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# ROBUST CONTROL FOR A CLASS OF LINEAR SYSTEMS

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**Abstract.** A robust digital controller design procedure for a class of linear systems is proposed following the modal control algorithm. This particular class of linear plants is introduced by the use of a linear-fractional mapping on the system parameter space which transforms the unit circle into itself. The proposed procedure is also suited for designing a controller for a plant with parameter uncertainties in a fixed direction.

## 1. Introduction

Much attention has been paid to the design problems of robust control systems and various design procedures have been proposed for different types of systems. A problem of current interest in the theory of robust control is the stability of linear dynamic systems with structured perturbations or uncertainties [1–3]. The Kharitonov's theorems [4], which give necessary and sufficient conditions for robust stability of Hurwitz polynomials, are mainly applicable to continuous interval systems. To obtain the robust stability conditions for discrete systems (Schur polynomials) starting from the corresponding conditions for Hurwitz polynomials, the bilinear (linear-fractional) mapping on the complex plane is used which transforms the left half plane into the unit circle.

In this paper we will use a linear-fractional mapping with a free parameter which transforms the unit circle into itself. On the one hand this mapping does not alter the stability properties of a discrete system [5]. But on the other hand the free parameter variation in a fixed region determines a class of admissible structured perturbations of the plant. Making use of the modal control algorithm, a robust digital controller will be designed for this particular class of linear plants. The direction of admissible variations of the plant parameters is determined by the closed-loop poles. So we have to solve a problem of choosing the desired closed-loop poles in accordance with the plant parameters uncertainties.

## 2. Problem statement

Let us consider a linear single-input plant in the state space form

$$x(t+1)=Ax(t)+bu(t) \qquad (1)$$
$$t=0,\ 1,\ 2,\ \ldots,$$

where $x(t)$ is the $n$-dimensional state vector, and $u(t)$ is the scalar input variable. For simplicity of presentation, let us assume that the plant model (1) is given in the controllable canonical form, i. e.

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$$A=\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ a_1 & a_2 & a_3 & \dots & a_n \end{bmatrix}, \qquad b=\begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}.$$

Let us assume that the plant parameters vector $a=(a_1, \dots, a_n)^T$ is uncertain in a fixed direction $c=(c_1, \dots, c_n)^T$, i. e.

$$a_i(\tau)=a_i+\tau c_i, \quad \tau\in R. \tag{2}$$

We have to find such a state feedback

$$u(t)=k^T x(t) \tag{3}$$

that the closed-loop system

$$x(t+1)=(A+bk^T)x(t) \tag{4}$$

will be stable for the nominal plant $a=a(0)$ and insensible against the plant parameters variations according to (2).

The idea of a solution to the above problem is the following. First of all we will introduce a mapping with a free parameter $\xi$ on the parameter space of the closed-loop system such that the stability is guaranteed for large variations of this free parameter. Next, we will define a class of linear plants generated by this free parameter $\xi$ and show that such a stationary state feedback exists that the closed-loop system will be stable for all this class of plants. The defined class of linear plants can be represented as a curve in the parameters space $a(\xi)\in R^n$. We have to choose such poles of the closed-loop system that the straight line (2) fits into the curve $a(\xi)$ for small $\xi$.

### 3. Stability of closed-loop systems

It is well known in the theory of discrete linear systems that the poles $\lambda_i$ of a stable system are placed inside the unit circle, i. e. $|\lambda_i|<1$, $i=1, \dots, n$. The linear-fractional mapping

$$\mu=e^{j\varphi}\frac{\lambda-\xi}{1-\bar{\xi}\lambda}$$

where $\lambda\in C$, $\mu\in C$, $\xi\in C$, $\varphi\in R$, $\bar{\xi}$ is the conjugate of $\xi$, $|\xi|<1$ transforms the unit circle into itself [6], i. e. $|\mu|<1$ if $|\lambda|<1$.

As the poles of a linear dynamic system must be placed symmetrically with respect to the real axis, we will use the mapping

$$\mu_i=\frac{\lambda_i-\xi}{1-\xi\lambda_i}, \quad i=1, \dots, n, \tag{5}$$

where $\xi\in(-1,1)$. Then to a real pole $\lambda_i$ corresponds a real $\mu_i$, and to a pair of conjugate poles $\lambda_i$ and $\bar{\lambda}_i$ corresponds also a pair of conjugates $\mu_i$ and $\bar{\mu}_i$.

Let

$$D(z)=\sum_{i=0}^{n} d_i z^i$$

be the characteristic polynomial of a stable closed-loop system. Then it can be shown by the use of the mapping (5) that the class of polynomials

$$E(z, \xi) = \sum_{i=0}^{n} e_i(\xi) z^i,$$

where

$$e_i(\xi) = \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{n-j}{i-k} \binom{j}{k} \xi^{i+j-2k} d_j, \tag{6}$$

$\binom{j}{k}$ is the binomial coefficient, $\xi \in (-1, 1)$ is also stable [5].

## 4. Robust controller for a class of linear plants

If the plant (1) is controllable, then such a state feedback (3) exists that an arbitrary prescribed set of closed-loop poles is available in the unit circle [7]. Starting from the characteristic polynomial of the closed-loop system $D(z)$ by $d_n = 1$ and the equation (4), we obtain for a plant in the controllable canonical form the following feedback gain vector

$$k_i = d_{i-1} - a_i, \quad i = 1, \ldots, n. \tag{7}$$

Starting from the class of stable (Schur) polynomials $E(z, \xi)$, $\xi \in (-1, 1)$, we obtain

$$k_i(\xi) = \bar{e}_{i-1}(\xi) - a_i,$$

where

$$\bar{e}_{i-1}(\xi) = \frac{e_{i-1}(\xi)}{e_n(\xi)}, \quad i = 1, \ldots, n$$

or, taking into account (6),

$$\bar{e}_i(\xi) = \frac{\sum_{j=0}^{n} \sum_{k=0}^{j} \binom{n-j}{i-k} \binom{j}{k} \xi^{i+j-2k} d_j}{\sum_{j=0}^{n} \xi^{n-j} d_j}, \quad i = 0, \ldots, n-1 \tag{8}$$

$$\bar{e}_n(\xi) = 1.$$

Let us now introduce a class of the following linear plants

$$a_i(\xi) = a_i - d_{i-1} + \bar{e}_{i-1}(\xi), \quad i = 1, \ldots, n. \tag{9}$$

$$\xi \in (-1, 1).$$

Taking account of (7) and (8), we can obtain that the feedback gain vector

$$k_i = d_{i-1} - a(0) \tag{10}$$

stabilizes all the class of plants (9) if only the polynomial $D(z)$ is stable. In other words, the controller (10) is robust against the plant parameters variation according to (9).

The proposed robust controller has poor practical importance because:

1) the class of plants (9) is very specific,

2) the polynomial $D(z)$ is not known,

3) the coefficients $d_0, \ldots, d_{n-1}$ must fit the plant parameters variation.

Therefore we are looking for an approximate robust controller starting from the known direction of the plant parameters variation.

### 5. Approximate robust controller design

Let us now consider the problem of the robust controller design stated in the second section. We know the nominal plant (1) and the direction of the plant parameters variation (2). We have to choose an approximate closed-loop polynomial $D(z)$ in order to design a robust modal controller for this plant.

We can find the tangent line to the curve (9) as a function of the closed-loop characteristic polynomial coefficients $d_0, \ldots, d_{n-1}$. Let us denote

$$\frac{da_i(\xi)}{d\xi} = \bar{d}_i(\xi;\ d_0, \ldots, d_{n-1}).$$

If the equalities

$$c_i = \bar{d}_i(0;\ d_0, \ldots, d_{n-1}) \tag{11}$$

hold, then the tangent line fits the line (2) because from (9) by $\xi=0$ we obtain $a_i = a_i(0)$. We have to solve the system of the equation (11) in respect to the coefficients $d_0, \ldots, d_{n-1}$ of the polynomial $D(z)$. If the solution represents a Schur polynomial, then the problem is solved and the controller can be found from the Eq. (10).

If none of the solutions to (11) is a Schur polynomial, we have to choose such a Schur polynomial $\tilde{D}(z)$ with coefficients $\tilde{d}_0, \ldots, \tilde{d}_{n-1}$ that the angle between the line (2) and the tangent line defined by $\tilde{d}_i = \bar{d}_i(0;\ \tilde{d}_0, \ldots, \tilde{d}_{n-1})$ is minimal. Then the feedback gain vector is

$$k_i = \tilde{d}_{i-1} - a_i(0), \quad i = 1, \ldots, n.$$

E x a m p l e. Let us have a second-order plant

$$x(t+1) = \begin{bmatrix} 0 & 1 \\ 2 & a_2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

with an uncertain parameter $a_2 \in (2{,}5;\ 3{,}5)$, i. e. $c^{\mathrm{T}} = [0\ \ c_2]$ and $a_2(\tau) = 3 + \tau c_2$.

From (9) we obtain

$$a_1(\xi) = \frac{d_0 + d_1\xi + \xi^2}{1 + d_1\xi + d_0\xi^2} - d_0,$$

$$a_2(\xi) = 3 - d_1 + \frac{d_1 + 2(d_0+1)\xi + d_1\xi^2}{1 + d_1\xi + d_0\xi^2}$$

and, according to (11), we have by $\xi=0$

$$\frac{da_1(\xi)}{d\xi} = d_1(1 - d_0) = 0,$$

$$\frac{da_2(\xi)}{d\xi} = 2(d_0+1) - d_1^2 = c_2.$$

The above equalities hold if $d_0 = 0{,}5c_2 - 1$, $d_1 = 0$.

The polynomial $D(z)$ is stable if $c_2 \in (0; 4)$. For $c_2 = 1$ we obtain from (10) the controller $k^{\mathrm{T}} = [-2{,}5;\ -3]$.

## 6. Conclusions

A robust digital controller design procedure for a class of linear systems has been proposed following the modal control algorithm. This particular class of linear plants has been introduced by the use of a linear-fractional mapping on the system parameters space. For the sake of simplicity, a plant in the controllable canonical form has been considered. The proposed procedure is also suited for designing a controller which is little sensible in respect to the plant parameters variation in a fixed direction.

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### ÜHE LINEAARSETE SÜSTEEMIDE KLASSI ROBUSTNE JUHTIMINE


On vaadeldud robustse diskreetse regulaatori sünteesi ülesannet ühe lineaarsete objektide klassi puhul. See klass on defineeritud murdlineaarse teisendusega süsteemi parameetrite (pooluste) ruumis. On näidatud, et murdlineaarses teisenduses esineva vaba parameetri $\xi$ muutused piirides $\xi \in (-1, 1)$ ei põhjusta suletud süsteemi ebastabiilsust.



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### РОБАСТНОЕ УПРАВЛЕНИЕ КЛАССОМ ЛИНЕЙНЫХ СИСТЕМ


Рассматривается задача синтеза робастного цифрового регулятора для одного класса линейных объектов. Этот класс линейных объектов определен с помощью дробно-линейного преобразования в пространстве параметров (полюсов) системы. Показано, что вариация свободного параметра $\xi$ дробно-линейного преобразования в пределах $\xi \in (-1, 1)$ не причиняет неустойчивости замкнутой системы.