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COMPLEXITY AND SIMPLICITY

An essay is by the definition a composition dealing with a subject from a personal point of view and without attempting completeness.

B. Mandelbrot

When I told a colleague of mine that I would like to write an essay on complexity and simplicity, to Nikolai Alumäe's memorial issue, then he, a well-read scientist with a broad outlook on science and life, responded: "It is not an easy task." I cannot agree with him more, but then this is exactly what Nikolai Alumäe would have liked. The problem on one's desk should always be intriguing [1], he used to say, and I dedicate this essay to the memory of my Teacher.

Of initial ideas

The eternal rivalry between complexity and simplicity has always been an intriguing problem to mankind. As our knowledge has been increasing, many seemingly complex phenomena have turned out to be rather simple when the reasons behind them have been cleared up. Take, for example, the motion of planets that ages ago needed special spheres and other geometrical structures to explain their motion with respect to the Earth. Yet, as soon as the heliocentric system was accepted by the general public, everything became clear and simple. So, the first lesson is that, given the reasons, seemingly complex things become rather simple.

However, the world around us is not only a hide-and-seek game — there are simple phenomena and there are complex phenomena in it. The knowledge that God does not play dice, as Albert Einstein has said, is pleasing but not very helpful. As usual, we first need clear definitions, at least for the working purpose. According to Chambers [2], *simple* is that which consists of one thing or element, *complex* is what is composed of more than one or of many parts. Taking these notions as basic, we later enlarge them intuitively, because, as a matter of fact, the categories involved are rather abstract.

Here, we shall start from examples demonstrating simplicity and/or complexity as we understand these phenomena (properties); and it is of fundamental importance to distinguish between the two, because contemporary theories tend to be very complex, so simplification seems to be a

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natural way to solve problems. Here lies a certain risk. A. Toffler mentioned in the Preface to I. Prigogine's and I. Stenger's monograph [3]: "Dissection is one of the most highly developed skills in contemporary western civilization: the split-up of problems into their smallest possible components. We are good at it. So good, we often forget to put the pieces back together again".

From this, an intriguing problem arises — how to put the pieces back again, i.e. how to build up complexity from simplicity. Below, we would like to analyze some steps on this long and difficult way, which may serve as a backbone for further research. And there are a lot of questions to be answered before complexity is understood more profoundly.

Of simplicity

According to what is said above, simple consists of one thing or element, which is a rather open definition. Here we have to rely more on the intuition that permits us to enlarge these notions without violating the basic idea. Simplicity in mathematics is related to simple rules, in physics — to linear causality like "if this is given then that follows", etc. Let us proceed with the aid of exemplification.

We are faced with simple rules quite often in our everyday practice. Engineers widely use Hooke's law stating that stress and deformation are proportional, Ohm's law — current and voltage are proportional, etc. These can be considered as simple laws emphasizing simple instantaneous effects — the larger one variable, the larger also the dependent variable. The cause and the result are nicely related to each other and proved to be correct in ever so many everyday examples. There are cases where only a simple model has opened the eyes of mankind to understand the rules of Nature. A brilliant example is the planetary model of the atom based on the ideas of E. Rutherford and N. Bohr.

There are two questions now: (i) is simple sometimes just simplified and (ii) what is the difference between simple and simple. The latter question seems to be a tautology but its essence is to widen the notion of simplicity.

It is quite clear that simple linear laws of physics are applicable only under special conditions (*ceteris paribus*). So, stress is proportional to deformation only if deformation is small. The real world is nonlinear and Hooke's law is only a simplified version of reality. This is easily understood when analyzing the properties of potential energy. In Fig. 1 potential energy U is depicted as a function of distance r between the

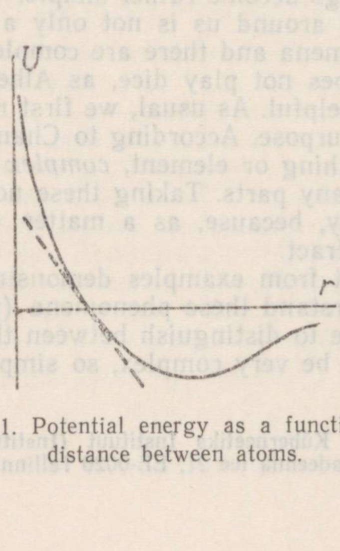

Fig. 1. Potential energy as a function of distance between atoms.

atoms. Around the equilibrium point O, the approximation by a linear function (dashed line) is justified, otherwise the energy function is essentially nonlinear (higher than quadratic).

In order to answer the second question, we need a certain measure. Not going into details, an intuitive approach may be used here. In a physicist's jargon, the word *deep* is used to describe the problems "that would not give way without long looks into Universe's bowels..." [4, p. 3]. Simplicity may also be deep. For example, we can turn to thermodynamics. Such notions as heat, energy, entropy and dissipation are simple but deep. Let us take the first of them — heat. This notion was introduced by J. J. Fourier in 1811. The essence of his idea is surprisingly simple and elegant — heat flow is proportional to the gradient of temperature. Said Prigogine and Stengers [3]: " the simplicity of Fourier's mathematical description of heat propagation stands in sharp contrast to the complexity of matter considered from the molecular point of view". However, the deepness of this new notion has enabled to build up thermodynamics based on simple principles. According to R. Feynmann, this is quite usual in Nature, which is actually a huge chess game: each move follows simple rules [5]. Nowadays we may ask, is such an approach still efficient. Some considerations concerning answers to this question will be given later.

Of complexity

The initial definition of complex (see above) is also certainly rather naive. One could ask, for example, what is the minimum of complexity involved that allows us to use the notion of complex. Intuitively we would like to add some other properties to the initial notion in order to grasp its fundamental essence.

According to R. Lee [5], contemporary understanding of complexity is closely related to K. Gödel's paper [6], "focussing on undecidability and the allied concepts of uncertainty and complexity". Nowadays our vision of Nature is undergoing radical changes towards the multiple, the temporal, and the complex.

Where does complexity come into the game? One should first distinguish between the structural and functional complexities [7]. The first depends on the number of interacting subunits and the second on the length of the algorithm needed to describe the entire behaviour of the system. Complexity could also arise from the interaction between the system and its observer [8]. G. Nicolis and I. Prigogine [9] stress the importance of natural complexity seen as part of everyday experience closely related to nonequilibrium states, nonlinear dynamical systems, predictability, and self-organization. E. W. Packel and J. F. Traub [10] show the importance of computational complexity. This short list may serve as evidence of many facets of complexity. Here, we concentrate our attention on natural complexity.

Again, as above, let us discuss some examples. When B. Mandelbrot in the seventies started to investigate a "simple" quadratic map

$$z_{n+1}=z_n^2+c \qquad (1)$$

with z_n, c complex, the question to him was, "Do you really expect to find anything new?" As we know now, the Mandelbrot set M (the domain of convergence of map (1) in the complex plane) is a very complicated structure, indeed (Fig. 2). It has rightly been described as the most complex mathematical shape ever invented [11]. Even more, the Mandelbrot set is another important milestone in the theory of complexity because from it the notion of fractals has started [12, 13]. According to

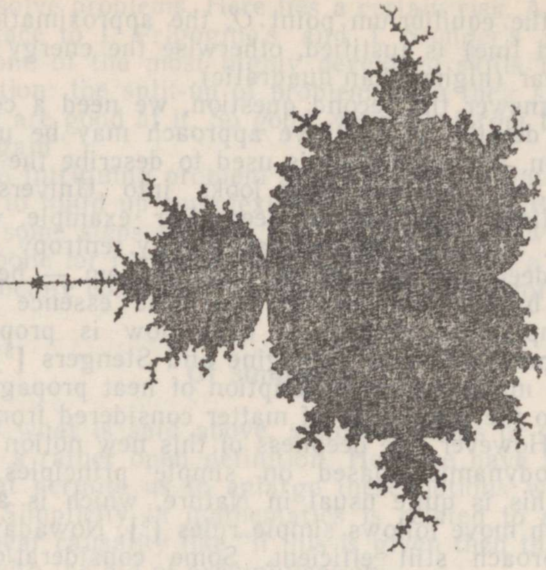

Fig. 2. The Mandelbrot set.

[9], the complexity of the Mandelbrot set is certainly natural. We shall later return to this example when we discuss the rules of creating complexity.

Generally speaking, fractals are complex structures like coastlines, rivers, blood vessel systems, neural networks, cauliflowers, diesel soots, polymer grains, colloid aggregates — this list could be prolonged, of course (see [14, 15]). Let us cite Mandelbrot [16]: "Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line... Nature exhibits not simply a higher degree but an altogether different level of complexity." Fractal geometry is progressing [17], discovering more and more rules in Nature.

Fractals depict the complexity of irregular and fragmented shapes in Nature. Besides them, there are many more complex structures and phenomena. In dynamics, there is a notion of complex dynamical systems built of units that are themselves simpler systems [18]. The behaviour of dynamical systems in multidimensional phase spaces may be complex (but not fractal!) [18, 19]. The ecological and human organizations are usually complex [3]. Or just an example in physics — gas as a purely deterministic aggregate of moving molecules that obey precise dynamical laws [11], and again it is a complex structure. The plane- and spacefilling tiling patterns are used not only for covering a floor or a wall with ceramic tiles but are of the utmost importance in crystallography [20]. The Penrose decagon tiling is shown in Fig. 3. What fascinates us in tiling (tessellation) is that Maurits Escher, the famous Dutch artist, has discovered many exact laws of tiling by just studying symmetry from an artist's viewpoint [21]. An example of his tiling is shown in Fig. 4.

All these examples still do not allow us to define complexity in a more definite way. We may list some of the characteristics of complexity:
— we do not easily understand the topological, geometrical a. o. properties of a complex structure;
— we do not ad hoc understand how a complex system is likely to respond to a given excitation (change of conditions).

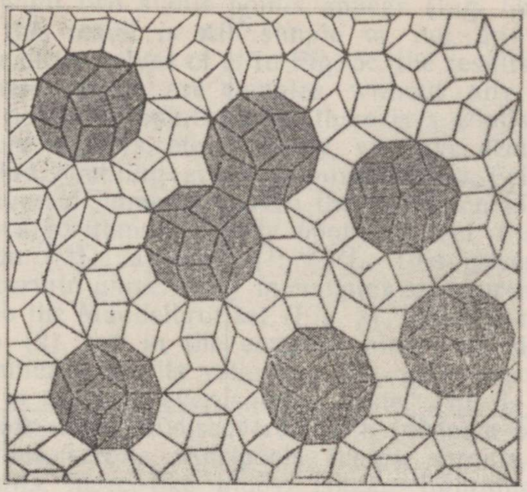

Fig. 3. Penrose decagon tiling [20].

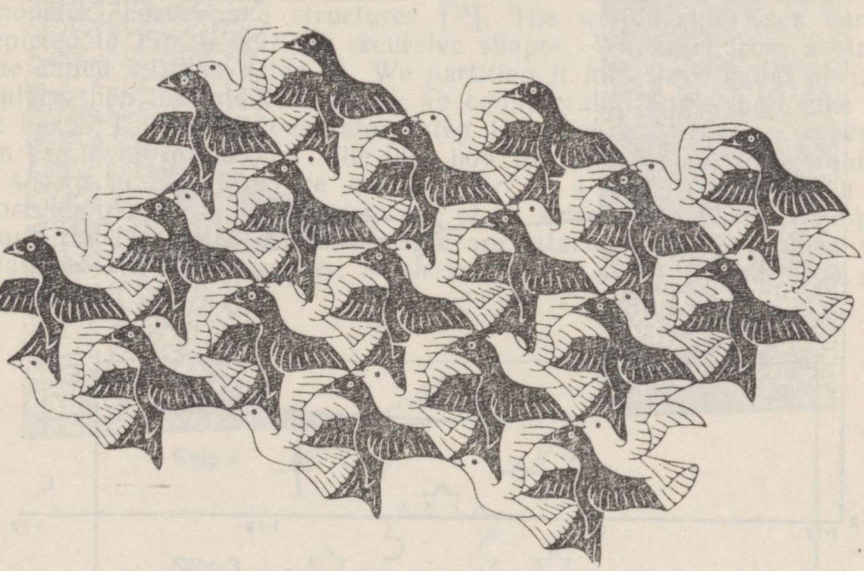

Fig. 4. Birds after M. Escher [21].

Of creating complexity

Naive argumentation puts some simple things together in order to get complex structures. Actually, this basic idea is realized in a more sophisticated way by means of the following procedures:
— building hierarchies;
— using recursive algorithms;
— using coupling between structural elements;
— taking into account memory effects.
This list is by no means complete.

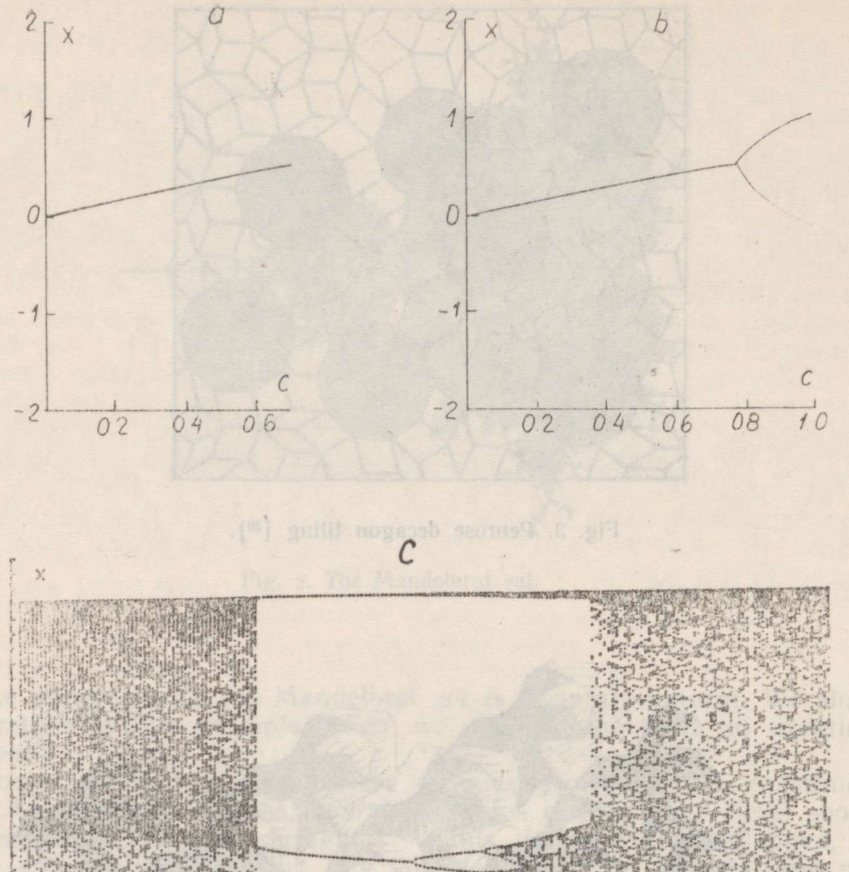

Fig. 5. Behaviour of stable points of quadratic map $x_{n+1}=c-x_n^2$:
a) $0<c<0.7$; b) $0<c<1.0$; c) $1.72<c<1.82$.

There is an important point to take notice of whatever the procedure to create complexity — the outcome may depend on a set of parameters. Let us take for example recursive algorithm in the form of a quadratic map

$$x_{n+1}=c-x_n^2 \tag{2}$$

which, contrary to (1), is used in the real domain only. If $c<0.7$, then it leads to a stable stationary point after a series of iterations (recurrent usage of map (2)). This is shown in Fig. 5a. Intuitively we may call this result simple. If $c\sim 0.78$, then a bifurcation appears, and instead of one stable point two stable points appear (Fig. 5b). Let us leave it open whether this result is still simple or not and proceed with the calculations for other values of c. In Fig. 5c, the results for $1.72 < c < 1.82$ are shown. These results are complex without any doubt! There are regions where no stable points occur, there is a window where a certain regular structure appears, etc. Actually, we have demonstrated here the celebrated period doubling scenario governed by Feigenbaum numbers [22]. These universal numbers govern the transition to chaos and do not depend on the algorithms but only upon the fact that period doubling (flip bifurcation [18]), appears [11, 19, 22]. This gives evidence of the existence of a certain universality in complex structures.

Returning to the Mandelbrot set (Fig. 2), which is related to the same quadratic map (2) but in the complex plane, we discover a striking quality of complexity — its richness. We only need to blow-up the structure, i. e. to move to ever smaller scale. This is a real Mathematical Zoo that can then be discovered with intricate beauty and endless variety: seahorses, scrolls, whirlpools, lumps, sprouts, burgeoning cacti, thin snakes, coils, insect-like blobs, zigzag lightnings, etc. [11]. The reader is referred to coloured plates, say in [17], in order to get some impression about the beauty of complexity. Lewis Carroll would have put them into his Jabberwocky, had he, an Oxford don and mathematician, only known about their existence.

Another example of recursive algorithms could be the construction of "monster" curves and structures [23]. The well-known Koch curve is depicted in Fig. 6 with its recursive shapes. We start from a straight line called initiator (step 0). We partition it into three equal parts and replace then the middle third by an equilateral triangle, and take away its basis. This is the generator (step 1) which is then used repeatedly (in Fig. 6 up to 4 steps). The idea how to construct a Pythagorean tree is shown in Fig. 7, where an initiator is a square to which a right isosceles triangle is attached to one of its sides, after which two squares along the free sides of the triangle are attached, etc. A product where triangles are non isosceles, is shown in Fig. 8. A cut-off of a cauliflower, isn't it?

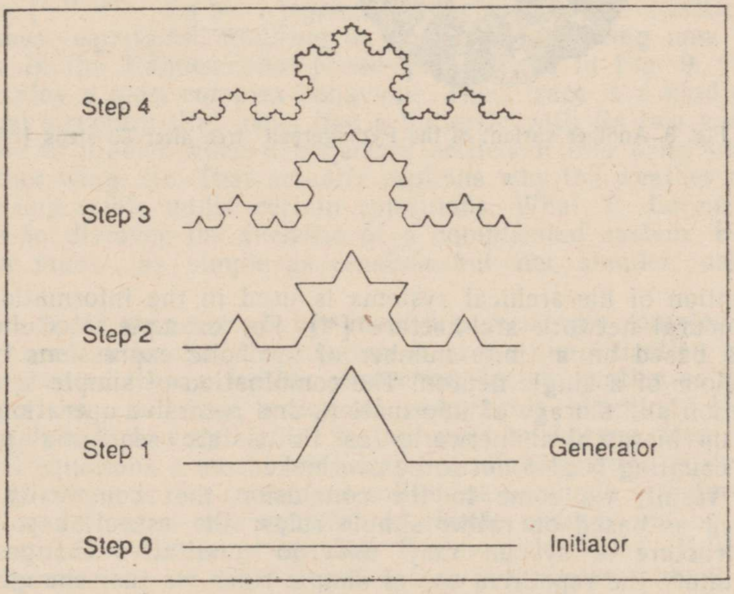

Fig. 6. Construction of the Koch curve [23].

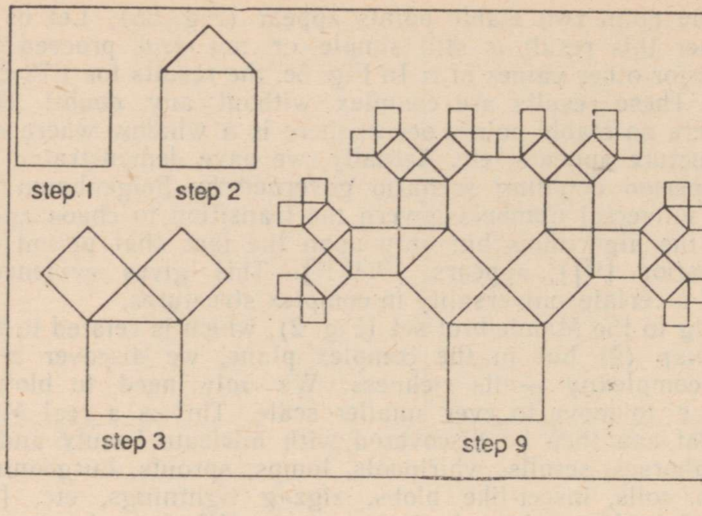

Fig. 7. A Pythagorean tree [23].

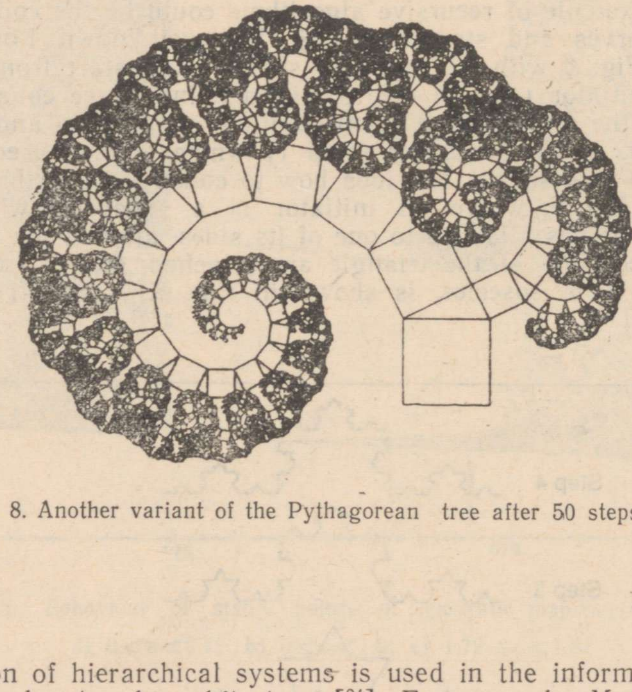

Fig. 8. Another variant of the Pythagorean tree after 50 steps [23].

The notion of hierarchical systems is used in the information theory [7] and neural network architecture [24]. For example, McCulloch-Pitts neuron is based on a finite number of symbolic expressions reflecting the behaviour of a single neuron. The combination of simple arithmetics, classification and storage of information, and recursive operation permits to build up hierarchical networks as finite-state sequential machines. Here the coupling is also not to be overlooked.

As a result, we come to the conclusion that complexity can be constructed as based on rather simple rules: the essential point being that a structure or system may undergo qualitative changes (bifurcations) under the repetitive use of simple rules, or just changing a set of parameters. Said Prigogine — complexity is created through mechanism of bifurcations [25].

Of main principles

We have seen that complexity can be created by means of simple rules. However, one essential keyword was only mentioned rather than stressed. This is *nonlinearity*, a peculiar notion emphasized by a negative prefix. Nonlinearity plays an important role in contemporary science and much has recently been written on the importance of being nonlinear [26]. Actually, the examples shown above (the Mandelbrot set and the quadratic map) involve nonlinearity.

To get rid of the habit of using the property of proportionality and the property of independence (additivity) as basic notions is not so easy because whole generations of scientists have been trained — in the sense of Leonardo da Vinci — to grasp the leading effects in such a convenient way. However, F. Hundertwasser said [17]: "...the straight line leads to the downfall of mankind. And that line is the rotten foundation of our doomed civilization." Let everyone decide him- or herself whether to agree or not to agree with F. Hundertwasser, but one thing is clear — with linearity there are no large-scale qualitatively new results.

There is a need to distinguish between two opposite routes: from simple to complex and from complex to simple. The first is usually called generalization. Sometimes it is said that mathematical research is largely a process of successive generalization. Starting from basic assumptions (axioms) geometry, rational mechanics and other sciences are built up. The second route is caused by the fact that contemporary science is often able to construct very complicated mathematical models that do not allow direct analyses resp. solutions. Then the only way to understand the process is to find out the main effects. There is certainly a possibility of losing something in the process of simplification. Nevertheless, the history of science knows many examples of how ingenious researchers have found brilliant outcomes. One of such examples is no doubt the Lorenz attractor. The starting point was based on Navier-Stokes equations modelling the circulation and convection of the atmosphere. E. Lorenz has found a simplified version of these equations in the form of a system of nonlinear ordinary differential equations [27] (see also [4, 11, 18]). To his surprise, all the richness of chaotic systems was in these equations, resulting in an attractor bearing now his name (Fig. 9). In the 3-dimensional phase-space shown in Fig. 9, the trajectories display a most complex behaviour. Their trace is a kind of a twofold spiral structure that looks like a butterfly with its two wings. It is impossible to predict when a trajectory settled at one wing jumps over to the other wing, etc. That actually explains why the weather prediction becomes impossible under certain conditions. What E. Lorenz actually did, was to discover the skeleton of a complicated system. Everything should be made as simple as possible but not simpler, said Albert Einstein.

Such a route does not always involve a seemingly complex structure like the Lorenz attractor, the outcome from the simplification could also be a complicated theory. In this connection, let us mention solitons. This is again a deep and rich notion of remarkable simplicity [28, 29]. The route to get to it is very similar to that used by Lorenz: from complicated initial equations a model nonlinear evolution equation is derived, the solution to which is the celebrated soliton as a simple structure. Again, Einstein was right — the equation is simple enough but not simpler! Usually a soliton is meant as the Korteweg-de Vries (KdV) soliton, i.e. the solution to the KdV equation that propagates without changing its shape with the amplitude-dependent velocity (Fig. 10). In addition, the KdV soliton behaves like a particle when colliding with another

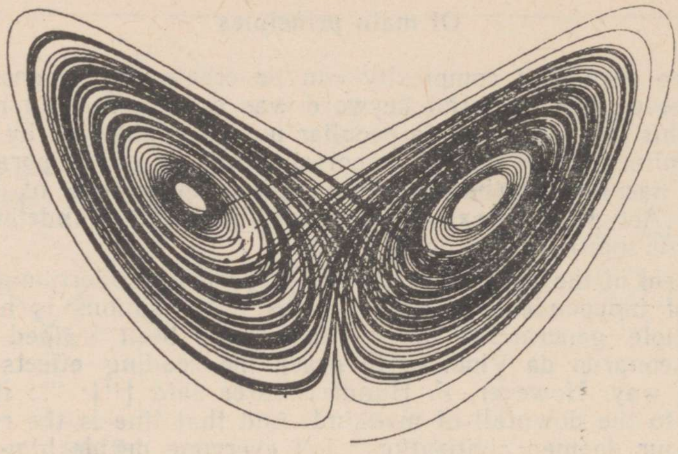

Fig. 9. The Lorenz attractor (calculated by V. Miider).

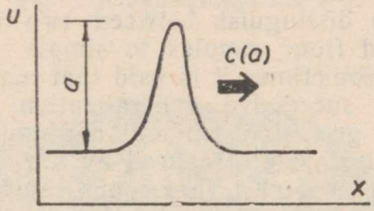

Fig. 10. The Korteweg-de Vries soliton.

soliton. The simplicity of solitons is deep, relating the governing equation to classical eigenvalue problems, and in this sense many facets of classical mechanics become intervowen and get new colour. That is the reason why the soliton concept is so widely used in contemporary hydrodynamics, plasma physics, solid mechanics, quantum mechanics, etc. The theories and methods in solitonics, like the inverse scattering theory, the Painlevé method, the Bäcklund transformation [29–31], are very complicated. The parallel between the concept of soliton and that of heat in the sense of the deepness of theories based upon these notions is obvious.

Let us point out that the Lorenz attractor represents more structural complexity, while solitons form a basis to functional complexity.

There are many ways to create complicated systems or structures (see above). Speaking about main principles, we have to point out one, which, at the first glance, may seem naive, but it is a powerful approach — namely, using analogies. Here is a simple example [11]:

— a line segment has 2 end points;
— a square has 4 corners;
— a cube has 8 corners;
— a four-dimensional hypercube has 16 corners;
— a five-dimensional supercube has 32 corners,

and so on. This analogy leads to generalizations in a multidimensional space. The real power of analogies is evident from the Nobel citation in Physics, 1991, stating that P.-G. de Gennes has got the Prize "for discovering that methods developed for studying order phenomena in simple systems can be generalized to more complex forms of matter, in particular to liquid crystals and polymers."

A natural question arises — how to measure complexity. Leaving aside the complexity of theories, which is, without any doubt, a fascinating philosophical problem, let us discuss physically tractable phenomena like solutions, maps, structures, etc., i. e. natural complexity. We may put questions like the following: is fractality of one structure larger than that of another structure, or is one attractor more chaotic than the other? Surprisingly enough there actually are answers to these questions. Fractality is measured by Hausdorff-Besicovich dimension [12, 14], the attractors are characterized by the spectrum of Lyapunov exponents [19] and chaotic systems by Kolmogorov entropy K [32]. For a regular system K is zero, for a chaotic system K is finite, and for a random (stochastic) system K is infinite. Said I. Stewart: "The brightest ray of light that chaos sheds focuses on the nature of complexity" [11].

For further research, a working hypothesis can be formulated:

World around us is based on simple but nonlinear rules which are applied repetitively and successively, mostly in time.

That is why we see Nature in its most complex way, and that is why nonlinear science (dynamics) has been so fast developing during the last decades.

Conclusions

What is said above is certainly not a theory of complexity but the ideas about its basic rules as the author sees them (cf. motto).

The philosophy of science pays a lot of attention to the essence of modern and postmodern (just contemporary) science. According to P. Rosenau [33], "modern science emphasizes parts rather than wholes, seeking to explain the totality by the sum of the parts..." On the other hand, there are two trends in postmodern science: affirmative and skeptical. Affirmative postmodernists concentrate upon an unbroken wholeness, emphasizing the elegance of complexity in the universe and the richness of difference. Skeptical postmodernists "conceive of the world as fragmented, disrupted, disordered, and in search of instabilities" [33]. These are actually two facets of the same core — complex Nature based upon holistic principles.

Concentrating upon complexity we may also ask whether it could be a paradigm in contemporary science. The answer is, however, negative — there can be no paradigm of the complex since this is a notion about science not a concept of science [34].

There is ever so much to be done in understanding complexity. Take for example the human brain with its 10^{11} neurons and 10^{15} synaptic junctions. There is an interesting phenomenon in modelling neural interactions — the usual equality of action and reaction encountered in physics does not hold in the neural network. To build up a fractal system of neural networks and to link it with cognitive behaviour represents an intriguing problem of complexity — perhaps one of the major scientific challenges of the 21st century [24].

The art of asking the right questions is more important than the art of solving them, said Georg Cantor, the inventor of a classical complex structure — the Cantor set. There are many questions about complexity yet to be formulated. The beauty of Nature lies in its complexity often made up of simple things.

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