Proc. Estonian Acad. Sci. Phys. Math. 1993, 42, 1, 7—13
https://doi.org/10.3176/phys.math.1993.1.01

UDC 548.552.24

Hillar ABEN * *and Alfred PURO* *

INTEGRATED PHOTOELASTICITY FOR AXISYMMETRIC FLOW BIREFRINGENCE STUDIES

1. Introduction

The birefringent fluid-flow method is one of the techniques for visually investigating complex fluid-flow problems. The majority of the existing visual techniques yield only qualitative information about the fluid motion [1]. The flow birefringent method has been used for quantitative studies of flow problems for almost forty years.

There are numerous publications related to flow birefringence. We refer to the fundamental study of the flow birefringence phenomenon by Pindera and Krishnamurthy [2], and to the review paper by Pih [3]. These papers also give rather complete bibliographies on the subject.

Most of the flow birefringence studies are devoted to the two-dimensional laminar flow. However, some results have also been obtained by investigating the two-dimensional turbulent flow [4].

As for the three-dimensional flow, the photoelastic scattered light method has been used to investigate it by several authors [5–7]. However, the drawback in connection with the scattered light method is that it is time-consuming; also it is difficult to make measurements automatically.

Integrated photoelasticity [8] is a nondestructive method for the three-dimensional stress analysis in transparent specimens. It is based on the measurement of the change in the polarization of light passed through a birefringent medium. In the general case, the theory of integrated photoelasticity is rather complicated. However, if birefringence is weak, simple integral relationships can be used in determining the stress field [9]. In the latter case, integrated photoelasticity may be interpreted as tensor field tomography [10, 11]. In comparison with the classical, scalar field tomography [12], tensor field tomography is more complicated, but also more informative.

Tomographic methods have also been used in investigating fluid and air flows [13–16]. Since in these investigations flow is regarded as a scalar field, only a few scalar characteristics of the flow, e. g. the distribution of the average (scalar) refractive index, or of the extinction coefficient, have been determined. An attempt to use tomography for the study of the flow as a vector field, has not been successful [17].

Integrated photoelasticity permits, in principle, to apply tomographic techniques for tensor fields. That has already been demonstrated in relation to axisymmetric and nonaxisymmetric stress tensor fields [10, 11; 18], also for the Kerr effect field [19, 20].

---

* Eesti Teaduste Akadeemia Küberneetika Instituut (Institute of Cybernetics, Estonian Academy of Sciences), Akadeemia tee 21, EE-0026 Tallinn. Estonia.

Since viscous flow is characterized by the distribution of the strain rate tensor, an attempt to determine this field using tensor field tomographic techniques is a natural development in experimental flow mechanics.

The aim of this paper is to show how integrated photoelasticity can be applied in investigating an axisymmetric laminar flow of weakly birefringent fluid.

2. Basic optical relationships

In the general case of passing the polarized light through a nonhomogeneous birefringent medium, the optical phenomena are complicated [8]. However, if birefringence of the medium is weak, optical relationships are much more simple [9]. Let us assume that the flow is only weakly birefringent. In the latter case, using methods of two-dimensional photoelasticity one can measure on each light ray the parameter of the isoclinic φ and optical retardation δ. These parameters are related to the components of the refractive index "tensor" n_1, n_2 and n_{12} (in the plane perpendicular to the wave normal) through the following relationships [9]:

$$\delta \cos 2\varphi = \int (n_{11} - n_{22})\, dl, \tag{1}$$

$$\delta \cos 2\varphi = 2 \int n_{12}\, dl. \tag{2}$$

Bearing in mind that birefringence of the flow is weak, these relationships can be written as

$$\delta \cos 2\varphi = \frac{1}{2n_0} \int (\varepsilon_{11} - \varepsilon_{22})\, dl, \tag{3}$$

$$\delta \sin 2\varphi = \frac{1}{n_0} \int \varepsilon_{12}\, dl, \tag{4}$$

where n_0 is the initial refractive index of the fluid, and ε_{ij} is the dielectric tensor.

Let the optical effect be a function of the strain rates $\dot{e}_{pq}$ [3, 21]. That is,

$$\frac{1}{2n_0} \varepsilon_{ij} = f_{ij}(\dot{e}_{pq}). \tag{5}$$

This can be expanded by the use of Taylor series expansion to give [21]

$$\frac{1}{2n_0} \varepsilon_{ij} = a_0 \delta_{ij} + a_1 \dot{e}_{ij} + a_2 \dot{e}_{ik} \dot{e}_{kj} + a_3 \dot{e}_{ik} \dot{e}_{km} \dot{e}_{mj} + \ldots, \tag{6}$$

where the a_i's are constants and repeated subscripts imply summation. This expansion can be contracted by the use of the Cayley-Hamilton theorem [22].

$$\frac{1}{2n_0} \varepsilon_{ij} = \alpha_0 \delta_{ij} + \alpha_1 \dot{e}_{ij} + \alpha_2 \dot{e}_{ik} \dot{e}_{kj}, \tag{7}$$

where α_i's are functions of the physical properties of the fluid and of the invariants formed from $\dot{e}_{ij}$. Practically α_i's can be assumed to be constants. They are named flow-optic constants and are determined experimentally.

Eq. (7) reveals [21]

$$\frac{1}{2n_0}(\varepsilon_{11}-\varepsilon_{22})=\alpha_1(\dot e_{11}-\dot e_{22})+\alpha_2[(\dot e_{11}+\dot e_{22})(\dot e_{11}-\dot e_{22})+\dot e_{13}^2-\dot e_{23}^2], \tag{8}$$

$$\frac{1}{n_0}\varepsilon_{12}=2\alpha_1\dot e_{12}+\alpha_2[2(\dot e_{11}+\dot e_{22})\dot e_{12}+2\dot e_{13}\dot e_{32}]. \tag{9}$$

Let us assume, in the first approximation, that $\alpha_2=0$. Then Eqs. (3) and (4) can be written as

$$\delta\cos 2\varphi=\alpha_1\int(\dot e_{11}-\dot e_{22})\,dl, \tag{10}$$

$$\delta\sin 2\varphi=2\alpha_1\int\dot e_{12}\,dl. \tag{11}$$

Eqs. (10) and (11) are the basic optical relationships we are going to use for determining the axisymmetric flow field.

3. Determining the axial velocity gradient

Fig. 1 shows geometrical notations in the cross-section of an axisymmetric flow (coordinate z is along the axis of the flow).

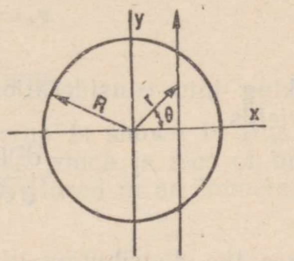

Fig. 1. Cross-section of an axisymmetric flow and geometrical notations.

Components of the strain rate tensor $\dot e_{ij}$ can be expressed as follows:

$$\dot e_{xx}=\frac{\partial v_x}{\partial x},\quad \dot e_{yy}=\frac{\partial v_y}{\partial y},\quad \dot e_{zz}=\frac{\partial v_z}{\partial z},\quad \dot e_{xz}=\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}, \tag{12}$$

where v_i are the components of the velocity vector.

Condition of incompressibility [23]

$$\operatorname{div}\bar v=0, \tag{13}$$

or

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=0 \tag{14}$$

can be written as

$$\dot e_{xx}+\dot e_{yy}+\dot e_{zz}=0, \tag{15}$$

or

$$\dot e_{xx}=-\dot e_{yy}-\dot e_{zz}. \tag{16}$$

For a light ray passing the cross-section parallel to the y axis (Fig. 1), we can write Eqs. (10) and (11) as

$$\delta\cos 2\varphi=\alpha_1\int(\dot e_{xx}-\dot e_{zz})\,dy, \tag{17}$$

$$\delta\sin 2\varphi=2\alpha_1\int\dot e_{xz}\,dy. \tag{18}$$

Relationships (16) and (17) reveal

$$\delta \cos 2\varphi = -\alpha_1 \int (\dot{e}_{yy} + 2\dot{e}_{zz})\, dy = -\alpha_1 \int \left(\frac{\partial v_y}{\partial y} + 2\frac{\partial v_z}{\partial z} \right) dy. \tag{19}$$

Since on the boundary $v_y = 0$, Eq. (19) reveals

$$\delta \cos 2\varphi = -2\alpha_1 \int \frac{\partial v_z}{\partial z}\, dy. \tag{20}$$

The latter equation shows that line integral of the velocity gradient $\partial v_z/\partial z$ can be calculated for any ray, using the measured values of φ and δ. Thus, determining $\partial v_z/\partial z$ has been reduced to a problem of scalar field tomography, and the field of $\partial v_z/\partial z$ can be determined by using the Radon inversion in Eq. (20).

4. Determining the velocity field in the plane of the cross-section of the flow

Let us assume that our flow is potential, i.e. irrotational. In this case the flow velocity potential Ψ exists, i.e. the velocity components in the cross-section can be expressed as

$$v_x = \frac{\partial \Psi}{\partial x}, \quad v_y = \frac{\partial \Psi}{\partial y}. \tag{21}$$

Taking into consideration Eq. (21), the incompressibility condition (14) yields

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = -\frac{\partial v_z}{\partial_z}. \tag{22}$$

Since the distribution of $\partial v_z/\partial z$ has been determined by using Eq. (20), the flow velocity potential Ψ can be determined by solving the Poisson Eq. (22) with the boundary condition $\partial\Psi/\partial r = 0$. The velocity components can be calculated from the relationships (21).

5. Determining the axial velocity component v_z

In the relationship (18) we have (Fig. 1)

$$\dot{e}_{xz} = \dot{e}_{rz} \cos \theta. \tag{23}$$

Now Eq. (18) can be written as

$$\delta \sin 2\varphi = 2\alpha_1 \int \dot{e}_{rz} \cos \theta\, dl. \tag{24}$$

From the latter equation, the distribution of $\dot{e}_{rz}$ can be determined by using algorithms elaborated in integrated photoelasticity [11, 24]. The shear strain rate $\dot{e}_{rz}$ can be expressed in the following form

$$\dot{e}_{rz} = \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}, \tag{25}$$

which can be written as

$$\frac{\partial v_z}{\partial r} = \dot{e}_{rz} - \frac{\partial v_r}{\partial z}. \tag{26}$$

Here

$$v_r = v_x \cos \theta + v_y \sin \theta. \tag{27}$$

Let us assume that photoelastic measurements are carried out in two adjacent sections, a distance Δz apart from each other. In the previous Paragraph we showed that velocity components v_x and v_y may be determined for any cross-section. Eq. (27) permits to calculate v_r. Now from Eq. (26) follows

$$\frac{\partial v_z}{\partial r} = \dot{e}_{rz} - \frac{\Delta v_r}{\Delta z}, \qquad (28)$$

where Δv_r is the difference of v_r in the two sections.

Since the right side of Eq. (28) is known, we can calculate the value of the axial velocity component v_z:

$$v_z = -\int_r^R \left(\dot{e}_{rz} - \frac{\partial v_r}{\partial z} \right) dr + C. \qquad (29)$$

The integration constant C is determined from the condition

$$v_z(R) = 0. \qquad (30)$$

Thus, the velocity vector field can be completely determined in the section under investigation. As a result of carrying out measurements in a number of sections, the three-dimensional flow velocity field can be reconstructed.

6. Experimental considerations

A schematic diagram of the experimental setup is shown in Fig. 2. The flow test channel must be made of glass which is free of birefringence. Besides, the flow test channel must be placed in an immersion bath to avoid refraction of light (Fig. 3).

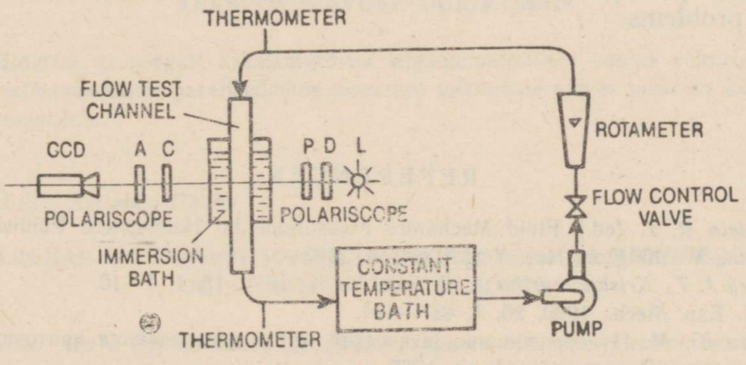

Fig. 2. The fluid circulation system and the polariscope: L — light source, D — diffusor, P — polarizer, C — compensator, A — analyzer, CCD — a CCD camera feeding data into a PC.

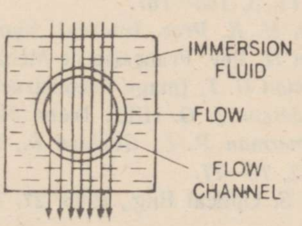

Fig. 3. The flow channel should be placed in an immersion bath to avoid refraction of light.

Different polariscopes can be used for recording the photoelastic data. For this purpose it is most convenient to use a computer controlled polariscope with a CCD camera.

The method described in this paper requires the measurement of the parameter of the isoclinic φ and of the optical retardation δ in sections under investigation. To calculate this data all over the field, in a polariscope with a CCD camera, light intensity patterns should be recorded for several positions of the polarizer, analyzer and compensator (this is the so-called phase-stepping method [25, 26]). These measurements take a few minutes. Therefore, the flow should be sufficiently stable.

Finding a suitable immersion fluid to match the refraction index of the flow channel glass, should present no problems. It may be more difficult to make birefringent fluid with the same index of refraction as that of the flow channel glass. In principle, it is possible to take into account the difference between the refractive indices of the fluid and of the flow channel [27].

7. Conclusions

This preliminary investigation has shown that integrated photoelasticity permits to determine the velocity field in an axisymmetric flow. The method described here can be generalized in various directions.

It is possible to take into consideration the nonlinear term in Eq. (7). A numerical algorithm can be elaborated which would permit to get rid of the assumption about weak birefringence. Bending of the light rays may be taken into account [23] as well. The method for axisymmetric flow can probably be expanded for flows of an arbitrary cross-section, analogously to determining stress in specimens of the arbitrary cross-section [18].

Finally, since flow birefringence has been observed in different molecular gases [28–31], integrated photoelasticity can be also used to solve air flow problems.

REFERENCES

1. *Goldstein R. J.* (ed.) Fluid Mechanics Measurements. Hemisphere Publishing Corporation, Washington, New York, London, 1983.
2. *Pindera J. T., Krishnamurthy A. R.* Exp. Mech., 1978, **18**, 1, 1—10.
3. *Pih H.* Exp. Mech., 1980, **20**, 4, 437—444.
4. *Бычков Ю. М.* Поляризационно-оптический метод исследования проточных частей насосов. Штиинца, Кишинев, 1975.
5. *Pindera J. T., Krishnamurthy A. R.* Exp. Mech., 1978, **18**, 2, 41—48.
6. *Ahimaz F. J.* Exp. Mech., 1970, **10**, 3, 133—134.
7. *McAfee W. J., Pih H.* Exp. Mech., 1974, **14**, 10, 385—391.
8. *Aben H.* Integrated Photoelasticity. McGraw-Hill Book Company, New York, 1979.
9. *Aben H. K., Josepson J. I., Kell K.-J. E.* Optics and Lasers in Engineering, 1989, **11**, 3, 145—157.
10. *Aben H. K.* Proc. Internat. Symp. Photoelasticity, Springer, Tokyo, 1986, 243—250.
11. *Aben H.* Rev. Française de Mécanique, 1989, 1, 121—130.
12. *Herman G. T.* Image Reconstruction by Projections. Academic Press, New-York, 1980.
13. *Sanderson J. G.* IEEE Trans. Nuclear Sci., 1979, **NS-26**, 2, 2685—2686.
14. *Emmerman P. J., Goulard R., Santoro R. J., Semerjian H. G.* J. Energy, 1980, **4**, 2, 70—77.
15. *Cha S.* Optical Eng., 1988, **27**, 7, 557—563.

16. *Cha S. S., Sun H.* Optical Eng., 1989, **28**, 11, 1241—1246.
17. *Kramer D. M., Lauterbur P. C.* IEEE Trans. Nuclear Sci., 1979, **NS-26**, 2, 2674—2677.
18. *Aben H., Idnurm S., Puro A.* Proc. Internat. Conf. Exp. Mech., Copenhagen, 1990, 2, 867—875.
19. *Hertz H. M.* Appl. Optics, 1986, **25**, 6, 914—921.
20. *Aben H. K.* Appl. Optics, 1987, **26**, 14, 2921—2924.
21. *Doyle J. F.* Exp. Mech., 1982, **22**, 1, 37—38.
22. *Rivlin R. S., Ericsen J. L.* J. Rat. Mech. Anal., 1965, **4**, 2, 323—425.
23. *Douglas J. F., Gasiorek J. M., Swaffield J. A.* Fluid Mechanics. Pitman, London, 1979.
24. *Aben H. K.* Proc. CSME Mechanical Engineering Forum, Toronto, 1990, 2, 143—146.
25. *Kihara T.* Proc. 9th Internat. Conf. Exp. Mech., Copenhagen, 1990, **2**, 821—827.
26. *Patterson E. A., Wang Z. F:* Proc. Conf. Appl. Solid Mech.-4, Leicester, 1991, 3—13.
27. *Абен X. К., Бросман Э. И., Саар А. Р.* Мат. VIII Всесоюзн. конф. по методу фотоупругости. Таллинн, 1979, 2, 12—15.
28. *Vest C. M.* Appl. Optics, 1985, **24**, 23, 4089—4094.
29. *Hess S.* Zeitschr. für Naturforschung, 1973, **28a**, 9, 1531—1532.
30. *Köhler W. E., Halbritter J.* Physica, 1974, **74**, 2, 294—306.
31. *Boyer G. R., Lamouroux B. F., Prade B. S.* J. Opt. Soc. Amer., 1975, **65**, 11, 1319—1320.

Received
September 9, 1992

Hillar ABEN, Alfred PURO

INTEGRAALNE FOTOELASTSUS TELGSÜMMEETRILISE VOOLU KAKSIKMURDVUSE UURIMISEKS

On näidatud, et nõrgalt kaksikmurdvas telgsümmeetrilises voolus võimaldab integraalne fotoelastsus kahes paralleellõikes teostatud mõõtmiste põhjal määrata kõiki kiirusvektori komponente.

Хиллар АБЕН, Альфред ПУРО

ИНТЕГРАЛЬНАЯ ФОТОУПРУГОСТЬ ДЛЯ ИССЛЕДОВАНИЯ ДВУПРЕЛОМЛЕНИЯ ОСЕСИММЕТРИЧНОГО ПОТОКА

Показано, что в случае слабо двупреломляющего осесимметричного потока интегральная фотоупругость позволяет определить все компоненты вектора скорости.