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MATHEMATICAL MODELLING OF THE HEARTBEAT

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Юри ЭНГЕЛЬБРЕХТ, Олав КОНГАС. МАТЕМАТИЧЕСКОЕ МОДЕЛИРОВАНИЕ ДИНАМИКИ СЕРДЦА

The heartbeat is one of the most familiar and steady rhythms known to mankind. An abnormally beating heart may display a complex variety of beat types, such as ventricular tachycardia and fibrillation. In terms of nonlinear mechanics some of the beats could be named chaotic — that explains the growing interest in these fascinating problems [1–3]. However, the mathematical modelling of such complicated biological processes as the heartbeat is not undertaken in order to predict all the possible phenomena but rather to use mathematics to open our eyes to possible physical mechanisms under consideration.

The normal pacemaker of the heart is the sino-atrial (SA) node that is actually a collection of cells with spontaneous automaticity [1]. The impulse then propagates through the atrioventricular (AV) node, which gives a certain delay and may affect the rhythm of the SA node (inverse interaction). Then the pulse propagates down to the ventricular muscles through the bundle of His (nerve fibres) which later takes the form of fractal-like Purkinje fibres. The latter conduct the electrical impulse to the myocardium, causing a nearly synchronized contraction of the ventricular muscle. What is usually measured, is the electrocardiogram (ECG) that gives a complex picture of dynamic activities in the heart. In physical terms, the rate of the pulse (action potential) is measured. A typical ECG is shown in Fig. 1 with traditional notations [3].

The first known mathematical model of the heartbeat was proposed in the late twenties by van der Pol and van der Mark [4], using a nonlinear ordinary differential equation of the form

$$\frac{d^2z}{dt^2} - F(z)\frac{dz}{dt} + \omega_0^2 z = 0, \tag{1}$$

where $z(t)$ is the voltage, ω_0 is the natural frequency and

$$F(z) = \alpha(1 - z^2), \quad \alpha = \text{const}. \tag{2}$$

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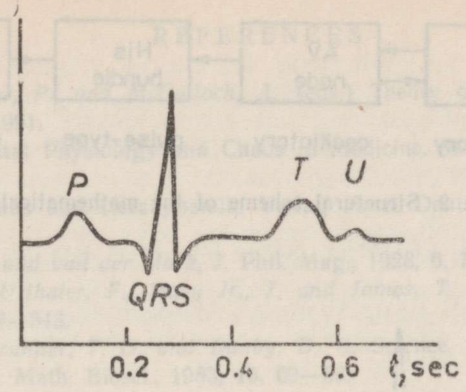

Fig. 1. Traditional ECG.

Actually, van der Pol and van der Mark constructed electronic circuits modelled by Eq. (1), and their physical model consisted of three nonlinear oscillators, i.e. three coupled electrical circuits corresponding to SA node, AV node and ventricles. Subsequent models have used the van der Pol relaxation oscillator, for example Katholi et al. [5] have used two weakly coupled van der Pol equations to model SA and AV nodes. Later however the main attention was directed to investigating the coupling effects and the type of equation was different [2, 6]. Within the framework of nonlinear dynamics, the importance of mappings (based on Poincaré sections) became clear, and this idea is also widely used to analyze heart dynamics (see, for example [7]).

In principle, there are two distinct physical mechanisms in the heart — the pacemaker and nerve-pulse transmission. The pacemaker is an oscillating structure (involving SA and AV nodes), and the signal is then transmitted through complicated nerve bundles and fibres (His bundle and Purkinje fibres) to myocardium. The mathematical models used up to now are based on equations of oscillatory character (involving limit cycles). Recent investigations have used for nerve pulse transmission the novel technique of evolution equations that has resulted in equations very close to the celebrated van der Pol equation. Namely, the nerve pulse transmission is governed by an evolution equation [8, 9]

$$\frac{\partial^2 z}{\partial\xi\,\partial x}+f(z)\frac{\partial z}{\partial\xi}+g(z)=0, \tag{3}$$

where z is the scaled action potential, ξ is the moving frame and x the axial coordinate. Functions $f(z)$ and $g(z)$ are responsible for attenuation and/or amplification. Function $g(z)$ is usually linear, but $f(z)$ is a quadratic polynomial with roots z_1, z_2 obeying

$$z_1>0, \quad z_2>0, \quad z_1\neq z. \tag{4}$$

When a constant profile is sought, then Eq. (3) is transformed into the O.D.E.

$$z''+f(z)z'+\theta^{-1}g(z)=0, \tag{5}$$

where $()'=d/d\eta$, $\eta=x+\theta\xi$, $\theta=\text{const}$. The celebrated van der Pol equation (cf. Eq. (1)) belongs to the same class of Liénard equations, but in this case the roots of $F(z)$ satisfy the conditions

$$z_1<0, \quad z_2>0. \tag{6}$$

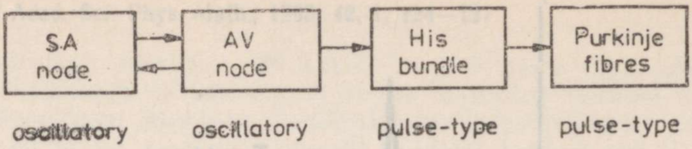

Fig. 2. Structural scheme of the mathematical model.

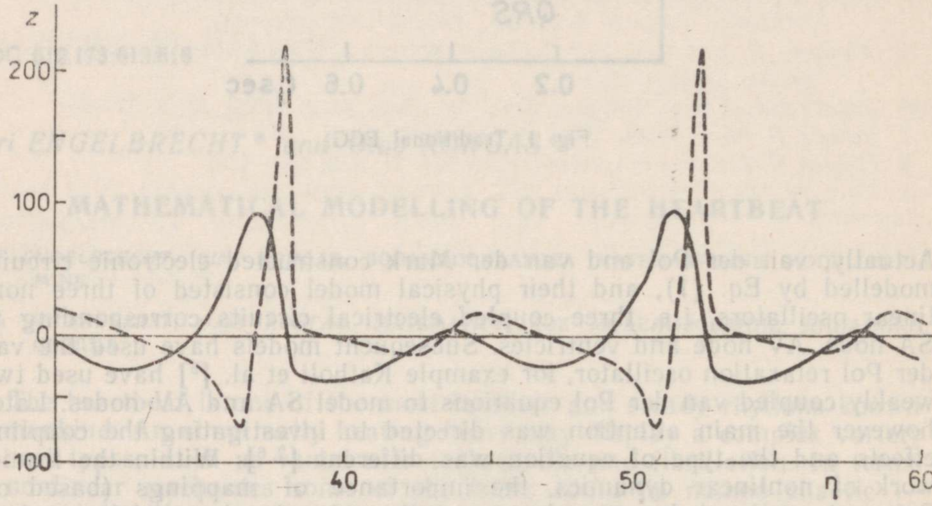

Fig. 3. Solution of Eq. (8). Solid line — solution, dashed line — its derivative.

The crucial difference between the values of roots (cf. conditions (4) and (6)) leads in one case to the limit cycle, i. e. to relaxation oscillations, in another case — to a single pulse. The reader is referred to [9] for details.

The main idea here is to construct a mathematical model of the heart as a system of coupled equations combining the equations with oscillatory-type solutions (SA and AV nodes) as well as the equations with pulse-type solutions. The principal scheme is shown in Fig. 2. Clearly, the model can be combined both with two elements (one oscillatory and one pulse-type) and with three or four elements.

The preliminary results are obtained using an equation with pulse-type solution with a periodical driving. The equation, when presented as a system of the first-order equations, is

$$z' = -y, \tag{7a}$$

$$y' = 0.13837(-25+1.5z-0.015z^2)y+z+A\sin Bt. \tag{7b}$$

The results of integration are shown in Fig. 3 for $A=100$, $B=0.87$ and $z(0)=0$, $y(0)=-80$. The resemblance of the derivative to the normal ECG (Fig. 1) is striking but in fact this diagram reflects 2 : 1 blocking. Actually, in the next stage the periodical driving will be replaced by the coupling term from the oscillatory-type equation modelling one of the nodes. Apart from the normal behaviour, attention must be paid to possible chaotic solutions giving evidence of arrhythmias. Here the coupling between the elements is of utmost importance. In Eq. (7), the driving was through acceleration (i. e. through force) but driving through velocity also needs attention [10].

This paper is the first in the series reflecting research with n-element models which is in progress.

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