On polynomials that are weakly uniformly continuous on the unit ball of a Banach space

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Abstract. We prove quantitative strengthenings of results on polynomials that are weakly uniformly continuous on the unit ball of a Banach space due to Aron, Lindström, Ruess, and Ryan (Proc. Amer. Math. Soc., 1999, 127, 1119–1125) and to Toma (Aplicações holomorfas e polinômios $\tau$-contínuos. 1993). Our method is based on the uniform factorization of compact sets of compact operators.

Key words: Banach spaces, uniform compact factorization, $n$-homogeneous polynomials.

1. INTRODUCTION

Let $X$ and $Y$ be Banach spaces over the same, either real or complex, field $\mathbb{K}$. We denote by $L(X,Y)$ the Banach space of all continuous linear operators from $X$ to $Y$, and by $K(X,Y)$ its subspace of compact operators.

Let $L^s(n)X$ denote the Banach space of continuous symmetric $n$-linear forms on $X$ and let $P(n)X$ denote the Banach space of continuous $n$-homogeneous polynomials on $X$. Then for each $P \in P(n)X$ there is a unique $A_P \in L^s(n)X$ satisfying $P(x) = A_P(x, \ldots, x)$ for each $x \in X$.

Recall that $P \in P(n)X$ is weakly uniformly continuous on the closed unit ball $B_X$ of $X$ if for each $\epsilon > 0$ there are $x_1^*, \ldots, x_n^* \in X^*$ and $\delta > 0$ such that if $x, y \in B_X$, $|x_i^*(x - y)| < \delta$ for $i = 1, \ldots, n$, then $|P(x) - P(y)| < \epsilon$. Let $P_{wu}(n)X$ denote the subspace of $P(n)X$ consisting of the polynomials that are weakly uniformly continuous on $B_X$. The corresponding subspace of $L^s(n)X$ is denoted by $L^s_{wu}(n)X$. Notice that $P_{wu}(n)X$, with the norm induced from $P(n)X$, is a Banach space (see [1], Proposition 2.4).
For each \( P \in \mathcal{P}(nX) \) there is a linear operator \( T_P : X \to \mathcal{L}^s(n^{-1}X) \) defined by 
\[
(T_P x_1)(x_2, \ldots, x_n) = A_P(x_1, x_2, \ldots, x_n).
\]
Clearly, the correspondence \( A_P \to T_P \) is linear and \( \|T_P\| = \|A_P\| \). According to \([1]\), \( P \in \mathcal{P}_{wu}(nX) \) if and only if \( T_P \in \mathcal{K}(X, \mathcal{L}^s(n^{-1}X)) \). Moreover, if \( P \in \mathcal{P}_{wu}(nX) \), then \( T_P \in \mathcal{K}(X, \mathcal{L}^s_{wu}(n^{-1}X)) \).

In 1999, Aron et al. (see \([2]\), Proposition 5) proved the following result.

**Theorem 1** \([2]\). Let \( X \) be a Banach space and let \( n = 2, 3, \ldots \). Let \( C_n \) be a relatively compact subset of the space \( \mathcal{K}(X, \mathcal{L}^s_{wu}(n^{-1}X)) \). Then there exists a compact subset \( C \) of \( X^* \) such that for all \( S \in C_n \) and all \( x \in X \)
\[
|(Sx)(x, \ldots, x)| \leq \sup_{x^* \in C} |x^*(x)|^n.
\]

Theorem 1 together with its proof in \([2]\) gives no information about the size of the set \( C \) corresponding to the size of \( C_n \).

The purpose of this article is to prove the following quantitative strengthening of Theorem 1. We denote \( |C| = \sup\{\|x\| : x \in C\} \), where \( C \) is a bounded set in a Banach space.

**Theorem 2.** Let \( X \) be a Banach space and let \( n = 2, 3, \ldots \). Let \( C_n \) be a relatively compact subset of the space \( \mathcal{K}(X, \mathcal{L}^s_{wu}(n^{-1}X)) \). Then there exists a compact circled subset \( C \) of \( X^* \) with \( |C| = \max\{|C_n|, 1\} \) such that for all \( S \in C_n \) and all \( x \in X \)
\[
|(Sx)(x, \ldots, x)| \leq \sup_{x^* \in C} |x^*(x)|^n.
\]

We use a standard notation. A Banach space \( X \) will be regarded as a subspace of its bidual \( X^{**} \) under the canonical embedding. The closure of a set \( A \subset X \) is denoted by \( \overline{A} \). The linear span of \( A \) is denoted by \( \text{span} \, A \) and the circled hull by \( \text{circ} \, A \).

### 2. PROOF OF THEOREM 2

The proof of Theorem 2 will be based on a factorization result that easily follows from

**Lemma 1.** Let \( X \) and \( Y \) be Banach spaces. For every relatively compact subset \( C \) of \( \mathcal{K}(X, Y) \), there exist a reflexive Banach space \( Z \), a linear mapping \( \Phi : \text{span} \, C \to \mathcal{K}(X, Z) \), and a norm one operator \( v \in \mathcal{K}(Z, Y) \) such that \( S = v \circ \Phi(S) \) for all \( S \in \text{span} \, C \). The mapping \( \Phi \) restricted to \( C \) is a homeomorphism and satisfies
\[
\|S\| \leq \|\Phi(S)\| \leq \min\{|C|, |C|^{1/2} b^{1/2} \|S\|^{1/2}\},
\]
\( S \in C \), where \( b \approx 2^{1/2} \) is an absolute constant.
Proof. Since \( \text{circ} \mathcal{C} \) is a compact subset of \( \mathcal{K}(X, Y) \), by [3], Theorem 6, there exist a reflexive Banach space \( Z \), a linear mapping \( \Phi : \text{span} \ C \rightarrow \mathcal{K}(X, Z) \), and a norm one operator \( v \in \mathcal{K}(Z, Y) \) such that \( S = v \circ \Phi(S) \), for all \( S \in \text{span} \ C \). Moreover, the mapping \( \Phi \) restricted to \( \text{circ} \mathcal{C} \) is a homeomorphism satisfying

\[
\|S\| \leq \|\Phi(S)\| \leq \min \left\{ \frac{d}{2}, \left( \frac{d}{2} \right)^{1/2} b^{1/2}\|S\|^{1/2} \right\},
\]

where \( d = \text{diam} \, \text{circ} \mathcal{C} \). To prove that it is also compact, let us fix an arbitrary \( \varepsilon > 0 \). Let \( \{\Phi(S_1), \ldots, \Phi(S_n)\} \), \( S_k \in C_2 \), be an \( \varepsilon \)-net in the relatively compact set \( \{\Phi(S) : S \in C_2\} \). Since \( \Phi(S_k) \) is a compact operator, \( (\Phi(S_k))^* \) is also a compact operator and therefore \( (\Phi(S_k))^*(B_{Z^*}) \) is a relatively compact set. Since \( \bigcup_{k=1}^n (\Phi(S_k))^*(B_{Z^*}) \) is clearly a relatively compact \( \varepsilon \)-net in the set \( \{(\Phi(S))^*(z^*) : S \in C_2, z^* \in B_{Z^*}\} \), this set is relatively compact. Hence, \( C_\Phi \) is a compact set.

Moreover, we get

\[
\|\Phi(S)x\| = \sup_{z^* \in B_{Z^*}} |z^*(\Phi(S)x)| = \sup_{z^* \in B_{Z^*}} |((\Phi(S))^*(z^*))(x)| \leq \sup_{x^* \in C_\Phi} |x^*(x)|
\]

for all \( S \in C_2 \) and all \( x \in X \).
Denoting \( C_v = \overline{v(B_Z)} \subset X^* \), we have that \( C_v \) is circled and compact, and
\[
\|v^* x\| = \sup_{z \in B_Z} |(v^* x)(z)| = \sup_{z \in B_Z} |(v z)(x)| \leq \sup_{x^* \in C_v} |x^*(x)|
\]
for all \( x \in X \).

Finally, let \( C = C_\Phi \cup C_v \). Then \( C \) is circled and compact, and
\[
|(Sx)(x)| \leq \|v^* x\| \|\Phi(S)x\| \leq \sup_{x^* \in C_v} |x^*(x)| \sup_{x^* \in C_\Phi} |x^*(x)| \leq \sup_{x^* \in C} |x^*(x)|^2
\]
for all \( S \in C_2 \) and all \( x \in X \).

By the definition of \( |C| \),
\[
|C| = \sup_{x^* \in C} \|x^*\| = \sup_{x^* \in C_\Phi \cup C_v} \|x^*\| = \max\{ \sup_{x^* \in C_\Phi} \|x^*\|, \sup_{x^* \in C_v} \|x^*\| \}
\]
\[
= \max\{|C_\Phi|, |C_v|\}.
\]
Let us first estimate
\[
|C_\Phi| = \sup_{x^* \in C_\Phi} \|x^*\| = \sup_{S \in C_2} \|\Phi(S)^*(z^*)\| = \sup_{S \in C_2} \|\Phi(S)^*\| = \sup_{S \in C_2} \|\Phi(S)\|.
\]

Using the conclusion of Lemma 1, we have for all \( S \in C_2 \),
\[
\|S\| \leq \|\Phi(S)\| \leq \sup_{S \in C_2} \|\Phi(S)\| = |C_\Phi|, \quad \text{and}
\]
\[
|C_\Phi| \leq |C_2|.
\]
Hence
\[
|C_2| \leq |C_\Phi| \leq |C_2|,
\]
meaning that \( |C_\Phi| = |C_2| \). Let us now compute
\[
|C_v| = \sup_{x^* \in C_v} \|x^*\| = \sup_{z \in B_Z} \|v z\| = \|v\| = 1.
\]
Consequently,
\[
|C| = \max\{|C_\Phi|, |C_v|\} = \max\{|C_2|, 1\}.
\]

Assume that the result is true for \( n - 1 \), where \( n \in \{3, 4, \ldots\} \). Let \( C_n \) be a relatively compact subset of the space \( \mathcal{K}(X, \mathcal{L}_W^{s,n-1}(X)) \). By Lemma 1 there exist a reflexive Banach space \( Z \), a linear mapping \( \Phi : \text{span} C_n \rightarrow \mathcal{K}(X, Z) \), and a norm
one operator $v \in \mathcal{K}(Z, \mathcal{L}^n_{wu}(n^{-1}X))$ such that $S = v \circ \Phi(S)$ for all $S \in \text{span } C_n$. Then for all $S \in C_n$ and for all $x \in X$, considering $(x, \ldots, x) \in (\mathcal{L}^n_{wu}(n^{-1}X))^*$ (note that if $A \in \mathcal{L}^n_{wu}(n^{-1}X)$, then $((x, \ldots, x), A) = A(x, \ldots, x)$),

$$|(Sx)(x, \ldots, x)| = |v(Sx)(x, \ldots, x)| = |(v^*(x, \ldots, x))\Phi(Sx)|,$$

hence

$$|(Sx)(x, \ldots, x)| \leq \|v^*(x, \ldots, x)\|\|\Phi(Sx)\|.$$ 

Put, as above,

$$C_\Phi = \{(\Phi(S))^*(z^*) : S \in C_n, z^* \in B_{Z^*}\} \subset X^*.$$ 

Then $C_\Phi$ is circled and compact, and we get

$$\|\Phi(S)x\| = \sup_{z^* \in B_{Z^*}} |z^*\Phi(S)x| = \sup_{z^* \in B_{Z^*}} |((\Phi(S))^*(z^*))(x)| \leq \sup_{x^* \in C_\Phi} |x^*(x)|$$

for all $S \in C_n$ and for all $x \in X$. Recall that $v(B_Z)$ is a relatively compact subset of $\mathcal{L}^n_{wu}(n^{-1}X)$. Hence

$$C_{n-1} := \{T_P : P \in \mathcal{P}_{wu}(n^{-1}X), A_P \in v(B_Z) \} \subset \mathcal{L}(X, \mathcal{L}^n(n^{-2}X))$$

is also relatively compact. According to [1], $C_{n-1} \subset \mathcal{K}(X, \mathcal{L}^n(n^{-2}X))$. Therefore, by the induction hypothesis, there is a circled and compact subset $C_v \subset X^*$ with $|C_v| = \max\{|C_{n-1}|, 1\}$ such that

$$|(T_Px)(x, \ldots, x)| \leq \sup_{x^* \in C_v} |x^*(x)|^{n-1}$$

for all $P \in \mathcal{P}_{wu}(n^{-1}X)$ with $A_P \in v(B_Z)$. Since $v(B_Z) \subset \mathcal{L}^n_{wu}(n^{-1}X)$, for all $z \in B_Z$ there exists $P \in \mathcal{P}_{wu}(n^{-1}X)$ such that $vz = A_P$. By definition, $A_P(x, x, \ldots, x) = (T_Px)(x, \ldots, x)$, $x \in X$. Hence, for all $z \in B_Z$ and all $x \in X$,

$$|(vz)(x, \ldots, x)| = |A_P(x, x, \ldots, x)| = |(T_Px)(x, \ldots, x)| \leq \sup_{x^* \in C_v} |x^*(x)|^{n-1}.$$ 

Therefore

$$\|v^*(x, \ldots, x)\| = \sup_{z \in B_Z} |(v^*(x, \ldots, x))(z)|$$

$$= \sup_{z \in B_Z} |(vz)(x, \ldots, x)| \leq \sup_{x^* \in C_v} |x^*(x)|^{n-1}.$$ 

Finally, let $C = C_\Phi \cup C_v$. Then $C$ is circled and compact, and

$$|(Sx)(x, \ldots, x)| \leq \|v^*(x, \ldots, x)\|\|\Phi(S)x\|$$

$$\leq \sup_{x^* \in C_v} |x^*(x)|^{n-1} \sup_{x^* \in C_\Phi} |x^*(x)| \leq \sup_{x^* \in C} |x^*(x)|^{n}$$

for all $S \in C_n$ and all $x \in X$. 

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To complete the proof, let us show that $|C| = \max \{|C_n|, 1\}$. Similarly to the case $n = 2$, we have

$$|C| = \sup_{x^* \in C} \|x^*\| = \sup_{x^* \in C_{\Phi} \cup C_v} \|x^*\| = \max \{\sup_{x^* \in C_{\Phi}} \|x^*\|, \sup_{x^* \in C_v} \|x^*\|\}$$

and

$$|C_{\Phi}| = \sup_{x^* \in C_{\Phi}} \|x^*\| = \sup_{S \in C_n} \|(\Phi(S))^*(z^*)\| = \sup_{S \in C_n} \|(\Phi(S))^*\| = \sup_{S \in C_n} \|\Phi(S)\|.$$ 

Using the conclusion of Lemma 1, we have for all $S \in C_n$,

$$\|S\| \leq \|\Phi(S)\| \leq |C_{\Phi}|$$

and

$$\|\Phi(S)\| \leq |C_n|.$$ 

Hence

$$|C_n| \leq |C_{\Phi}| \leq |C_n|,$$

meaning that $|C_{\Phi}| = |C_n|$. Let us show that $|C_v| = 1$. Recall that $|C_v| = \max\{|C_{n-1}|, 1\}$. Since

$$|C_{n-1}| = \sup_{T_p \in C_{n-1}} \|T_p\| = \sup_{A_p \in v(BZ)} \|A_p\| \leq \sup_{z \in BZ} \|vz\| = \|v\| = 1,$$

we clearly have $|C_v| = 1$. □

3. APPLICATION TO POLYNOMIALS

The next theorem is proved by Toma [4] (an alternative proof is given in [2]).

**Theorem 3** [4]. Let $X$ be a Banach space, let $n = 2, 3, \ldots$, and let $P \in \mathcal{P}^{(n)}(X)$. The polynomial $P \in \mathcal{P}_{wu}^{(n)}(X)$ if and only if there exists a compact subset $C$ of $X^*$ such that for all $x \in X$

$$|P(x)| \leq \sup_{x^* \in C} |x^*(x)|^n.$$

The following is a quantitative version of Theorem 3.

**Corollary 1.** Let $X$ be a Banach space, let $n = 2, 3, \ldots$, and let $P \in \mathcal{P}^{(n)}(X)$. The following are equivalent:

(a) $P \in \mathcal{P}_{wu}^{(n)}(X)$,

(b) there exists a compact subset $C$ of $X^*$ such that for all $x \in X$

$$|P(x)| \leq \sup_{x^* \in C} |x^*(x)|^n,$$
(c) there exists a compact circled subset $C$ of $X^*$ with

$$\max\{\|P\|, 1\} \leq |C| \leq \max\left\{\frac{n^n}{n!}\|P\|, 1\right\}$$

such that for all $x \in X$

$$|P(x)| \leq \sup_{x^* \in C} |x^*(x)|^n.$$

Proof. (a) $\Rightarrow$ (c). Let $P \in \mathcal{P}_{wu}(nX)$, then $\{T_P\} \subset \mathcal{K}(X, \mathcal{L}_{wu}(n^{-1}X))$. Applying Theorem 2 to $C_n = \{T_P\}$, we get that there is a compact circled subset $C$ of $X^*$ with $|C| = \max\{\|T_P\|, 1\}$ such that for all $x \in X$

$$|P(x)| = |A_P(x, x, \ldots, x)| = |(T_P x)(x, \ldots, x)| \leq \sup_{x^* \in C} |x^*(x)|^n.$$

Applying the polarization formula (see, for example, [\ref{5}], Theorem 1.7), we have

$$\|P\| \leq \|T_P\| \leq \frac{n^n}{n!}\|P\|.$$

Hence $\max\{\|P\|, 1\} \leq |C| \leq \max\{\frac{n^n}{n!}\|P\|, 1\}$.

(c) $\Rightarrow$ (b). Obvious.

(b) $\Rightarrow$ (a). Follows immediately from Theorem 3. $\square$

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REFERENCES

Banachi ruumi ühikkeral nõrgalt ühtlaselt pidevate polünoomidest

Kristel Mikkor

On tõestatud Aroni-Lindströmi-Ruessi-Ryani \(^2\) ja Toma \(^4\) teoreemide kvantitatiivsed versioonid Banachi ruumi ühikkeral nõrgalt ühtlaselt pidevate polünoomide kohta. Tõestusmeetod tugineb kompaktsete operaatorite kompaktsete hulkade ühtlasele faktorisatsioonile.