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ON PRINCIPLES OF AFTEREFFECTS AND NONLOCALITY IN CONTINUUM MECHANICS

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Мати КУТСЕР. О ПРИНЦИПАХ ПОСЛЕДЕЙСТВИЯ И НЕЛОКАЛЬНОСТИ В МЕХАНИКЕ СПЛОШНОЙ СРЕДЫ

(Presented by J. Engelbrecht)

Several new scientific ideas have been formed at seminars chaired by Nikolai Alumäe in the Institute of Cybernetics. The idea of classification of principles for generalization of constitutive equations belongs to Uno Nigul, who has used it in his unpublished lecture notes. The author of this note feels his duty to explain this idea to larger audience in more detail.

Generally speaking, the conventional constitutive equations in continuum mechanics emphasize the instantaneous dependencies. As a result, the equations of motion are then of differential type.

In this note, the main principles for generalization of constitutive equations and respective equations of motion are presented. It is shown that there are two main classes to be analyzed both leading to integral (or integro-differential) type of governing expressions.

Wave motion of solids is governed by the conservation law of momentum that actually is Newton's second law. This is, however, expressed in terms of stress and deformation. Consequently, in order to define a full governing system, a constitutive law is needed relating stress to deformation. A similar situation occurs also, for example, in electrodynamics, where a constitutive law connects voltage and current.

The constitutive equation is generally an empirical relation based on experiments. In mechanics of solids, usually Hooke's law is used emphasizing the proportionality of stress to deformation. In general terms

$$A=EB. \qquad (1)$$

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In this expression variables A, B and constant E are meant to have the instantaneous values. Actually, in reality, this dependence may be much more complicated, the variables being dependent on time and/or space coordinates.

Up to now, there is no clear distinction between possible variants of constitutive equations. Here, in order to get more generality, instead of algebraic equations like (1) emphasizing instantaneous effects, we shall use integral (or integro-differential) relations with certain kernels.

There are two basic principles how the variables $A(x,t)$ and $B(x,t)$ can be related:

1) the *principle of aftereffects:* characteristic parameters of process in a certain point of space at a given moment of time t depend on the history of changing values of these parameters in the same point of space (time-dependent processes);

2) the *principle of nonlocality:* characteristic parameters of the process at a certain moment of time t depend on the values of these parameters in other points of space at the same moment of time t (space-dependent processes).

The constitutive equations constructed according to principle 1 are widely known in mechanics of solids, particularly in the theory of viscoelasticity with several types of kernels [1–4], but there are much fewer examples corresponding to principle 2 [5–7]. Even more, one could intuitively join both the principles but no serious investigations in this field are known to the author.

The basic idea of the theory using principle 1 may be presented by expression [2]

$$A(x,t) = \mathrm{E}(x,t)\{B(x,t) - R(t) * B(x,t)\}, \tag{2}$$

where $A(x,t)$ is a one-dimensional variable (for example force or mechanical stress σ), $B(x,t)$ is a dimensionless variable (for example relative deformation ε), $R(t)$ denotes the relaxation kernel, positive function $\mathrm{E}(x,t)$ has the same dimensions as $A(x,t)$, and the following type of notation of the convolution integrals is used

$$F(t) * G(x,t) = \int_0^t F(t-\tau)\, G(x,\tau)\, d\tau. \tag{3}$$

Then expression (2) may be presented as

$$A(x,t) = \mathrm{E}(x)\{B(x,t) - \int_0^t R(t-\tau) B(x,t)\, d\tau\}. \tag{4}$$

Single-integral models represented by integral equations (2) or (4) give an opportunity to calculate $A(x,t)$, knowing the «history» of $B(x,t)$, with a certain weight. In wave propagation and oscillation problems such single-integral models also describe the influence of dissipative effects.

The equation of motion governing one-dimensional waves in solids may be presented in the form

$$\frac{\partial^2 U(x,t)}{\partial t^2} - \frac{\partial \sigma(x,t)}{\partial x} = 0 \tag{5}$$

with the initial conditions

$$U(x,0) = 0, \quad \frac{\partial U(x,0)}{\partial x} = 0, \quad \sigma(x,0) = 0. \tag{6}$$

Here U denotes the displacement.

Denoting in the constitutive equation (2)

$$A(x,t)=\sigma(x,t), \quad B(x,t)=\frac{\partial U(x,t)}{\partial x}=\varepsilon(x,t), \tag{7}$$

one can get the equation of motion in terms of $U(x,t)$. However, in order to draw later parallels with equations derived according to principle 2, we use the factorization that permits to split the second-order Eq. (5) into the first-order equations describing single waves. For the wave propagating in the positive direction of x, we get [2]

$$\frac{\partial U(x,t)}{\partial t}+c\left\{\frac{\partial U(x,t)}{\partial x}-\int_0^t R_1(t-\tau)\frac{\partial U(x,\tau)}{\partial x}d\tau\right\}=0. \tag{8}$$

Here $R_1(t)$ denotes the modified kernel function which, due to factorization, is related to $R(t)$ by

$$R(t)=R_1(t)-\frac{1}{4}R_1(t)*R_1(t). \tag{9}$$

We assume that $R_1(t)$ (like $R(t)$) is a positive function bounded in the interval $0<t<\infty$ and satisfying the conditions

$$\lim_{t\to 0}\int_0^t R_1(\tau)\,d\tau=0, \tag{10}$$

$$R_1(t)\to 0 \quad \text{if} \quad t\to\infty. \tag{11}$$

It means that relaxation kernel $R_1(t)$ is an integrable function in all its domain of determination. In most cases of real processes the relaxation kernel $R_1(t)$ (or $R(t)$) is a monotonously descending function in the whole domain of definition, and it corresponds to the fading memory principle in the models describing aftereffects [1,2]. So the event in the very past has little impact on the process in the present.

The principle of nonlocality (principle 2) may be demonstrated on the basis of one-dimensional Whitham equation [5]

$$\frac{\partial\varphi(x,t)}{\partial t}+\int_{-\infty}^{\infty}K(x-\xi)\frac{\partial\varphi(\xi,t)}{\partial\xi}d\xi=0. \tag{12}$$

Here $K(x)$ is a given function. This equation has an elementary solution

$$\varphi=Ae^{i\varkappa x-i\omega t} \tag{13}$$

in the case when

$$-i\omega e^{i\varkappa x}+\int_{-\infty}^{\infty}K(x-\xi)\,i\varkappa e^{i\varkappa\xi}\,d\xi=0. \tag{14}$$

From Eq. (14) the phase velocity may be calculated easily in the form

$$c=\frac{\omega}{\varkappa}=\int_{-\infty}^{\infty}K(\zeta)\,e^{-i\varkappa\zeta}\,d\zeta. \tag{15}$$

The right-hand side of (15) is the Fourier transformation of the kernel $K(x)$, and therefore we have

$$K(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}c(\varkappa)\,e^{i\varkappa x}\,d\varkappa. \tag{16}$$

So we may construct the equation of the type (12) for arbitrary phase velocity or, which is the same, for arbitrary dispersion relation, using the Fourier transformation for obtaining the kernel $K(x)$. For example, if we take the phase velocity in the form of series

$$c(\varkappa)=c_0+c_2\varkappa^2+\ldots+c_{2m}\varkappa^{2m}, \tag{17}$$

the kernel

$$K(x)=c_0\delta(x)-c_2\delta''(x)+\ldots+(-1)^m c_{2m}\delta^{(2m)}(x) \tag{18}$$

substituted into (12) yields

$$\frac{\partial\varphi}{\partial t}+c_0\frac{\partial\varphi}{\partial x}-c_2\frac{\partial^3\varphi}{\partial x^3}+\ldots+(-1)^m\frac{\partial^{2m+1}\varphi}{\partial x^{2m+1}}=0. \tag{19}$$

Here $\delta(x)$ denotes the delta-function. The first three terms in (19) give the linearized Korteweg-de Vries type equation, i.e. a differential equation. In this case (simple kernel (18)) nonlocal interactions are not taken into account.

Denoting $\varphi=U$, equation (12) can be rewritten

$$\frac{\partial U(x,t)}{\partial t}+\int\limits_{-\infty}^{\infty}K(x-\xi)\frac{\partial U(\xi,t)}{\partial\xi}d\xi=0 \tag{20}$$

that should be compared to the Eq. (8). The basic difference of the kernels demonstrates the difference between the principles of aftereffects and nonlocality.

The dispersion relations may be of a very complicated form and it is sometimes a rather difficult task to find out the exact kernel function. As shown in [6], asymptotic analysis may still lead to acceptable governing equations. In most cases, the kernel function is an even function and decays if $x\to\infty$.

Nonlocal interaction is of importance for describing geometrical dispersion caused by internal (layered) inhomogeneities. Acoustic waves in a fluid layer are described in [6], and in magnetic flux tubes (slabs or cylinders) in [7]. A model theory of gravitating particles in quantum mechanics leads to a certain nonlinear Schrödinger equation with an integral part [8] that again describes nonlocal interaction.

It can be easily concluded that a constitutive equation describing nonlocalities should be presented as

$$A(x,t)=\mathrm{E}(x)\{B(x,t)+\int\limits_{-\infty}^{\infty}K(x-\xi)B(\xi,t)d\xi\}. \tag{21}$$

Here

$$A(x,t)=\frac{\partial\varphi}{\partial t},\quad B(x,t)=\frac{\partial\varphi}{\partial x}. \tag{22}$$

Of course, using constitutive equation of type (21) leads to a more complicated equation of motion than (12).

It may be interesting to construct the constitutive equations which take into consideration both principles — the aftereffects and the nonlocality. In other words, the effects based on the «memory» of the media and the dispersion of the energy could be simultaneously accounted for.

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