Complex interpolation of compact operators: 
an update

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Abstract. After 41 years it is still not known whether an operator acting on a Banach pair and which acts compactly on one or both of the “endpoint” spaces also acts compactly on the complex interpolation spaces generated by the pair. We report some recent steps towards solving this and related problems.

Key words: complex interpolation, compact operator.

1. INTRODUCTION

The year 1964 saw the appearance of two remarkable and fundamental papers in the theory of interpolation spaces. Jacques-Louis Lions and Jaak Peetre \cite{1} introduced their “real method” spaces $(A_0, A_1)_{\theta, p}$ and Alberto Calderón \cite{2} introduced his “complex method” spaces $[A_0, A_1]_\theta$. Both papers provide numerous important results about their respective interpolation spaces, including some compactness theorems.

Now, more than 41 years after the appearance of these papers, we are still unable to answer the following very natural question which is asked implicitly in Calderón’s paper, and answered affirmatively there in an important special case.

Question C: Suppose that $A_0$ and $A_1$ are compatible Banach spaces, i.e., they form a Banach pair, and that so are $B_0$ and $B_1$. Suppose that $T : A_0 + A_1 \rightarrow B_0 + B_1$ is a linear operator such that $T : A_0 \rightarrow B_0$ compactly and $T : A_1 \rightarrow B_1$ boundedly.
Does it follow that $T$ maps the complex interpolation space $[A_0, A_1]_\theta$ into the complex interpolation space $[B_0, B_1]_\theta$ compactly for each $\theta \in (0, 1)$?

We cannot even answer the following question which could be expected to be somewhat easier.

**Question C$_2$:** This is the same as Question C, but under the stronger hypothesis that $T : A_1 \to B_1$ is also compact.

In addition to the case considered in [2], there are also quite a number of other special cases in which Question C, and therefore also Question C$_2$ have since been discovered to have affirmative answers. Most of the relevant papers for such results are mentioned on p. 262 of [3] and p. 353 of [4]. We refer also to [5,6], and the website http://www.math.technion.ac.il/~mcwikel/compact.

By contrast, the analogues of Questions C and C$_2$ in which the spaces $[A_0, A_1]_\theta$ and $[B_0, B_1]_\theta$ are replaced by $(A_0, A_1)_{\theta,p}$ and $(B_0, B_1)_{\theta,p}$ are apparently somewhat easier to answer. In fact the analogue of Question C$_2$ was answered affirmatively [7] already in 1969, and the analogue of Question C was answered affirmatively [8,9] in 1992.

In this short note we will report on some recent developments related to Questions C and C$_2$ and consider possible approaches towards answering them. We will assume that the reader is familiar with most earlier papers treating this topic, and also with the alternative definitions in [10] of complex interpolation spaces via minimal (orbit) and maximal (co-orbit) functors applied to pairs of weighted sequence spaces $(FL^p_0, FL^p_1)$, $p = 1, \infty$.

Since notation varies slightly from paper to paper, we should specify that here, for $p \in [1, \infty]$, we shall let $FL^p$ denote the space of complex sequences $\{\lambda_k\}_{k \in \mathbb{Z}}$ which arise as Fourier coefficients of some element of $L^p(\mathbb{T})$ with the norm induced by the norm of $L^p(\mathbb{T})$. Analogously, $FC$ is the closed subspace of $FL^\infty$ of sequences of Fourier coefficients of continuous functions. Then, for each $\alpha \in \mathbb{R}$, we let $FL^p_\alpha$ denote the space of sequences $\{\lambda_k\}_{k \in \mathbb{Z}}$ such that $\{e^{\alpha k}\lambda_k\}_{k \in \mathbb{Z}} \in FL^p$ with the obvious norm, and $FC_\alpha$ is defined analogously. Finally, for any Banach space $A$, we use the usual notation $(\ell^p(A))$ for the space of all $A$ valued sequences $\{a_n\}_{n \in \mathbb{N}}$ for which the norm $\|\{a_n\}_{n \in \mathbb{N}}\|_{\ell^p(A)} := \|\{\|a_n\|_A\}_{n \in \mathbb{N}}\|_{\ell^p}$ is finite.

2. SOME SIGNIFICANT CHOICES OF THE “RANGE” PAIR $(B_0, B_1)$

As proved in [4] (cf. also [9]), the problem of answering Question C can be reduced to the problem of answering any one of a number of special cases of Question C. Among those reductions, the following one, suggested by various ideas in [11] and [10] (cf. Proposition 3 of [4], p. 356 and Step 1 of the proof of Theorem 2.1 of [9], pp. 339–340), is particularly relevant for our discussion here:

**Proposition 2.1.** In order to answer Question C, it suffices to resolve it in the special case where the “domain” pair $(A_0, A_1)$ is $(\ell^1(FL^0_0), \ell^1(FL^1_1))$ and
the “range” pair \((B_0, B_1)\) is either \((\ell_\infty (FL_0^\infty), \ell_\infty (FL_1^\infty))\) or \((\ell_\infty (FC_0), \ell_\infty (FC_1))\).

Let us put this into perspective with what is known so far. The case where \((A_0, A_1)\) is an arbitrary pair and \((B_0, B_1)\) is \((FC_0, FC_1)\) can be resolved affirmatively as an immediate corollary of known results. In fact, using Fejér’s classical theorem about Fourier series, it is clear that the pair \((FC_0, FC_1)\) satisfies the special approximation condition required for Calderón’s partial answer to Question C in \([2]\).

Some months ago we reasoned that, until such time as someone sees how to resolve the case where \((B_0, B_1)\) is enlarged from \((FC_0, FC_1)\) to \((\ell_\infty (FC_0), \ell_\infty (FC_1))\) or to \((\ell_\infty (FL_0^\infty), \ell_\infty (FL_1^\infty))\), a reasonable intermediate step would be to consider the case where \((B_0, B_1) = (FL_0^\infty, FL_1^\infty)\). This pair apparently does not have any of the properties which would enable it to be treated by known theorems which resolve various special cases of Question C.

Recently \([12]\) we have been able to resolve this intermediate case:

**Theorem 2.2.** Suppose that \((A_0, A_1)\) is an arbitrary Banach pair and that \(T : A_0 + A_1 \rightarrow FL_0^\infty + FL_1^\infty\) is a bounded operator such that \(T : A_0 \rightarrow FL_0^\infty\) compactly and \(T : A_1 \rightarrow FL_1^\infty\) boundedly. Then \(T : [A_0, A_1]_\theta \rightarrow [FL_0^\infty, FL_1^\infty]_\theta\) compactly for each \(\theta \in (0, 1)\).

We remark that \([FL_0^\infty, FL_1^\infty]_\theta = FC_\theta\). (Cf. \([10,4]\]).

**3. NECESSARY AND SUFFICIENT CONDITIONS IN TERMS OF INFINITE MATRICES MAPPING \((\ell_1^j (FL_0^\infty), \ell_1^j (FL_1^\infty))\) into \((\ell_\infty (FL_0^\infty), \ell_\infty (FL_1^\infty))\)**

In the light of Proposition 2.1, one way of trying to answer Question C is to study various properties of the operators which map \(\ell_1^j (FL_0^\infty)\) into \(\ell_\infty^j (FL_1^\infty)\) for \(j = 0, 1\). These operators can be realized as infinite matrices, or rather as infinite matrices each of whose entries is itself an infinite matrix. These matrices were used in Theorem 2.1 of \([9]\), p. 339, to show that if, for any given pairs \((A_0, A_1)\) and \((B_0, B_1)\), Question C has an affirmative answer for one particular value of \(\theta \in (0, 1)\), then this implies an affirmative answer for all \(\theta \in (0, 1)\) for those pairs.

As time passes it seems that we should give more consideration also to the possibility of a negative answer to Question C. Indeed, this possibility is also raised by the result to be mentioned in the next section. In this section we briefly describe how a more careful examination of the above mentioned infinite matrices enables one to formulate questions about them, which may ultimately provide an affirmative or negative answer to Question C.

Since our exposition here needs to be short, we will refer the reader to Section 2 of \([9]\) as a point of departure and source of notation and more details for much of what we want to say. (Given more space, we would have preferred to present these
things in a slightly different way.) Thus we are dealing with the spaces $E_{\alpha}$ and $F_{\alpha}$
defined, for each $\alpha \in \mathbb{R}$, by $E_{\alpha} = \ell_1^U(FL_{\alpha}^1)$ and $F_{\alpha} = \ell_1^V(FL_{\alpha}^\infty)$, and we are
considering an operator $T$ which maps $E_j$ into $F_j$ boundedly for $j = 0, 1$. We also
assume that $T$ maps $E_0$ into $F_0$ compactly for at least one value $\theta_0$ of $\theta$. But here, in
contrast to $[9]$, we take $\theta_0 = 0$. In $[9]$ the index sets $U$ and $V$ may be uncountable.
But, as is clear from $[4]$ (see the proof of Proposition 3 on pp. 356–357), we in fact
only need to consider the case where $U$ and $V$ are countable.

As explained in $[9]$, we can realize $T$ via a matrix of operators $\{T_{uv}\}_{u \in U, v \in V}$,
where each $T_{uv}$ is a bounded map of $FL_{\alpha}^1$ into $FL_{\alpha}^\infty$ for all $\alpha \in [0, 1]$. More
explicitly, we have

$$
Tx = \left\{ \sum_{u \in U} T_{uv}x_u \right\}_{v \in V}
$$

for each element $x = \{x_u\}_{u \in U}$ in $E_{\alpha} = \ell_1^U(FL_{\alpha}^1)$. 

(1)

(Of course, there seems to be a typographical error in (1). But this is only because
our notation here has been kept consistent with some slightly unsuccessful notation
used in $[9]$.) Furthermore, for each fixed $u \in U$ and $v \in V$, the operator $T_{uv}$ can be represented as an infinite matrix (of complex numbers) $\{t_{jk}(u, v)\}_{j, k \in \mathbb{Z}}$. In
other words, we can completely specify the action of $T$ in terms of the “matrix of
matrices” $\{t_{jk}(u, v)\}_{j, k \in \mathbb{Z}, u \in U, v \in V}$.

Now let $T_{uv}$ be the operator represented by the diagonal matrix obtained by
replacing all nondiagonal elements of the preceding matrix by 0, i.e. the matrix
$\{\delta_{jk}t_{jk}(u, v)\}_{j, k \in \mathbb{Z}}$. More generally, for each $n \in \mathbb{Z}$, let $T_{uvn}$ be the operator
represented by the “$n$-displaced” diagonal matrix $\{\delta_{jk+n}\}t_{jk}(u, v)\}_{j, k \in \mathbb{Z}}$. The next
step is to introduce the “diagonal” operator $T_n$ for each fixed $n \in \mathbb{Z}$, which is
specified, analogously to above, via the matrix of operators $\{T_{uvn}\}_{u \in U, v \in V}$, i.e. by the
“matrix of matrices” $\{\delta_{jk+n}\}t_{jk}(u, v)\}_{j, k \in \mathbb{Z}, u \in U, v \in V}$.

It can be shown (cf. $[9]$) that $T_n$ maps $E_{\alpha}$ boundedly into $F_{\alpha}$ for each $\alpha \in \mathbb{R}$
and each $n \in \mathbb{Z}$. Furthermore, if $0 < \alpha < 1$, then the series $\sum_{n \in \mathbb{Z}} T_n$ converges in
the norm topology of the Banach space of bounded operators mapping $E_{\alpha}$ into $F_{\alpha}$,
and the sum of this series is our original operator $T$. Thus, if we wish to deduce
that $T : E_0 \to F_0$ is compact for some $\theta \in (0, 1)$, it will suffice to show that

$$
T_n \text{ maps } E_\theta \text{ into } F_\theta \text{ compactly for each } n \in \mathbb{Z}.
$$

(2)

In $[9]$ it is shown that (2) holds for all $\theta \in (0, 1)$ whenever $T : E_0 \to F_0$ is
compact for (at least) one value of $\theta$ in $(0, 1)$. It is also rather straightforward to
see, because of the “diagonal” structure of $T_n$, that the condition (2) is equivalent
for all real values of $\theta$, i.e. if it holds for any particular $\theta$, then it holds for all $\theta \in \mathbb{R}$.
We now point out that all this shows that the answer to Question C, regardless of
whether it is yes or no, hinges inevitably on the question of the compactness of the
“diagonal” operators $T_n$. Furthermore, the crucial things happen for values of $\theta$
which may be assumed to be arbitrarily close to 0. (Here we are also using the
reiteration formula for the complex method.) In fact the answer to Question C is the same as the answer to the following question:

**Question ∆:** Suppose that $T$ is an arbitrary compact operator from $E_0$ into $F_0$ with the additional property that $T : E_0 \rightarrow F_0$ is bounded for some $\alpha \neq 0$. Does it follow that the “diagonal” operator $T_n$ maps $E_0$ into $F_0$ compactly for all $n \in \mathbb{Z}$?

**Remark 3.1.** By considering compositions of $T$ and of $T_n$ with suitable shift operators, it is not hard to see that the answer to Question ∆ is the same as the answer to the corresponding question about $T_n$ for just one value of $n$, say $n = 0$.

4. A RELATED QUESTION ABOUT INFINITE FAMILIES OF COMPACT SUBSETS OF $\ell^\infty$

In this section we consider a different kind of question. It is a simpler version of the question appearing on page 362 of [4]. Our main reason for considering this question is that, by using arguments similar to those given in [4], it can be shown that an affirmative answer to it would suffice to imply an affirmative answer to Question C.

**Question CKM**: Suppose that, for each $\theta \in [0, 2\pi)$, we are given a subset $M(e^{i\theta})$ of the unit ball of $\ell^\infty$. Define the set $M(0)$ to consist of all elements $\alpha = \{\alpha_n\}_{n \in \mathbb{N}}$ of $\ell^\infty$, which are of the form $\{f_n(0)\}_{n \in \mathbb{N}}$ for some sequence of functions $f_n$, which are continuous on the closed unit disk and analytic in its interior and for which $\{f_n(e^{i\theta})\}_{n \in \mathbb{N}} \subseteq M(e^{i\theta})$ for each $\theta \in [0, 2\pi)$. If $M(e^{i\theta})$ is compact for every $\theta \in [0, 2\pi)$, does it follow that $M(0)$ is contained in a compact subset of $\ell^\infty$?

In fact, an affirmative answer to Question C would also follow from an affirmative answer to a special case of Question CKM, in the case where one makes the additional assumption that the sets $M(e^{i\theta})$ are “uniformly compact” on $[0, 2\pi)$, i.e., that for each $\varepsilon > 0$ there exists an integer $N(\varepsilon)$ such that, for each $\theta \in [0, 2\pi)$, the set $M(e^{i\theta})$ is contained in the union of $N(\varepsilon)$ balls in $\ell^\infty$ of radius $\varepsilon$.

If the compactness of the set $M(0)$ defined in Question CKM follows when the condition imposed in Question CKM is weakened so that $M(e^{i\theta})$ is assumed to be compact only for every $\theta$ in some fixed subset $E$ of $[0, 2\pi)$ with positive measure, then this suffices to give an affirmative answer to Question C. In fact, we only need to consider sets $E$ of a very particular form.

Here, in contrast to the partial affirmative result of Theorem 2.2, we report a result in a negative direction, namely the following remarkable example obtained by Fedor Nazarov, which suggests the possibility of a negative answer to Question CKM.

**Example.** For each $\varepsilon > 0$, there exists a positive integer $N(\varepsilon)$ and a collection of subsets $\{M(e^{it})\}_{t \in [0, 2\pi)}$ of the unit ball such that, for each $t \in [0, 2\pi)$, the set $M(e^{it})$ is contained in the union of $N(\varepsilon)$ balls in $\ell^\infty$, each of radius $\varepsilon$, but the
set \( M(0) \), defined as in Question CKM\(_2\), contains a sequence \( \{e_n\}_{n \in \mathbb{N}} \) for which 
\[ \|e_n - e_m\|_{\ell^\infty} = 1 \]
for all \( m \neq n \).

The details of this example can be found on the website http://www.math.technion.ac.il/~mcwikel/compact.

**REFERENCES**


**Operaatorite komplekssest interpolatsioonist: täiendus**

Michael Cwikel ja Svante Janson

Juba 41 aastat pole ikka veel teada, kas operaator, mis toimib teatud Banachi paari peal ja on kompaktne ühes või mõlemas lõpp-punktis, toimib kompaktsesti selle paari poolt tekitatud interpolatsiooni ruumide peal. On antud ülevaade selle ja sarnaste probleemide lahendamise mõnest täiendusest.