda_1(\tilde{x}_1)$ is not identically zero, because otherwise (3) would be solvable for $u(t)$.

Step 2. Assume that the thus obtained additional constraint $\lambda_1(\tilde{x}_1)$ has constant rank s_1 around $\tilde{x}_1^0$. Define $\sigma_1:=s_0+s_1$. Then, locally, $L_1:=\lambda_1^{-1}(0)$ is a $(n-\sigma_1)$-dimensional subset of L_0. Note that L_1 contains x^0: $\lambda_1(\tilde{x}_1^0)=0$.

Choose the local coordinates $\tilde{x}_1=(x_1', \tilde{x}_2)$ on L_0 with $\dim x_1'=s_1$ such that locally $L_1=\{x\,|\,x_0'=0,\ x_1'=0\}$, and let f be partitioned accordingly:

$$f(x, u, w) = \begin{bmatrix} f'_0(x'_0, x'_1, \tilde{x}_2, u, w) \\ f'_1(x'_0, x'_1, \tilde{x}_2, u, w) \\ f_2(x'_0, x'_1, \tilde{x}_2, u, w) \end{bmatrix}. \tag{6}$$

The constraints $y(t)=0$, $\lambda_1(\tilde{x}_1(t))=0$ for all t now imply $x(t) \in L_1$ for every t, that is $x'_0(t)=0$, $x'_1(t)=0$ for all t and hence

$$x'_1(t+1) = f'_1(0, 0, \tilde{x}_2(t), u(t), w(t)) = 0,$$
$$x'_0(t+1) = f'_0(0, 0, \tilde{x}_2(t), u(t), w(t)) = 0,$$

or in the matrix form,

$$x^*_1(t+1) = F_1(0, 0, \tilde{x}_2(t), u(t), w(t)) = 0. \tag{7}$$

Assume that the rank of the matrix

$$\frac{\partial}{\partial u} F_1(0, 0, \tilde{x}_2, u, w)$$

is constant, say r_1, around $(\tilde{x}^0_2, u^0, w^0)$.

If the Eq. (7) can be solved for $u(t)$ around (x^0, u^0, w^0) (the sufficient condition for this is $r_1 = \sigma_1$)

$$u(t) = \varphi(\tilde{x}_2(t), w(t)), \tag{8}$$

then the algorithm stops, because L_1 (around x^0) is clearly the set we are looking for. Actually, the feedback control (8) is such as to keep in L_1 the trajectory starting from any point of L_1, and the system

$$\tilde{x}_2(t+1) = f_2(0, 0, \tilde{x}_2(t), \varphi(\tilde{x}_2(t), w(t)), w(t))$$

characterizes the zero-output-constrained dynamics. In this case we may still set, formally, $L_2 = L_1$.

If the Eq. (7) cannot be solved for $u(t)$, then permute temporarily, if necessary, the components of $F_1(\cdot)$ so that the first r_1 rows of the matrix $\partial F_1(\cdot)/\partial u$ are linearly independent. Decompose $x^*_1(t+1)$ and $F_1(\cdot)$ accordingly:

$$x^*_1(t+1) = \begin{bmatrix} x^*_{11}(t+1) \\ x^*_{12}(t+1) \end{bmatrix}, \quad F_1(\cdot) = \begin{bmatrix} F_{11}(\cdot) \\ F_{12}(\cdot) \end{bmatrix},$$

where $x^*_{11}(t+1)$ and $F_{11}(\cdot)$ consist of the first r_1 elements. Since the last rows of $\partial F_1(\cdot)/\partial u$ are linearly dependent on the first r_1 rows, we can write

$$x^*_{11}(t+1) = F_{11}(0, 0, \tilde{x}_2(t), u(t), w(t)),$$
$$x^*_{12}(t+1) = R_1(x^*_{11}(t+1), \tilde{x}_2(t), w(t)).$$

So (7) clearly implies that for every $w(t) \in W^0$

$$R_1(0, \tilde{x}_2(t), w(t)) = 0. \tag{9}$$

If it is possible to find a solution with respect to $\tilde{x}_2$ of the equation

$$R_1(0, \tilde{x}_2, w) = 0$$

which does not depend on w, then define $\lambda_2(\tilde{x}_2)$ as the left-hand side of this solution in the implicit form $\lambda_2(\tilde{x}_2) = 0$.

If it is not possible to find such a solution, then the system (1) with disturbances does not have the zero dynamics and the algorithm stops.

If the solution $\lambda_2(\tilde{x}_2)$ exists, it can be found in the following way. In a neighbourhood W^0 of w^0, the components R_1^s $(s=1, \ldots, s_0+s_1-r_1)$ of the nonlinear vector-valued function R_1 can be represented in terms of Taylor series expansion

$$R_1^s(0, \tilde{x}_2, w) = R_{10}^s(\tilde{x}_2) + \sum_{l \geqslant 1} R_{1l}^s(\tilde{x}_2)(w-w^0)^{[l]}.$$

Define the vector-valued function $\lambda_2(\tilde{x}_2)$ as the one formed from the independent (over the field of analytic functions of $\tilde{x}_2$) components of R_{10}^s, R_{1l}^s, $l \geqslant 1$, $s=1, \ldots, s_0+s_1-r_1$.

Note that $\lambda_2(\tilde{x}_2)$ is not identically zero because otherwise (7) would be solvable for $u(t)$.

Step k. Suppose that in step $k-1$, λ_{k-1} has been defined so that $\lambda_{k-1}(\tilde{x}_{k-1})$ has constant rank s_{k-1} around x_{k-1}^0. Define $\sigma_{k-1} := = s_0+s_1+\ldots+s_{k-1}$. Then, locally, $L_{k-1} := \lambda_{k-1}^{-1}(0)$ is a $(n-\sigma_{k-1})$-dimensional subset of L_{k-1}. Note that L_{k-1} contains $x^0 : \lambda_{k-1}(\tilde{x}_{k-1}^0) = 0$.

Choose local coordinates $\tilde{x}_{k-1} = (x'_{k-1}, \tilde{x}_k)$ on L_{k-1} with $\dim x'_{k-1} = = s_{k-1}$ such that locally $L_{k-1} = \{x \mid x'_0 = 0, \ x'_1 = 0, \ldots, x'_{k-1} = 0\}$, and let f be partitioned accordingly,

$$f(x, u, w) = \begin{bmatrix} f'_0(x'_0, x'_1, \ldots, x'_{k-1}, \tilde{x}_k, u, w) \\ f'_1(x'_0, x'_1, \ldots, x'_{k-1}, \tilde{x}_k, u, w) \\ \ldots\ldots\ldots\ldots\ldots \\ f'_{k-1}(x'_0, x'_1, \ldots, x'_{k-1}, \tilde{x}_k, u, w) \\ f_k(x'_0, x'_1, \ldots, x'_{k-1}, \tilde{x}_k, u, w) \end{bmatrix}.$$

The constraints $y(t)=0$, $\lambda_1(\tilde{x}_1(t))=0, \ldots, \lambda_{k-1}(\tilde{x}_{k-1}(t))=0$ for all t now imply $x(t) \in L_{k-1}$ for every t, that is $x'_0(t)=0$, $x'_1(t)=0, \ldots$ $\ldots, x'_{k-1}(t)=0$ for all t and hence

$$x'_0(t+1) = f'_0(0, \ldots, 0, \tilde{x}_k(t), u(t), w(t)) = 0,$$

$$x'_1(t+1) = f'_1(0, \ldots, 0, \tilde{x}_k(t), u(t), w(t)) = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x'_{k-1}(t+1) = f'_{k-1}(0, \ldots, 0, \tilde{x}_k(t), u(t), w(t)) = 0,$$

or in the matrix form,

$$x_{k-1}^*(t+1) = F_{k-1}(0, \ldots, 0, \tilde{x}_k(t), u(t), w(t)) = 0, \tag{10}$$

where $x_{k-1}^*(t+1)$ and F_{k-1} have $\sigma_{k-1} := s_0+s_1+\ldots+s_{k-1}$ elements and $\tilde{x}_k$ denotes coordinates on L_{k-1}. Assume that

$$\frac{\partial}{\partial u} F_{k-1}(0, \ldots, 0, \tilde{x}_k, u, w)$$

has constant rank r_{k-1} around $(\tilde{x}_k^0, u^0, w^0)$.

If the Eq. (10) can be solved for $u(t)$ (the sufficient condition for this is $r_{k-1} = \sigma_{k-1}$)

$$u(t) = \varphi(\tilde{x}_k(t), w(t)), \tag{11}$$

then the algorithm stops, because L_{k-1} (around x^0) is clearly the set we are looking for. Actually, the feedback control (11) is such as to keep in L_{k-1} the trajectory starting from any point of L_{k-1}, and the system

$$\tilde{x}_k(t+1)=f_k(0, \ldots, 0, \tilde{x}_k(t), \varphi(\tilde{x}_k(t), w(t)), w(t))$$

characterizes the zero-output constrained dynamics. In this case we may still set, formally, $L_k=L_{k-1}$.

If the Eq. (10) cannot be solved for $u(t)$, then permute temporarily, if necessary, the components of $F_{k-1}(\cdot)$ so that the first r_{k-1} rows of the matrix $\partial F_{k-1}(\cdot)/\partial u$ are linearly independent. Decompose $x^*_{k-1}(t+1)$ and $F_{k-1}(\cdot)$ accordingly:

$$x^*_{k-1}(t+1)=\begin{bmatrix} x^*_{k-1,1}(t+1) \\ x^*_{k-1,2}(t+1) \end{bmatrix}, \quad F_{k-1}(\cdot)=\begin{bmatrix} F_{k-1,1}(\cdot) \\ F_{k-1,2}(\cdot) \end{bmatrix},$$

where $x^*_{k-1,1}(t+1)$ and $F_{k-1,1}(\cdot)$ consist of the first r_1 elements. Since the last rows of $\partial F_{k-1}(\cdot)/\partial u$ are linearly dependent on the first r_{k-1} rows, one can write

$$x^*_{k-1,1}(t+1)=F_{k-1,1}(0, \ldots, 0, \tilde{x}_k(t), u(t), w(t)),$$

$$x^*_{k-1,2}(t+1)=R_{k-1}(x^*_{k-1,1}(t+1), \tilde{x}_k(t), w(t)).$$

So (10) clearly implies that

$$R_{k-1}(0, \tilde{x}_k(t), w(t))=0 \qquad (12)$$

for every $w(t) \in W^0$.

If it is possible to find a solution with respect to $\tilde{x}_k$ of the equation

$$R_k(0, \tilde{x}_k, w)=0,$$

which does not depend on w, then define $\lambda_k(\tilde{x}_k)$ as the left-hand side of this solution in the implicit form $\lambda_k(\tilde{x}_k)=0$.

If it is not possible to find such a solution, then the system (1) with disturbances does not have the zero dynamics, and the algorithm stops.

If the solution $\lambda_k(\tilde{x}_k)$ exists, it can be found in the following way. In a neighbourhood W^0 of w^0, the components R^s_{k-1} $(s=1, \ldots, \sigma_{k-1}-r_{k-1})$ of the nonlinear vector-valued function R_{k-1} can be represented in terms of Taylor series expansion

$$R^s_{k-1}(0, \tilde{x}_k, w)=R^s_{k-1,0}(\tilde{x}_k)+\sum_{l\geqslant 1} R^s_{k-1,l}(\tilde{x}_k)(w-w^0)^{[l]}.$$

Define the vector-valued function $\lambda_k(\tilde{x}_k)$ as the one formed from the independent (over the field of analytic functions of $\tilde{x}_2$) components of $R^s_{k-1,0}$, $R^s_{k-1,l}$, $l\geqslant 1$, $s=1, \ldots, \sigma_{k-1}-r_{k-1}$.

Note that $\lambda_k(\tilde{x}_k)$ is not identically zero because otherwise (10) would be solvable for $u(t)$. If one assumes that λ_k has constant rank s_k around $\tilde{x}^0_k$ and defines $\sigma_k := s_0+s_1+\ldots+s_k$ then this constraint yields locally the $(n-\sigma_k)$-dimensional subset $L_k := \lambda_k^{-1}(0)$. End of the kth step.

We summarize the constant rank assumptions made in the above algorithm in

Definition 4. *An isolated equilibrium point* (x^0, u^0, w^0) *with* $h(x^0)=0$ *is a regular point for the zero-dynamics algorithm if for each* $k\geqslant 0$ *the mapping* λ_k *and the matrix* $\partial F_k(\cdot)/\partial u$ *have constant* rank *around* (x^0, u^0, w^0).

If (x^0, u^0, w^0) is a regular equilibrium point for the zero-dynamics algorithm, then it easily follows that the algorithm terminates after $k^* < n$ iterations. Actually, if at a kth step λ_k is not identifically zero on L_{k-1} and $L_k = \lambda_k^{-1}(0)$ is a smooth manifold, then $\dim L_k < \dim L_{k-1}$.

The algorithm stops if one of the following cases occurs.

1. $R_{k^*}(0, \tilde{x}_{k^*}, w) = 0$ is not solvable.

Then the system (1) does not have the zero dynamics.

2. $L_{k^*} = \{x^0\}$.

Then the system (1) has the trivial zero dynamics.

3. $L_{k^*} = L_{k^*-1}$ with $\dim L_{k^*} \neq 0$.

Then L_{k^*} is the zero-dynamics submanifold.

The zero dynamics is either an autonomous system or a control system on L_{k^*}, depending on the number of solutions of the equation

$$F_{k^*-1}(0, \ldots, 0, \tilde{x}_{k^*}, u, w) = 0 \qquad (13)$$

with respect to u. If the Eq. (13) has a unique solution $\varphi(\tilde{x}_{k^*}(t), w(t))$, then the zero dynamics are formed by the following autonomous system

$$\tilde{x}_{k^*}(t+1) = f_{k^*}(0, \ldots, 0, \tilde{x}_{k^*}(t), \varphi(\tilde{x}_{k^*}(t), w(t)), w(t)).$$

If the solution of the Eq. (13) can be written as a family of functions $\varphi(\tilde{x}_{k^*}(t), w(t), v(t))$ parametrized by a vector $v(t) \in R^{m_2}$, $m_2 \leqslant m$, then the zero dynamics are formed by the following control system

$$\tilde{x}_{k^*}(t+1) = f_{k^*}(0, \ldots, 0, \tilde{x}_{k^*}(t), \varphi(\tilde{x}_{k^*}(t), w(t), v(t)), w(t))$$

with control inputs $v(t)$. The latter can happen, for example, in the case of a nonsquare (i. e. the system of less outputs than inputs) system.

4. Examples

We illustrate the results of this paper with the aid of some simple examples.

Example 1. Consider a system of the form (1) with 2 control inputs, 1 disturbance input and 2 outputs, defined on R^4 with

$$f(x, u, w) = \begin{bmatrix} x_2 + w \\ x_3 u_1 \\ x_4 \\ u_2 \end{bmatrix}$$

$$h_1(x) = x_1,$$
$$h_2(x) = x_3.$$

Proceeding with the zero-dynamics algorithm, we see that $\lambda_0(x) = h(x)$ has rank 2 for all x. Thus $s_0 = 2$ and

$$L_0 = \{x \in R^4 \mid x_1 = x_3 = 0\}.$$

The constraint $y(t) = 0$ for all t implies that the following equalities have to hold for every $w(t)$:

$$x_2(t) + w(t) = 0,$$
$$x_4(t) = 0.$$

Of course, it is not possible to find a solution of this system of equations with respect to x_2 and x_4, which does not depend on w, and the algorithm terminates. Consequently, this system does not have the zero dynamics.

Example 2. Consider a system of the form (1) with 2 control inputs, 2 disturbance inputs and 2 outputs, defined on R^4 with

$$f(x, u, w)=\begin{bmatrix} x_2w_1+u_1 \\ x_2x_3+x_4w_2+x_3u_1 \\ u_2 \\ x_3 \end{bmatrix}$$

$$h_1(x)=x_1,$$
$$h_2(x)=x_2.$$

Proceeding with the zero-dynamics algorithm, we find $s_0=2$ and

$$L_0=\{x\in R^4\,|\,x_1=x_2=0\}.$$

The constraint $y(t)=0$ for all t implies that the following equalities have to hold for every $w(t)$:

$$x_1(t+1)=u_1(t)=0,$$
$$x_2(t+1)=x_4(t)w_2(t)+x_3(t)u_1(t)=0.$$

The matrix

$$\frac{\partial F_0(\cdot)}{\partial u}=\begin{bmatrix} 1 & 0 \\ x_3 & 0 \end{bmatrix}$$

has $\operatorname{rank} r_0=1$ for all $x\in L_0$ and the algorithm can be continued. Only $x_1(t+1)$ can be made zero by the suitable choice of control $u_1(t)=0$, but to make $x_2(t+1)$ equal to zero, the equality

$$x_4(t)w_2(t)=0$$

has to hold for every $w(t)$. Thus $\lambda_1=x_4$, $s_1=1$ and

$$L_1=\{x\in R^4\,|\,x_1=x_2=x_4=0\}.$$

The constraints $y(t)=0$ and $\lambda_1(t)=0$ for all t imply that the following equalities have to hold

$$x_1(t+1)=u_1(t)=0,$$
$$x_2(t+1)=x_3(t)u_1(t)=0,$$
$$x_4(t+1)=x_3(t)=0.$$

The matrix

$$\frac{\partial F_1(\cdot)}{\partial u}=\begin{bmatrix} 1 & 0 \\ x_3 & 0 \\ 0 & 0 \end{bmatrix}$$

has still $\operatorname{rank} r_1=1$ for all $x\in L_1$. Provided $x(t)\in L_1$, $x_1(t+1)$ and $x_2(t+1)$ can be made equal to zero by the suitable choice of control, but to make $x_4(t+1)$ equal to zero, $x_3=0$ has to hold.

Thus $\lambda_2=x_3$, $s_2=1$,

$$L_2=\{x\in R^4\,|\,x_1=x_2=x_3=x_4=0\}=\{0\}$$

and therefore the algorithm terminates. The system has the trivial zero dynamics.

Example 3. Consider a system of the form (1) with 2 control inputs, 1 disturbance input and 2 outputs, defined on R^4 with

$$f(x,u,w)=\begin{bmatrix} x_3+u_1+\ln u_2 \\ x_4-u_1-\ln u_2 \\ wx_1+\sin x_4 \\ x_3-x_1+u_1+2\ln u_2+w \end{bmatrix}$$

$$h_1(x)=x_1,$$
$$h_2(x)=x_2.$$

Proceeding with the zero-dynamics algorithm, we find $s_0=2$ and

$$L_0=\{x\in R^4\,|\,x_1=x_2=0\}.$$

The constraint $y(t)=0$ for all t implies that the following equalities have to hold

$$x_1(t+1)=x_3(t)+u_1(t)+\ln u_2(t)=0,$$
$$x_2(t+1)=x_4(t)-u_1(t)-\ln u_2(t)=0.$$

The matrix

$$\frac{\partial F_0(\cdot)}{\partial u}=\begin{bmatrix} 1 & \frac{1}{u_2} \\ -1 & \frac{-1}{u_2} \end{bmatrix}$$

has rank $r_0=1$ for all $x\in L_0$ and the algorithm can be continued. Only $x_1(t+1)$ can be made zero by the suitable choice of the control, $u_1(t)+\ln u_2(t)=-x_3(t)$, but to make $x_2(t+1)$ equal to zero, the equality

$$x_4(t)+x_3(t)=0$$

has to hold. Thus $\lambda_1=x_3+x_4$, $s_1=1$ and in the new coordinates ($z_2=x_1$, $z_2=x_2$, $z_3=x_3+x_4$, $z_4=x_4$)

$$L_1=\{z\in R^4\,|\,z_1=z_2=z_3=0\}.$$

The constraints $y(t)=0$ and $\lambda_1(t)=0$ for all t imply that the following equalities have to hold

$$z_1(t+1)=-z_4(t)+u_1(t)+\ln u_2(t)=0,$$
$$z_2(t+1)=z_4(t)+u_1(t)-\ln u_2(t)=0,$$
$$z_3(t+1)=-z_4(t)+\sin z_4(t)+u_1(t)+2\ln u_2(t)+w(t)=0.$$

The matrix

$$\frac{\partial F_1(\cdot)}{\partial u}=\begin{bmatrix} 1 & \frac{1}{u_2} \\ -1 & -\frac{1}{u_2} \\ 1 & \frac{2}{u_2} \end{bmatrix}$$

has now rank $r_1=2$ for all $z\in L_1$ and the algorithm stops, because L_1 is the zero-dynamics submanifold. The system

$$z_4(t+1)=\sin z_4(t)$$

characterizes the zero-output constrained dynamics.

E x a m p l e 4. Consider a system of the form (1) with 2 control inputs, 1 disturbance input and 2 outputs, defined on R^4 with

$$f(x, u, w)=\begin{bmatrix} x_2+u_1 \\ x_2x_3+x_3u_1 \\ u_2 \\ x_3+w \end{bmatrix}$$

$$h_1(x)=x_1,$$

$$h_2(x)=x_2.$$

Proceeding with the zero-dynamics algorithm, we find $s_0=2$ and

$$L_0=\{x\in R^4 \mid x_1=x_2=0\}.$$

The constraint $y(t)=0$ for all t implies that the following equalities have to hold

$$\begin{aligned} u_1(t)&=0, \\ x_3(t)u_1(t)&=0. \end{aligned} \tag{14}$$

In spite of the fact that the matrix

$$\frac{\partial F_0}{\partial u}=\begin{bmatrix} 1 & 0 \\ x_3 & 0 \end{bmatrix}$$

has the rank $r_0=1$, the equations (14) can be solved for $u(t)$ (not uniquely): $u_1(t)=0$, $u_2(t)$ is arbitrary. Therefore, the algorithm stops, because L_0 is the zero-dynamics submanifold. The system

$$x_3(t+1)=u_2(t),$$

$$x_4(t+1)=x_3(t)+w(t)$$

with the arbitrary $u_2(t)$ characterizes the zero-output constrained dynamics.

REFERENCES

1. *Byrnes, C. I., Isidori, A.* Proc. 23rd IEEE Conf. on Decision and Control. Las Vegas, NV, 1984, 1569—1573.
2. *Monaco, S. and D. Normand-Cyrot.* Proc. 26th IEEE Conf. on Decision and Control, Los Angeles, 1987, 979—986.
3. *Isidori, A., Moog, C. H.* Lect. Notes in Control and Inf. Sci., 1988, 105.
4. *Di Benedetto, M. D., Slotine, J.-J. E.* Proc. 27th IEEE Conf. on Decision and Control. Austin, Texas, 1988, 915—920.
5. *Isidori, A., Byrnes, C. I.* IEEE Trans. Autom. Control, 1990, 35, 2, 131—140.
6. *Byrnes, C. I., Isidori, A.* Systems and Control Letters, 1988, 11, 9—17.
7. *Van der Schaft, A. J.* Proc. M.T.N.S. Phoenix, AZ, 1988, 499—506.
8. *Di Benedetto, M. D.* IEEE Trans. Autom. Control, 1990, 35, 12, 1351—1355.
9. *Di Benedetto, M. D., Grizzle, J. W.* Asymptotic and exact model matching for nonlinear systems. The University of Michigan. Dept. of Electrical Eng. and Comp. Sci., 1991. Report No CGR-91-1.
10. *Kotta, Ü.* Prepr. 2nd IFAC Symp. on Nonlinear Control Systems Design, Bordeaux, 1992, 563—568.

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Ülle KOTTA

DISKREETSE MITTELINEAARSE SÜSTEEMI NULLDÜNAAMIKAST SISENDHÄIRINGUTE OLEMASOLUL

On käsitletud diskreetse ajaga mittelineaarse süsteemi väljundite nulliga võrdsustamise ülesannet juhul, kui süsteemi mõjutavad mõõdetavad sisendhäiringud, ning uuritud võimalust leida juhttoime tagasiside kujul oleku järgi, mis tagaks suletud süsteemi väljundi võrdumise nulliga. Sel eesmärgil on üldistatud nulldünaamika algoritm diskreetse ajaga mittelineaarsetele häiringutega süsteemidele. Algoritmi abil on leitud nn. (lokaalne) nulldünaamika alammuutkond ja vaadeldava ülesande lokaalne lahend süsteemi tasakaalupunkti ümbruses. Ülesanne on lahenduv, kui süsteemi algolek asub nimetatud alammuutkonnal. Ka tagasiside võrrandid on leitud nulldünaamika algoritmi abil.

Юлле КОТТА

О НУЛЕВОЙ ДИНАМИКЕ НЕЛИНЕЙНОЙ СИСТЕМЫ ДИСКРЕТНОГО ВРЕМЕНИ ПРИ ВХОДНЫХ ВОЗМУЩЕНИЯХ

Рассматривается задача обнулевания выхода нелинейной системы дискретного времени при постоянно действующих измеряемых входных возмущениях. Изучается возможность построения управления в виде обратной связи по состоянию, обеспечивающего тождественно равный нулю выход замкнутой системы. С этой целью обобщается алгоритм нулевой динамики для названного класса систем и на основе его находится локальное решение задачи в окрестности точки равновесия системы. С помощью алгоритма строится т. н. (локальное) подмногообразие нулевой динамики. Изучаемая задача имеет решение, если начальное состояние системы принадлежит к подмногообразию нулевой динамики. Уравнения обратной связи найдены также как результат применения алгоритма.