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S. B. FILIPPOV and P. E. TOVSTIK**

TURNING POINTS AND LINES IN LINEAR PROBLEMS OF FREE VIBRATIONS AND BUCKLING OF THIN SHELLS

(Presented by J. Engelbrecht)

The equations for thin elastic shells contain a natural small parameter $\mu>0$ connected with the relative shell thickness h/R. So, in solving these equations it is convenient to use the asymptotic integration methods. Some problems of shell vibrations and buckling may be reduced to linear ordinary differential equations. In other cases, we have to consider the equations with partial derivatives. If the coefficients of these equations are variable, then the construction of the asymptotic expansions of their solutions may be complicated by the turning points or caustics.

The first important results for the turning point problem in the thin-shell theory were obtained by N. A. Alumäe. This paper presents a review of some works in which the results of N. A. Alumäe on the axisymmetric conical-shell vibrations and cylindrical and conical shell buckling are developed.

Ordinary differential equations

1. The free axisymmetric vibrations for a shell of revolution may be described by equation [1], as follows:

$$(L_0+\mu^4L_1+\Lambda)\,U=0. \tag{1.1}$$

Here L_0, L_1 are the linear ordinary differential operators with the variable coefficients, $\mu^2\sim h/R$ is a small parameter; $U=(u,w)$, where u and w are the displacement projections of the middle surface points to the directions of the generatrix and the normal, Λ is the unknown frequency parameter.

System (1.1) may be reduced to an equation of the sixth order:

$$-\mu^4\sum_{k=1}^{6}a_k(x)\frac{d^kw}{dx^k}+\sum_{k=0}^{2}b_k(x)\frac{d^kw}{dx^k}=0, \tag{1.2}$$

$$a_6=1,\qquad b_2=\Lambda-k_2^2(x),$$

where $x\in[\alpha,\beta]$ is the generatrix length, $k_2(x)=B^{-1}\sqrt{1-B'^2}$ is the middle surface curvature, $B(x)$ is the distance between the point on the middle surface and the axis of symmetry.

We suppose that $B(x)$ is a regular function. Then all coefficients in (1.1) and (1.2) are also regular. The expressions of these coefficients in terms of B are given in [1].

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* Санкт-Петербургский университет (St. Petersburg University). Старый Петергоф, Санкт-Петербург, Россия.

In the intervals of x where $b_2(x) \neq 0$ Eq. (1.2) has four solutions with a large index of variation. These solutions may be represented as the following asymptotic series:

$$w_m(x, \mu) \simeq \sum_{k=0}^{\infty} \mu^k A_{km}(x) \exp\left[\frac{1}{\mu}\int_{x_0}^{x} q_m(t)\,dt\right], \tag{1.3}$$

$$A_{0m} = B^{-1/2} q_m^{-3/2}, \quad q_m = b_2^{1/4} \exp\frac{(m-1)\pi i}{2}, \quad m = 1, 2, 3, 4.$$

The other two solutions have asymptotic expansions by the powers of μ^4

$$w_n(x, \mu) \simeq \sum_{k=0}^{\infty} \mu^{4k} w_{kn}(x), \quad n = 5, 6, \tag{1.4}$$

where w_{05} and w_{06} are the solutions of the membrane equation

$$b_2 \frac{d^2 w}{dx^2} + b_1 \frac{dw}{dx} + b_0 w = 0. \tag{1.5}$$

The asymptotic expansions (1.3) and (1.4) are not valid near the turning point $x = x_0$, where $b_2(x_0) = 0$. For $x = x_0$ the coefficient A_{0m} goes to infinity and the solution w_{05} has a singular point.

In 1960 N. A. Alumäe [2] got the main terms of the asymptotic expansions of the solutions in the neighbourhood of the turning point for the conical shell. In 1965, the same results were obtained for the arbitrary shell of revolution [3].

Using the universal set of standard functions, the following terms of asymptotic expansions of solutions (1.1) have been found [4, 5]. If the turning point is simple $b_2'(x_0 \neq 0)$, then we may use as standard functions the solutions $v_0^{(i)}$, $i = 1, 2, \ldots, 5$ of the equation

$$\frac{d^5 v}{d\eta^5} - \eta \frac{dv}{d\eta} - v = 0.$$

In this case the asymptotic expansions of the solutions of Eq. (1.2) have the form

$$w^{(i)} \simeq \sum_{k=0}^{\infty} \varepsilon^k A_k(x)\, v_k^{(i)}(\eta) + \varepsilon \delta_{i5} w^*(x, \mu), \quad i = 1, 2, 3, 4, 5, \quad w^{(6)} = w_6, \tag{1.6}$$

$$\varepsilon = \mu^{4/5}, \quad \eta = \varepsilon^{-1}\left(\frac{5}{4}\int_{x_0}^{x} b_2^{1/4}\,dx\right)^{4/5}, \quad v_{k+1}^{(i)} = \int v_k^{(i)}\,d\eta, \quad k = 0, 1, \ldots$$

The functions A_k, w^*, w_{k6} and η are regular at $x = x_0$.

The solutions w_i form the fundamental set of the solutions of Eq. (1.2), hence the solutions $w^{(i)}$ may be represented as linear combinations of functions w_i. These linear combinations are different for $x < x_0$ and $x > x_0$. If $b_2'(x_0) > 0$ the relation between the solutions $w^{(i)}$ and w_i is given by formulas

$$\begin{gathered} D_1(w_1 - w_4) \leftarrow w^{(1)} \rightarrow 2D_1 w_1, \\ D_1(w_3 - w_2) \leftarrow w^{(2)} \rightarrow 2D_1(w_4 - w_2), \\ -iD_1(w_2 - w_3) \leftarrow w^{(3)} \rightarrow -iD_1 w_3, \\ iD_1(w_1 + w_4) \leftarrow w^{(4)} \rightarrow iD_1(w_2 + w_4) + D_2 w_6, \\ D_3 w_5 + D_4 w_6 \leftarrow w^{(5)} \rightarrow D_3 w_5 + D_4 w_6 + D_1(w_4 - w_2), \\ w_6 \leftarrow w^{(6)} \rightarrow w_6. \end{gathered} \tag{1.7}$$

The left sides of (1.7) correspond to the case $x<x_0$ and the right sides correspond to the case $x>x_0$. To get (1.7) the asymptotic expansions of standard functions, $v_k^{(i)}(\eta)$, as $|\eta|\to\infty$ are used. The constants D_i are given in [1,4]. The formulas for the case $b_2'(x_0)<0$ may be derived from (1.7) by the substitution of $-x$ for x. For the unknown function u similar formulas may be written. Functions (u_{50}, w_{50}) and (u_{60}, w_{60}) are the solutions of the membrane system

$$(L_0+\Lambda)U=0. \tag{1.8}$$

For the spheroidal shell there may also exist a double turning point at which $b_2(x_0)=b_2'(x_0)=0$, $b_2''(x_0)\neq 0$. The asymptotic solutions in the neighbourhood of x_0 may be built by means of the standard equation [5]

$$\frac{d^6v}{d\eta^6}=\eta^2\frac{d^2v}{d\eta^2}+4\eta\frac{dv}{d\eta}+\gamma v,$$

where γ is the given constant.

2. The expansions of the asymptotic solutions have been used for the approximate solving of the boundary problems for the axisymmetrical vibrations of the shells of revolution. Let the shell edges be clamped

$$u=w=w'=0 \quad \text{for} \quad x=\alpha, \quad x=\beta. \tag{2.1}$$

If

$$\Lambda<\Lambda^-, \quad \Lambda^-=\min_{x\in[\alpha,\beta]} k_2^2(x), \tag{2.2}$$

then $b_2(x)<0$ and interval $[\alpha, \beta]$ does not contain turning points. We find frequencies and vibration modes substituting the linear combination of solution (1.3) and (1.4)

$$u=\sum_{n=1}^{6} C_n u_n, \quad w=\sum_{n=1}^{6} C_n w_n \tag{2.3}$$

into boundary conditions (2.1). The equality to zero of the determinant $\Delta(\Lambda, \mu)$ of the received linear system with respect to constants C_n yields the frequency equation

$$\Delta(\Lambda,\mu)=\Delta_0(\Lambda)+O(\mu)=0, \quad \Delta_0(\Lambda)=u_{50}(\alpha)u_{60}(\beta)-u_{60}(\alpha)u_{50}(\beta).$$

Therefore, the eigenvalues Λ which satisfy unequality (2.2) differ from the eigenvalues of membrane system (1.8) with the boundary conditions $u(\alpha)=u(\beta)=0$ by the order of μ. This fact is caused by the stress-strain state of the shell to be represented for $\Lambda<\Lambda^-$ as a sum of main membrane state (1.4) and edge effect (1.3). In this case the degeneration of the boundary problem (1.1), (2.1) into the membrane boundary problem is regular [6].

If the unequality

$$\Lambda>\Lambda^+, \quad \Lambda^+=\max_{x\in[\alpha,\beta]} k_2^2(x),$$

is fulfilled, then $b_2(x)>0$ and the representation (2.3) may be also used as a general solution of the system (1.1). In this case only two solutions (1.3) are the functions of the edge effect. Two other functions (1.3) quickly oscillate. The frequency equation has the form

$$\Delta_0(\Lambda)\cos z+O(\mu)=0, \quad z(\Lambda,\mu)=\frac{1}{\mu}\int_{\alpha}^{\beta} b_2^{1/4}\,dx.$$

In the case $\Lambda^- \leqslant \Lambda \leqslant \Lambda^+$ $(\Lambda^- < \Lambda^+)$ the interval $[\alpha, \beta]$ contains the turning points. Let us suppose that $b'(x) > 0$ for $x \in [\alpha, \beta]$. For example, this supposition is valid for the conical shell. Then only one simple turning point $x = x_0$ $(\alpha \leqslant x_0 \leqslant \beta)$ exists.

Let us represent the general solution of (1.1) in the form

$$u = \sum_{n=1}^{6} C_n u^{(n)}, \quad w = \sum_{n=1}^{6} C_n w^{(n)}.$$

If the turning point is far from the shell edges

$$x_0 - \alpha \gg \varepsilon, \quad \beta - x_0 \gg \varepsilon, \tag{2.4}$$

then for $x = \alpha$ and $x = \beta$ solutions $(u^{(i)}, w^{(i)})$ ought to be replaced by the linear combinations of solutions (u_i, w_i) in accordance with (1.7). In this case the frequency equation has the form [1], as follows:

$$\Delta_0(\Lambda) \sin z_0 + G(\Lambda) \cos z_0 + O(\mu^{1/2}) = 0, \quad z_0 = \frac{1}{\mu} \int_{x_0}^{x} b_2^{1/4}\, dx.$$

Function $G(\Lambda)$ depends on the solutions of membrane system (1.8).

For $x < x_0$, the vibration mode is a sum of the membrane solutions and the edge effect solutions. For $x > x_0$, it contains two quickly oscillating solutions. The principal behaviour of the vibration mode component — $w(x)$ — is shown in Fig. 1.

If Λ is close to Λ^- or Λ^+, the problem of the approximate eigenvalues determination is more complex. In this case, the turning point lies near one of the edges of $[\alpha, \beta]$, and at this edge it is necessary to use expansions (1.6).

Let us now study the shell of revolution similar to the spheroidal shell for which the curvature $k_2(x)$ has for $x \in [\alpha, \beta]$ the single extremum in $x = x_0$ $(k_2''(x_0) \neq 0)$. The case $k_2''(x_0) > 0$ corresponds to the elongated spheroid, and the case $k_2''(x_0) < 0$ corresponds to the compressed spheroid. We suppose that $k_2(\beta) > k_2(\alpha)$. If $k_2''(x_0) > 0$, then for $\Lambda^- < \Lambda \leqslant k_2^2(\alpha)$ the interval $[\alpha, \beta]$ contains two simple turning points, and for $k_2^2(\alpha) < < \Lambda \leqslant \Lambda^+ = k_2^2(\beta)$ there is only one simple turning point. The value of $\Lambda = \Lambda^-$ corresponds to the double turning point. If $k_2''(x_0) < 0$, the two simple turning points exist for $k_2^2(\beta) < \Lambda \leqslant \Lambda^+$, and the value of $\Lambda = \Lambda^+$ corresponds to the double turning point.

In [7, 8] the approximate frequency equations have been got for the case when there were two simple turning points x_1 and x_2 which were sufficiently far from each other, namely:

$$x_2 - x_1 \gg \mu^{2/3}. \tag{2.5}$$

If condition (2.5) is fulfilled, the solution (u_i, w_i) may be used in the interval (x_1, x_2). We also suppose that for both turning points unequality (2.4) is fulfilled. Using twice formulas (1.7) (at points $x = x_1$ and $x = x_2$) we find the relations between the solutions at $x = \alpha$ and $x = \beta$ and use these relations for the construction of the approximate frequency equation.

For the shell of the elongated spheroidal type $(k_2''(x_0) > 0)$ this equation has the form:

$$P(\Lambda) \cos z_* + Q(\Lambda) \sin z_* + R(\Lambda) = 0, \quad z_* = \frac{1}{\mu} \int_{x_1}^{x_2} b^{1/4}\, dx.$$

The vibration mode quickly oscillates in the interval (x_1, x_2).

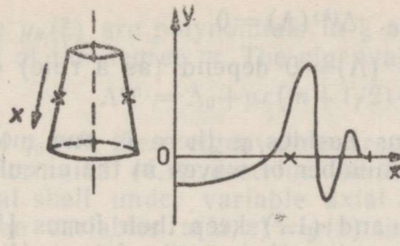

Fig. 1.

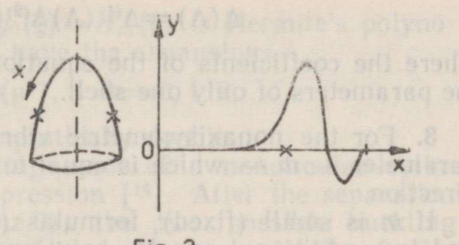

Fig. 2.

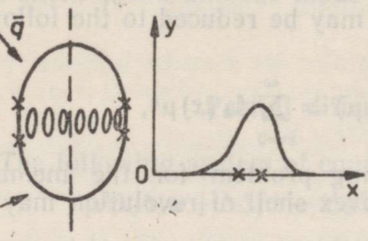

Fig. 3.

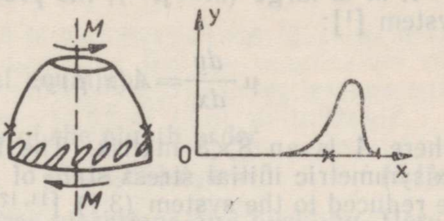

Fig. 4.

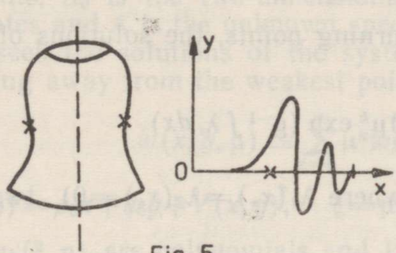

Fig. 5.

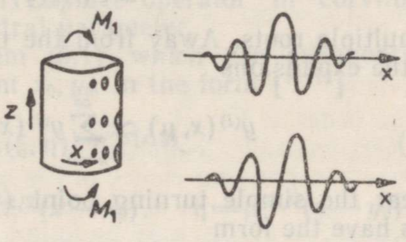

Fig. 6.

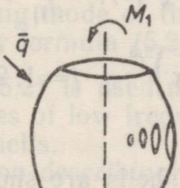

Fig. 7.

Fig. 8.

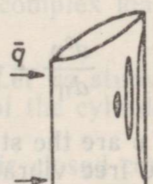

Fig. 9.

For the compressed spheroidal shell ($k_2''(x_0) < 0$) in the first approximation we have

$$K(\Lambda)\sin z_1 \sin z_2 + L(\Lambda)\sin z_2 \cos z_1 + M(\Lambda)\sin z_1 \cos z_2 + N(\Lambda)\cos z_1 \cos z_2 = 0,$$

$$z_1 = \frac{1}{\mu}\int_{\alpha}^{x_1} b^{1/4}\,dx, \qquad z_2 = \frac{1}{\mu}\int_{x_2}^{\beta} b^{1/4}\,dx.$$

The vibration modes oscillate in the intervals (α, x_1) and (x_2, β).

By using the asymptotic integration method the axisymmetric vibrations of the joined shells of revolution have been studied in the case with/without turning points [9]. In most cases the frequency equation in the first approximation has the form

$$\Delta(\Lambda)=\Delta^{(1)}(\Lambda)\Delta^{(2)}(\Lambda)\dots\Delta^{(k)}(\Lambda)=0,$$

where the coefficients of the equation $\Delta^{(i)}(\Lambda)=0$ depend (as a rule) on the parameters of only one shell.

3. For the nonaxisymmetric vibrations besides μ there is one more parameter — m — which is equal to the number of waves in the circular direction.

If m is small (fixed), formulas (1.6) and (1.7) keep their forms [10] and two additional regular solutions (u_7, w_7) and (u_8, w_8) of type (1.4) exist.

If m is large $(m\sim\mu^{-1})$, the problem may be reduced to the following system [1]:

$$\mu\frac{dy}{dx}=A(x,\mu)y, \quad A(x,\mu)=\sum_{k=0}^{\infty}A_k(x)\mu^k, \tag{3.1}$$

where A is an 8×8 matrix. The buckling problem for the membrane axisymmetric initial stress state of a convex shell of revolution may also be reduced to the system (3.1) [11, 12].

At the turning points $x=x^*$, the equation

$$\det(A_0(x)-\lambda E)=0 \tag{3.2}$$

has multiple roots. Away from the turning points, the solutions of (3.1) have the expansions

$$y^{(i)}(x,\mu)\simeq\sum_{k=0}^{\infty}y_k^{(i)}(x)\mu^k\exp\left(\mu^{-1}\int\lambda_i\,dx\right).$$

Near the simple turning point (where $\lambda_1(x_*)=\lambda_2(x_*)=0$) two solutions have the form

$$y^{(j)}(x,\mu)\simeq\sum_{k=0}^{\infty}a_k^{(j)}(x)\mu^k v(\eta)+\mu^{1/3}\sum_{k=0}^{\infty}b_k^{(j)}(x)\mu^k\frac{dv}{d\eta}, \tag{3.3}$$

$$\frac{d^2v}{d\eta^2}-\eta v=0, \qquad \eta=\mu^{-2/3}\left(3/2\int_{x^*}^{x}i\lambda_j\,dx\right)^{2/3}, \quad j=1,2,$$

where v are the standard Airy's functions.

The free vibrations and buckling modes of shells are shown in Figs. 2, 3, 4. More specifically, Fig. 2 shows vibrations of the paraboloid, Fig. 3 shows the buckling of a spheroidal shell under external pressure, and Fig. 4 shows the buckling of a convex shell under torsion. Crosses indicate turning points.

Let us consider a shell of revolution the curvature of which changes sign at $x=x_*$, as shown in Fig. 5. For the vibrations and buckling of such a shell, four solutions have expansions (3.1), where $\lambda_1=\lambda_2$ and $\lambda_3=\lambda_4$ for all x and $\lambda_1(x_*)=\lambda_3(x_*)$. Near the point x_* the solutions have the same form (3.3) [13].

4. Expansion (3.3) is useless if the distance between two turning points is small (i.e. $|x_*^{(2)}-x_*^{(1)}|=O(\mu^{1/2})$, see Fig. 3). Let system (3.1) contain the unknown spectral parameter Λ. We seek the eigenvalues Λ for which this system has solutions exponentially decreasing away from the interval $[x_*^{(1)}-x_*^{(2)}]$.

Let $x_*^{(1)}=x_*^{(2)}=x_0$ for $\Lambda=\Lambda_0$ and equation (3.2) have the multiple root $\lambda=iq_0$. Then the solution has the expansion [11]

$$y(x,\mu)\simeq\sum_{k=0}^{\infty}\mu^{k/2}y_k(\xi)e^{\mu^{-1}(iq_0x+a\xi^2)}, \quad \xi=\mu^{-1/2}(x-x_0), \quad \Re a<0. \tag{4.1}$$

Here $y_k(\xi)$ are polynomials in ξ and $y_0(\xi)=H_m(\xi)$ is Hermite's polynomial of the degree m. The eigenvalues have the expansions

$$\Lambda^{(m)}=\Lambda_0+\mu r(m+1/2)+O(\mu^2), \quad m=0, 1, 2, \dots .$$

If $q_0\neq 0$, these eigenvalues are asymptotically double.

Formulas (4.1) describe the buckling mode of a noncircular cylindrical shell under variable axial compression [14]. After the separation of the variables $w(x,z)=y(x)\sin(n\pi z/l)$, the two possible buckling modes corresponding to the same critical load are shown in Fig. 6. For a short shell, $q_0=0$ and the mode is similar to the one shown in Fig. 3.

Partial differential equations

5. The following system of equations of the eighth order

$$\mu^2\Delta_0\Delta_0 w+\Lambda\Delta_1 w-\Delta_2\Phi=0, \quad \mu^2\Delta_0\Delta_0\Phi+\Delta_2 w=0 \tag{5.1}$$

may be used to solve some problems of free vibrations and buckling. Here Δ_i are the linear differential operators of the second order with variable coefficients, Δ_0 is the two-dimensional Laplace operator in curvilinear coordinates and Λ is the unknown spectral parameter.

We seek the solutions of the system (5.1) which are exponentially decreasing away from the weakest point x_0, y_0, in the form [15, 16]

$$w(x,y,\mu)\simeq\sum_{k=0}^{\infty}\mu^k w_k(\xi,\eta)e^{i\mu^{-1}S(x,y)}, \tag{5.2}$$

$$S(x,y)=p_0x+q_0y+V(x,y), \quad \xi=\mu^{-1/2}(x-x_0), \quad \eta=\mu^{-1/2}(y-y_0),$$

where $w_k(\xi,\eta)$ are polynomials and $V(x,y)$ is a homogeneous quadratic form in $x-x_0$ and $y-y_0$ and the imaginary part of V is positive definite.

The buckling mode of the spheroidal shell under the complex loading is described by formula (5.2) (see [17] and Fig. 7).

6. Mode (5.2) is used mainly for the convex shells. Let us study the localized modes of low frequency vibrations or buckling of the cylindrical and conical shells.

The equation describing the buckling of the thin elastic closed conical shell has the form:

$$(L+\mu^4L_\mu+\Lambda L_T)U=0, \quad \Gamma U=0. \tag{6.1}$$

Here L and L_μ are the linear differential operators describing the elongation and the bending-twisting of the shell, L_T is a linear differential operator caused by the prebuckling tangential stresses, Λ is the unknown parameter of the critical load, $U=(u,v,w)$ is a displacement vector. Operator Γ provides the linear boundary conditions on the shell edges $x=x_1(y)$ and $x=x_2(y)$.

If the free vibration were analysed, L_TU in (6.1) ought to be replaced by U. In this case Λ is the unknown frequency parameter.

The solution of boundary-value problem (6.1) can be expressed as

$$\Lambda\simeq\eta_1^6\sum_{n=0}^{\infty}\mu_1^n\Lambda_n, \quad w\simeq w^{(0)}+\mu_1^2 w^{(e)},$$

$$u\simeq\mu_1^2u^{(0)}+\mu_1^4u^{(e)}, \quad v\simeq\mu_1 v^{(0)}+\mu_1^3 v^{(e)}, \quad \mu_1^2=\mu. \tag{6.2}$$

Here $(u^{(0)}, v^{(0)}, w^{(0)})$ is the main solution. The edge effect functions $u^{(e)}$, $v^{(e)}$, $w^{(e)}$ decrease exponentially away from the shell edges $x=x_1$ and $x=x_2$.

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In the works by N. A. Alumäe [18, 19], the expansions (6.2) have been used for the analysis of the cylindrical and conical shells buckling in the case when variables x and y in (6.1) were separated. We study the problems in which the exact separation of the variables in (6.1) is impossible, yet their asymptotic separation is possible.

The main solution is sought in the form [20], as follows:

$$w(x, y, \mu_1) \simeq \sum_{k=0}^{\infty} \mu_1^{k/2} w_k(x, \xi) e^{i\mu_1^{-1} q_0 y + a\xi^2}, \tag{6.3}$$

$$\xi = \mu_1^{-1/2}(y - y_0), \quad \Im q_0 = 0, \quad \Re a < 0.$$

Here $w_k(x, \xi)$ are polynomials in ξ. Formulas for u and v may be received by replacing w in (6.3) by u and v. Solution (6.3) decreases away from the weakest generatrix $y = y_0$ (see Fig. 8).

Substituting (6.3) in (6.1) one can find unknown constants y_0, q_0, a and functions v_k, u_k and w_k. First-order approximation yields

$$\frac{k^2(y)}{q_0^4} \frac{\partial^2}{\partial x^2}\left(x^3 \frac{\partial^2 w_0}{\partial x^2}\right) + \frac{q_0^4}{x^3} w_0 + \Lambda_0 N w_0 = 0, \tag{6.4}$$

where $k(y) = xk_2$, $Nw_0 = -q_0^2 T_2 x^{-1} w_0$, T_2 is membrane initial stress-resultant. In the problem of free vibrations $Nw_0 = -xw_0$, and $\Lambda \simeq \simeq \mu_1^4 \Lambda_0 + \ldots$.

The order of Eq. (6.4) makes it possible to satisfy at each edge $x = x_k$ only two boundary conditions. The other two conditions may be satisfied by the corresponding choice of the edge-effect solutions.

The eigenvalue Λ_0 is a function of variables q_0 and y_0: $\Lambda_0 = f(q_0, y_0)$. The values q_0 and y_0 may be found from the conditions

$$\frac{\partial f}{\partial q_0} = 0, \quad \frac{\partial f}{\partial y_0} = 0. \tag{6.5}$$

The parameter $a_0 = ia/2$ is the root of the square equation

$$a_0^2 \frac{\partial^2 f}{\partial q_0^2} + 2a_0 \frac{\partial^2 f}{\partial q_0 \partial y_0} + \frac{\partial^2 f}{\partial y_0^2} = 0.$$

From the condition of solvability of the second-order approximation, it follows [20]:

$$w_0^{(m,n)}(x, \xi) = w_0^n(x) H_m(\xi),$$

$$\Lambda_1^{(m,n)} = -i\left(m + \frac{1}{2}\right)\left(a_0^{(n)} \frac{\partial^2 f^{(n)}}{\partial q_0^2} + \frac{\partial^2 f^{(n)}}{\partial q_0 \partial y_0}\right), \tag{6.6}$$

$$m = 0, 1, 2, \ldots; \quad n = 1, 2, 3, \ldots$$

Here $H_m(\xi)$, the Hermite's polynomial of degree m, $w_0^{(n)}$, is the eigenfunction of boundary problem (6.4) corresponding to $\Lambda_0^{(n)}$.

The eigenvalues $\Lambda^{(m,n)} = \Lambda_0^{(n)} + \mu_1 \Lambda_1^{(m,n)} + \ldots$ of the boundary problem (6.1) are asymptotically double.

Formulas (6.6) may be obtained by using a system of the shallow-shells Eqs. (5.1), but for the construction of the following terms of the asymptotic series it is necessary to use a more exact system (6.1).

This asymptotic method may also be used for the study of the cylindrical shell vibrations and buckling. In this case the multiplier x ought to be replaced by 1. For a cylindrical shell, the analytical approximate solutions for some problems should be obtained.

As an example, let us study the buckling of a circular cylindrical shell with the slanted freely-supported edges under external pressure. In this case $k_2=T_2=1$. The boundary conditions for Eq. (6.4) have the form

$$w_0=w_0''=0 \quad \text{for} \quad x=x_1, \quad x=x_2. \tag{6.7}$$

Let $x_1(\varphi)=0$ and $x_2(\varphi)=l+\tan\beta\cos\varphi$, where $\beta\neq 0$ is the edge inclination angle. Then the boundary problem (6.4), (6.7) solution has the form

$$w_0^{(n)}=\sin\frac{\pi n x}{x_2(y)}, \quad f^{(n)}=q_0^2+\frac{\pi^4 n^4}{x_2^4(y)\,q_0^6}, \quad n=1, 2, \ldots$$

For the buckling problem we seek the lowest eigenvalue corresponding to $n=1$. By solving the system (6.5), when $n=1$, we get

$$y_0=0, \quad q_0^2=3^{1/4}\pi/l_0, \quad l_0=x_2(0).$$

Therefore the weakest generatrix is the longest cylinder generatrix $y=0$ (see Fig. 9). The first approximation to the eigenvalue $\Lambda_0^{(1)}=4\pi\cdot 3^{-3/4}/l_0$ corresponds to the critical pressure, and it coincides with that given by Southwell-Papkovich [21] for the shell of the length l_0 and straight edges.

It is easy to see that

$$\frac{\partial^2 f^{(1)}}{\partial q_0^2}=16, \quad \frac{\partial^2 f^{(1)}}{\partial q_0\,\partial y_0}=0, \quad \frac{\partial^2 f^{(1)}}{\partial y_0^2}=\frac{4\pi\tan\beta}{4^{3/4} l_0}.$$

The second approximation correction, which takes into account the slanted-edge effect, is given by formula

$$\Lambda_1^{(0,1)}=\frac{1}{2}\sqrt{\frac{\partial^2 f^{(1)}}{\partial q_0^2}\frac{\partial^2 f^{(1)}}{\partial y_0^2}}=2\sqrt{\Lambda_0^{(1)}s_0}, \quad s_0=l_0^{-1}\tan\beta.$$

The next approximation correction may be found from the condition of solvability for the next approximation

$$\Lambda_2^{(0,1)}=-\frac{5}{6}+\frac{4}{3}\alpha_0^2+\frac{22}{27}s_0+\frac{\sqrt{2}\,\lambda_0^{(1)}s_0^2}{4\alpha_0^2}, \quad \alpha_0=\frac{\pi}{l_0}.$$

The localization of vibration modes near the cylindrical and conical shell generatrix is caused by the variability of geometrical and physical shell parameters, by the slanted-edge effect and by the initial stress resultants variability. In the papers [7, 14, 22, 23] and others, such problems have been analysed by the asymptotic method described above.

The buckling form localized in the neighbourhood of the asymptotic line of the shell of negative Gaussian curvature has been constructed in [24].

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S. B. FILIPPOV, P. E. TOVSTIK

ÕHUKESTE KOORIKUTE OMAVÕNKUMISTE JA STABIILSUSE LINEAARSETE ÜLESANNETE HARGNEMISPUNKTID JA HARGNEMISJOONED

On esitatud ülevaade uuringutest õhukeste elastsete koorikute teoorias, kus oluline roll on hargnemispunktide analüüsil. N. Alumäe esitatud meetodit on edasi arendatud hargnemispunktide analüüsis, rakendatuna telgsümmeetriliste kooniliste koorikute omavõnkeprobleemi lahendamiseks ning silindriliste ja kooniliste koorikute stabiilsuse uurimiseks.

С. Б. ФИЛИППОВ, П. Е. ТОВСТИК

ТОЧКИ И ЛИНИИ РАЗВЕТВЛЕНИЯ В ЛИНЕЙНЫХ ЗАДАЧАХ СВОБОДНЫХ КОЛЕБАНИЙ И УСТОЙЧИВОСТИ ТОНКИХ ОБОЛОЧЕК

Дан обзор работ, посвященных анализу точек разветвления в задачах свободных колебаний и устойчивости упругих тонких оболочек. Развиты ранние результаты Н. А. Алумяэ по анализу колебаний осесимметричных конических оболочек и устойчивости цилиндрических и конических оболочек.