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CONTROL THEORY

Feedback linearization of discrete-time nonlinear control systems: computational aspects

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Abstract. An alternative solution of the static state feedback linearization problem for the discrete-time case is given. This solution is based on the sequence of distributions, whose computation requires only the knowledge of the backward shift equations. This computational method is especially suitable for the class of discrete-time systems, obtained from the implicit Euler discretization of continuous-time systems. As a practical example the implicit Euler discretization of hydraulic press equations is considered.

Key words: feedback linearizability, discrete-time systems, vector fields, implicit Euler method.

1. INTRODUCTION

The static state feedback linearization of nonlinear control systems, including the discrete-time case, is one of the most studied problems in nonlinear control. We examine in this paper only the discrete-time systems, for which the question of the existence of a regular static state feedback and the state transformation, allowing transforming the discrete-time system equations into the Brunovsky form, has been studied, for instance, in [1]–[11]. The necessary and sufficient conditions have been formulated in many different ways and various linearization algorithms have been introduced in those papers.

Note that the existing methods, except the one introduced in [1], require the use of both forward and backward shifts in computations. The practical application of such solutions has limitations in the case when the system is described in terms of backward shift equations and not via classical forward shift equations as usual. This is especially true when the backward shift equations cannot be easily transformed into the classical form. That case requires solving a system of n nonlinear algebraic equations if the system is state reversible or $n+m$ equations if the system is only submersive. One typical subclass of systems described in terms of backward shift equations results from sampling when one prefers the implicit Euler discretization scheme to the explicit scheme in order to enlarge the numerical stability region [6]. The paper [12] shows that the explicit Euler method has certain drawbacks for global approximation of homogenous systems (taking an intermediate place between linear and nonlinear systems) with nonzero degrees, whereas the implicit Euler scheme ensures convergence of the approximating solutions. For the additional advantages

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of the implicit Euler method, see [12] and the references therein. As for the method from [1], note that it uses the forward shift equations only, but requires also the inverse of the Jacobi matrix of the system, whose computation can be difficult.

In this paper we suggest an alternative approach to solve the feedback linearization problem which needs only backward shifts and is based on the results of [5]. Strictly speaking, we do not suggest a new method but rather show that the solvability conditions of the solution in terms of certain distributions in [5] can be replaced by conditions in terms of different but related distributions, the computation of which can be completed in terms of backward shifts only. The idea of avoiding forward shifts relies on the application of the concept of distribution invariants and their relative degrees in computations. Our approach is somewhat similar to the method from [6] to check the linearizability property and define the coordinate transformation. The method introduced in [6] is also based on the invariants of the distributions, but in this paper we do not compute the forward shifts of these invariants directly. Our method is therefore easier to apply, when the forward shift equations are difficult to find.

Finally, note that the results of this paper are generic, i.e. valid for almost every point. Since we look at dimensions (or ranks) over the field of functions, not over $\mathbb{R}$, there is no point about constant dimensionality of the distributions. A generic rank is a maximal rank on an open and dense set. The rank may drop on some subset. Reducing the set, one can always achieve a constant rank over $\mathbb{R}$, see more in [13].

2. FEEDBACK LINEARIZATION: STANDARD SOLUTION

Consider the *extended* discrete-time nonlinear control system [5]

$$x^{\langle 1\rangle}(t) = \bar{\Phi}\left(x(t),u(t)\right), \quad z(t) = \chi\left(x(t),u(t)\right), \tag{1}$$

where $x^{\langle 1\rangle}(t) := x(t+1)$, $t \in \mathbb{Z}$, the variables $x(t) \in \bar{X} \subset \mathbb{R}^n$, $u(t) \in U \subset \mathbb{R}^m$, $z(t) \in Z \subset \mathbb{R}^m$, and the state transition map $\bar{\Phi} : \bar{X} \times U \to \bar{X}$ is supposed to be analytic. Both $\bar{X}$ and U are assumed to be open sets. The variable $z(t) \in \mathbb{R}^m$ is chosen so that the extended map $\Phi = [\bar{\Phi}^T, \chi^T]^T$ has the global analytic inverse $x = \Lambda(x^{\langle 1\rangle}, z)$, $u = \lambda(x^{\langle 1\rangle}, z)$, defined on its image $\Phi(\bar{X} \times U)$. System (1) defines the inversive difference field $\mathscr{K}$ of meromorphic functions of a finite number of variables from the set $C = \{x, u^{\langle k\rangle}, z^{\langle -l\rangle}, k \geq 0, l \geq 1\}$. Here $u^{\langle k\rangle}$ denotes the kth-order forward shift of x and $z^{\langle -l\rangle}$ the lth-order backward shift of z. The 1st-order forward shift of variable x is defined by equations (1) and the 1st-order backward shifts by

$$x^{\langle -1\rangle} = \Lambda(x, z^{\langle -1\rangle}), \quad u^{\langle -1\rangle} = \lambda(x, z^{\langle -1\rangle}). \tag{2}$$

The higher-order shifts are defined recursively, see more in [5]. The backward shift can be extended to the vector fields[1]

$$\Xi = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} \eta_j \frac{\partial}{\partial u_j} \tag{3}$$

by

$$\Xi^{\langle -1\rangle} = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} b_j \frac{\partial}{\partial z_j^{\langle -1\rangle}}, \tag{4}$$

where

$$a_i = \left\langle \mathrm{d}x_i^{\langle 1\rangle}, \Xi \right\rangle^{\langle -1\rangle}, \quad b_j = \left\langle \mathrm{d}\chi_j, \Xi \right\rangle^{\langle -1\rangle}. \tag{5}$$

[1] Note that in [5] more general formulae are given for forward and backward shifts of vector fields, having also the components in directions $\partial/\partial u^{\langle k\rangle}$, $k > 0$, and $\partial/\partial z^{\langle -l\rangle}$, $l > 0$.

The projection of $\Xi^{\langle -1\rangle}$ is the vector field

$$\Xi^{\langle -1\rangle\pi} = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}. \tag{6}$$

Note that the backward shift and projection operators do not commute.

Definition 1. *The relative degree of a function $\varphi(x)$ is the smallest positive integer r such that*

$$\frac{\partial \varphi^{\langle l\rangle}}{\partial u} \equiv 0, \quad \forall l = 0,...,r-1, \quad \frac{\partial \varphi^{\langle r\rangle}}{\partial u} \not\equiv 0. \tag{7}$$

Definition 2. *[3] A regular static state feedback is an analytic map $\alpha : \bar{X} \times V \to U$*

$$u = \alpha(x,v) \tag{8}$$

such that $\text{rank}_{\mathscr{K}}(\partial\alpha/\partial v) = m$*, and $V \in \mathbb{R}^m$.*

Definition 3. *System (1) is said to be (generically) linearizable by a regular static state feedback if there exists a state diffeomorphism $X = \Psi(x)$ and a regular static state feedback $u = \alpha(x,v)$, such that in the new coordinates one has m $(i = 1,...,m)$ independent chains of forward shifts*

$$X_{i,1}^{\langle 1\rangle} = X_{i,2}, \; ..., \quad X_{i,r_i-1}^{\langle 1\rangle} = X_{i,r_i}, \quad X_{i,r_i}^{\langle 1\rangle} = v_i, \tag{9}$$

where r_i is the relative degree of $X_{i,1}$. The form (9) is called the Brunovsky form.

The standard linearizability conditions are formulated in terms of the non-decreasing stabilizing sequence of distributions $\mathscr{D}_k \subseteq \text{span}_{\mathscr{K}}\{\partial/\partial u, \partial/\partial z^{\langle -1\rangle}\}$, defined as

$$\mathscr{D}_k = \text{span}_{\mathscr{K}} \left\{ \frac{\partial}{\partial z^{\langle -1\rangle}}, \left(\frac{\partial}{\partial u}\right)^{\langle -l\rangle\pi}, l = 1,...k \right\}, \tag{10}$$

where $(\partial/\partial u)^{\langle -l\rangle\pi}$ denotes the projection of the lth-order backward shift of the vector field $\partial/\partial u$. Denote by $k^\star$ the smallest integer such that $\mathscr{D}_1 \subset ... \subset \mathscr{D}_{k^\star-1} \subset \mathscr{D}_{k^\star} = \mathscr{D}_{k^\star+1}$. Note that $k^\star \leq n$, because all $\mathscr{D}_k$'s belong to the $(n+m)$-dimensional space and, according to (10), $\dim_{\mathscr{K}} \mathscr{D}_k > m$ for all $k > 0$. Consequently, the maximal number of independent $\mathscr{D}_k$'s cannot be greater than n, and $k^\star$ is the first step, at which the sequence $\mathscr{D}_k$ stabilizes.

Definition 4. *A distribution $\mathscr{D}$ is called involutive if for two arbitrary vector fields $\Xi_1, \Xi_2 \in \mathscr{D}$ also $[\Xi_1, \Xi_2] \in \mathscr{D}$.*

Theorem 5. *[5] System (1) is (generically) static state feedback linearizable if and only if*
(i) all $\mathscr{D}_k$, $k > 0$, are involutive and
(ii) $\dim_{\mathscr{K}} \mathscr{D}_{k^\star} = n + m$*.*

Observe that the direct application of Theorem 5 requires the means to compute explicitly both the backward and forward shifts, see (4), (5).

3. DISCRETE-TIME MODELS OF CONTINUOUS-TIME CONTROL SYSTEMS

In most cases it is impossible to find the exact discrete-time model of a nonlinear continuous-time system [14]. In general, the discretization of continuous-time state equations

$$\dot{x} = f(x,u) \tag{11}$$

requires approximation. The simplest approach is to use the explicit Euler discretization scheme which converts equations (11) into the form

$$x^{\langle 1\rangle} = x + f(x,u)T. \tag{12}$$

A disadvantage of this scheme is the small region of numerical stability. In order to increase this region, the *implicit* (alternatively called the backward) Euler discretization scheme is preferred:

$$x^{\langle 1\rangle} = x + f(x^{\langle 1\rangle},u)T. \tag{13}$$

The scheme (13) also allows one to use larger sampling times T [6]. In order to obtain the system description as in (1), one has to solve equations (13) with respect to $x^{\langle 1\rangle}$. Even in case of a relatively simple form of f it may lead to very complicated equations in terms of the explicit forward shift operator, as shown in [6]. Therefore, in such cases it is preferable to use the system description in terms of the explicit backward shift operator, obtained from (13) simply as

$$x^{\langle -1\rangle} = x - f(x,z^{\langle -1\rangle})T =: \Lambda(x,z^{\langle -1\rangle}), \quad z^{\langle -1\rangle} = u^{\langle -1\rangle}. \tag{14}$$

4. THE MAIN RESULT

The goal of this paper is to introduce an alternative method to check feedback linearizability and to define the coordinate transformation for the static state feedback linearization that relies only on the backward shift operator. The idea that allows us to avoid the application of the forward shift is based on the concept of distribution invariants, in particular on Theorem 6 below.

Theorem 6. *[5] The following statements are equivalent:*
(i) a function $\varphi_k(x)$ is an invariant of $\mathscr{D}_k$, i.e.

$$\langle \mathrm{d}\varphi_k, \Xi\rangle \equiv 0, \quad \forall \Xi \in \mathscr{D}_k, \tag{15}$$

(ii) the relative degree of $\varphi_k(x)$ is at least $k+1$.

4.1. Computation of distributions Δ_k

In order to check the static state feedback linearizability, we introduce, instead of $\mathscr{D}_k$, the non-decreasing sequence of distributions Δ_k, $k > 0$. The reason is that the computation of Δ_k requires only the knowledge of the backward shift equations. Note that the computation of the basis vector fields $(\partial/\partial u)^{\langle -l\rangle\pi}$ of $\mathscr{D}_k$'s in (10) requires the knowledge of both the forward and backward shift equations, see (4) and (5). Or, alternatively, one can compute $\mathscr{D}_k$ with the help of the forward shift equations and the inverse of the corresponding Jacobi matrix, whose columns (interpreted as vector fields) can be used for computations of $(\partial/\partial u)^{\langle -l\rangle\pi}$. Lemma 8 below shows that if Δ_k is involutive, then Δ_k and $\mathscr{D}_k$ coincide.

In this subsection we present Algorithm 1 to compute, step by step, the distributions Δ_k, $k > 0$, that rely only on the backward shift operator. Each step of the algorithm uses the invariants of the distribution, obtained at the previous step. These invariants will be shifted backward and then, with the help of these backward shifts the basis vector fields of the next distribution are defined.

Denote by $I_k = \varphi_k(x)$ a complete set of independent functions, whose relative degree is at least $k+1$. Denote the number of independent invariants I_k by n_k. Note that in Algorithm 1 below we do not compute directly the forward shifts of the functions $\varphi_k(x)$ but just use the fact that they exist in principle, and the fact that the relative degree of $\varphi_k(x)$ is at least $k+1$.

Algorithm 1. Computation of distributions Δ_k and their invariants.

Step 0. $I_0 := x$, $\phi_0 := \Lambda(x, z^{\langle -1 \rangle})$, $n_0 = n$.

Step k $(k \geq 1)$**.** Suppose that we have a complete set of independent functions $I_{k-1} = \varphi_{k-1}(x)$ with relative degrees being at least k. This step (i) computes their backward shifts $I_{k-1}^{\langle -1 \rangle}$, (ii) defines with their help the next distribution Δ_k, and (iii) finds its invariants I_k.

Define the map $\phi_{k-1} = I_{k-1}^{\langle -1 \rangle} : (\bar{X} \times \mathbb{R}^m) \to \mathbb{R}^{n_{k-1}}$, based on backward shift equations (2) as follows:

$$\phi_{k-1}(x, z^{\langle -1 \rangle}) := \varphi_{k-1}(\Lambda(x, z^{\langle -1 \rangle})). \tag{16}$$

Introduce the kernel of the Jacobi matrix

$$T\phi_{k-1} = \frac{\partial \phi_{k-1}(x, z^{\langle -1 \rangle})}{\partial (x, z^{\langle -1 \rangle})} \tag{17}$$

as a distribution

$$\operatorname{Ker} T\phi_{k-1} = \operatorname{span}_{\mathscr{K}}\{\Xi_{k-1}\} : \langle \mathrm{d}\phi_{k-1}, \Xi_{k-1} \rangle \equiv 0. \tag{18}$$

Introduce the distribution

$$\Delta_k = \operatorname{Ker} T\phi_{k-1} \cup \operatorname{span}_{\mathscr{K}} \left\{ \frac{\partial}{\partial z^{\langle -1 \rangle}} \right\}. \tag{19}$$

Find all independent invariants $I_k = \varphi_k(x)$ of Δ_k as the functions satisfying the conditions

$$\langle \mathrm{d}I_k, \Xi_{k-1} \rangle \equiv 0, \quad \left\langle \mathrm{d}I_k, \frac{\partial}{\partial z^{\langle -1 \rangle}} \right\rangle \equiv 0. \tag{20}$$

Due to Lemma 7 below, the set I_k is also a complete set of all independent functions with the relative degree at least $k+1$:

$$\frac{\partial I_k^{\langle l \rangle}}{\partial u} \equiv 0, \quad \forall\, l = 0, \ldots, k. \tag{21}$$

If $\Delta_k = \Delta_{k-1}$, the algorithm stops.

Lemma 7. *The following statements are equivalent:*
(i) the elements of I_k are the invariants of Δ_k, i.e. (20) holds,
(ii) the elements of I_k have the relative degree at least $k+1$, i.e. (21) holds.

Proof. Show first (20) ⇒ (21). Recall again that although we use in the proof the forward shifts $I_k^{\langle 1 \rangle}$ of I_k, we actually do not compute them with the help of equations (1), which we do not know. We only use the fact that $I_k^{\langle 1 \rangle}$ exists.

According to (18) and (20), all the elements of I_k are the invariants of $\operatorname{Ker} T\phi_{k-1}$. Then, due to (16) and (18), $\mathrm{d}I_k \in \operatorname{span}_{\mathscr{K}}\{\mathrm{d}I_{k-1}^{\langle -1 \rangle}\}$. Shifting the last relation forward yields $\mathrm{d}I_k^{\langle 1 \rangle} \in \operatorname{span}_{\mathscr{K}}\{\mathrm{d}I_{k-1}\}$. Because the relative degree of I_{k-1} is at least k due to its definition, also $I_k^{\langle 1 \rangle}$ has the relative degree at least k. That is, the relative degree of I_k is really at least $k+1$ and (21) holds.

Next show (21) ⇒ (20). If the relative degree of I_k is at least $k+1$, then its forward shift $I_k^{\langle 1 \rangle}$ has the relative degree at least k and, according to the definition of I_{k-1}, $\mathrm{d}I_k^{\langle 1 \rangle} \in \operatorname{span}_{\mathscr{K}}\{\mathrm{d}I_{k-1}\}$. Shifting this formula back gives $\mathrm{d}I_k \in \operatorname{span}_{\mathscr{K}}\{\mathrm{d}I_{k-1}^{\langle -1 \rangle}\}$. From (16) and (18) it follows then $\langle \mathrm{d}I_k, \Xi_{k-1} \rangle \equiv 0$ for all $\Xi_{k-1} \in \operatorname{Ker} T\phi_{k-1}$ and, because I_k depends only on x, also $\langle \mathrm{d}I_k, \partial/\partial z^{\langle -1 \rangle} \rangle \equiv 0$. This means that (20) really holds. □

4.2. Linearizability conditions

In this subsection we will reformulate the necessary and sufficient linearizability conditions in terms of distributions Δ_k. We first examine the relationship between the distributions Δ_k as in (19) and $\mathscr{D}_k$ as in (10). Denote by $\bar{\mathscr{D}}_k$ and $\bar{\Delta}_k$ the involutive closures of $\mathscr{D}_k$ and Δ_k, respectively.

Lemma 8. $\bar{\Delta}_k = \bar{\mathscr{D}}_k$, $k = 1, ..., k^\star - 1$.

Proof. According to Theorem 6, the set of functions I_k is a complete set of independent invariants of $\mathscr{D}_k$. Due to Lemma 7, I_k is also a complete set of independent invariants of Δ_k. As shown in [5], the invariants of a distribution are also the invariants of its involutive closure and vice versa. Consequently, $\bar{\Delta}_k$ and $\bar{\mathscr{D}}_k$ have the same set of independent invariants, i.e. they are both the annihilators of an integrable codistribution $\mathrm{span}_{\mathscr{K}}\{\mathrm{d}I_k\}$ and, therefore, are equal. □

Theorem 9. *System (2) is static state feedback linearizable if and only if*
(i) all Δ_k, $k > 0$, *are involutive and*
(ii) there exists an index $k^\star$ *such that* $\dim_{\mathscr{K}} \Delta_{k^\star} = n + m$.

Proof. Follows directly fromTheorem 5 and Lemma 8. □

4.3. State transformation

Suppose that the system, described in terms of backward shift equations (2), is static state feedback linearizable, i.e. the conditions of Theorem 9 hold. Then with the help of Algorithm 2 the linear chains of backward shifts can be constructed, analogous to the Brunovsky chains (9).

Algorithm 2. Finding the state transformation.

Initialization. Consider the distributions Δ_k, $k = 1, ..., k^\star$, computed by Algorithm 1, whereby $\dim_{\mathscr{K}} \Delta_{k^\star} = n + m$. Then $\Delta_{k^\star - 1}$ is the largest distribution, which has non-zero invariants. According to Lemma 7, the relative degree of these invariants is $k^\star$, being the highest r_i in (9).

Step 1. Find (a) the variables $X_{i_1,1}$ in (9), having relative degree $k^\star$, and (b) their forward shifts $X_{i_1,1}^{\langle 1 \rangle} = X_{i_1,2}$.

(a) Using (20), compute the independent invariants $I_{k^\star-1} = \varphi_{k^\star-1}(x)$ of $\Delta_{k^\star-1}$. Their number is $n_{k^\star-1}$, all with relative degree $k^\star$. Take

$$X_{i_1,1} := I_{k^\star-1,i_1} = \varphi_{k^\star-1,i_1}(x), \quad i_1 = 1, ..., n_{k^\star-1}. \tag{22}$$

(b) Since the forward shift equations are unknown, one cannot compute the shifts directly, but has to use an indirect method. The relative degree of $X_{i_1,2}$ is obviously $k^\star - 1$. Therefore one can express them in terms of invariants $I_{k^\star-2} = \varphi_{k^\star-2}(x)$ of $\Delta_{k^\star-2}$, which can be computed via (20) while performing Algorithm 1. So, there exist functions $\psi_{k^\star-1,i_1}$ such that $X_{i_1,2} = \psi_{k^\star-1,i_1}(I_{k^\star-2})$ and one has to find these functions. For this purpose shift the last relation back by one step:

$$X_{i_1,1} = \psi_{k^\star-1,i_1}(I_{k^\star-2}^{\langle -1 \rangle}), \tag{23}$$

and compute the backward shifts of $I_{k^\star-2} = \varphi_{k^\star-2}(x)$ using (2):

$$I_{k^\star-2}^{\langle -1 \rangle} = \varphi_{k^\star-2}(\Lambda(x, z^{\langle -1 \rangle})). \tag{24}$$

Now, in order to express $X_{i_1,1}$ in terms of $I_{k^\star-2}^{\langle -1 \rangle}$, eliminate from the system of equations (22) and (24) the variables $z^{\langle -1 \rangle}$ and x. This yields $n_{k^\star-1}$ implicit functions

$$F_1(X_{i_1,1}, I_{k^\star-2}^{\langle -1 \rangle}) = 0. \tag{25}$$

The solution of (25) with respect to $X_{i_1,1}$ gives functions in (23) in the explicit form. Observe that solving (25) is, in general, much simpler than solving equations (2) to obtain forward shift equations (1). In the first case one has $n_{k^\star-1}$ equations, in the second case $n+m$ equations. Next, shift (23) forward by one step to get $X_{i_1,2} = \psi_{k^\star-1,i_1}(I_{k^\star-2})$. To express $X_{i_1,2}$ in terms of x, substitute $I_{k^\star-2} = \varphi_{k^\star-2}(x)$ into the last formula

$$X_{i_1,2} = \psi_{k^\star-1,i_1}(\varphi_{k^\star-2}(x)). \tag{26}$$

Check if the number of coordinates obtained at this step equals n. If yes, then stop the algorithm, if no, go to the next step.

Step 2.[2] Find (a) the variables $X_{i_2,1}$ having relative degrees $k^\star-1$ (if they exist), and (b) $X_{i_1,2}^{\langle 1\rangle} = X_{i_1,3}$, $X_{i_2,1}^{\langle 1\rangle} = X_{i_2,2}$.

(a) Using (20), one can compute the independent invariants $I_{k^\star-2} = \varphi_{k^\star-2}(x)$ of $\Delta_{k^\star-2}$. Their number is $n_{k^\star-2}$, the relative degrees are at least $k^\star-1$. Because the set of invariants is not uniquely defined, one can express them as follows: first $2n_{k^\star-1}$ invariants $X_{i_1,1}$ from (22) and $X_{i_1,2}$ from (26) and, if $2n_{k^\star-1} < n_{k^\star-2}$, then add $n_{k^\star-2} - 2n_{k^\star-1}$ additional independent invariants

$$X_{i_2,1} = I_{k^\star-2,i_2} = \varphi_{k^\star-2,i_2}(x), \tag{27}$$

where $i_2 = n_{k^\star-1}+1,\ldots,n_{k^\star-2}$. That is, the independent invariants of $\Delta_{k^\star-2}$ are

$$I_{k^\star-2} = \{X_{i_1,1}, X_{i_1,2}, X_{i_2,1}\}. \tag{28}$$

(b) Because the relative degree of $X_{i_1,3}$ as well as of $X_{i_2,2}$ is $k^\star-2$, one can express them in terms of the invariants $I_{k^\star-3} = \varphi_{k^\star-3}(x)$ of $\Delta_{k^\star-3}$. That is, there exist the functions $\psi_{k^\star-2,i_1}$ and $\psi_{k^\star-2,i_2}$ such that $X_{i_1,3} = \psi_{k^\star-2,i_1}(I_{k^\star-3})$ and $X_{i_2,2} = \psi_{k^\star-2,i_2}(I_{k^\star-3})$. In order to compute $X_{i_1,3}$ and $X_{i_2,2}$, one has to find these functions. For this purpose shift the last relations back by one step:

$$X_{i_1,2} = \psi_{k^\star-2,i_1}(I_{k^\star-3}^{\langle -1\rangle}), \quad X_{i_2,1} = \psi_{k^\star-2,i_2}(I_{k^\star-3}^{\langle -1\rangle}), \tag{29}$$

and compute the backward shifts of $I_{k^\star-3} = \varphi_{k^\star-3}(x)$ using (2):

$$I_{k^\star-3}^{\langle 1\rangle} = \varphi_{k^\star-3}(\Lambda(x, z^{\langle -1\rangle})). \tag{30}$$

Now, in order to express $X_{i_1,2}$ and $X_{i_2,1}$ in terms of $I_{k^\star-3}^{\langle 1\rangle}$, eliminate from the system of equations (26), (27), and (30) the variables $z^{\langle -1\rangle}$ and x, resulting in $n_{k^\star-1}+n_{k^\star-2}$ implicit functions

$$F_2(X_{i_1,2}, X_{i_2,1}, I_{k^\star-3}^{\langle -1\rangle}) = 0. \tag{31}$$

After solving (31) with respect to $X_{i_1,2}$ and $X_{i_2,1}$ we get the functions (29) in the explicit form. Next shift (29) forward by one step to get $X_{i_1,3} = \psi_{k^\star-2,i_1}(I_{k^\star-3})$ and $X_{i_2,2} = \psi_{k^\star-2,i_2}(I_{k^\star-3})$. To express $X_{i_1,3}$ and $X_{i_2,2}$ in terms of x, substitute $I_{k^\star-3} = \varphi_{k^\star-3}(x)$ into the last formulae:

$$X_{i_1,3} = \psi_{k^\star-2,i_1}(\varphi_{k^\star-3}(x)), \quad X_{i_2,2} = \psi_{k^\star-2,i_2}(\varphi_{k^\star-3}(x)). \tag{32}$$

Check if the number of coordinates obtained at this step equals n. If yes, then stop the algorithm, if no, go to the next step.

Step k. Find (a) the variables $X_{i_k,1}$ in (9) with relative degree $k^\star-k+1$ (if they exist), and (b) $X_{i_1,k}^{\langle 1\rangle} = X_{i_1,k+1}$, $X_{i_2,k-1}^{\langle 1\rangle} = X_{i_2,k}$, ..., $X_{i_k,1}^{\langle 1\rangle} = X_{i_k,2}$.

[2] Although Step 2 is a special case of Step k for $k=2$, we decided to add it for readability and better understanding of Example.

(a) Using (20), one computes the independent invariants $I_{k^\star-k} = \varphi_{k^\star-k}(x)$ of $\Delta_{k^\star-k}$ as follows.

• $X_{i_1,1}$ and their forward shifts up to order $k-1$, i.e. altogether kn_{k-1} invariants:

$$X_{i_1,1} = \varphi_{k^\star-1,i_1}(x), \quad X_{i_1,2} = \psi_{k^\star-1,i_1}(\varphi_{k^\star-2}(x)), \ldots, \quad X_{i_1,k} = \psi_{k^\star-k+1,i_1}(\varphi_{k^\star-k}(x)); \tag{33}$$

• $X_{i_2,1}$ and their forward shifts up to order $k-2$, i.e. altogether $(k-1)n_{k^\star-2}$ invariants:

$$X_{i_2,1} = \varphi_{k^\star-2,i_2}(x), \quad X_{i_2,2} = \psi_{k^\star-2,i_2}(\varphi_{k^\star-3}(x)), \ldots, \quad X_{i_2,k-1} = \psi_{k^\star-k+2,i_2}(\varphi_{k^\star-k}(x)); \tag{34}$$

$\vdots$

• $X_{i_{k-1},1}$ and their first-order forward shifts, i.e. altogether $2n_{k^\star-k+1}$ invariants:

$$X_{i_{k-1},1} = \varphi_{k^\star-k+1,i_{k-1}}(x), \quad X_{i_{k-1},2} = \psi_{k^\star-k+1,i_{k-1}}(\varphi_{k^\star-k}(x)). \tag{35}$$

If the number of these invariants is smaller than the total number $n_{k^\star-k}$ of independent invariants of $\Delta_{k^\star-k}$, then add $n_{k^\star-k} - 2n_{k^\star-k+1} - \ldots - (k-1)n_{k^\star-2} - kn_{k-1}$ invariants

$$X_{i_k,1} = I_{k^\star-k,i_k} = \varphi_{k^\star-k,i_k}(x) \tag{36}$$

in order to get a complete set of independent invariants of $\Delta_{k^\star-k}$:

$$I_{k^\star-k} = \{X_{i_1,1}, \ldots, X_{i_1,k}, X_{i_2,1}, \ldots, X_{i_2,k-1}, \ldots, X_{i_k,1}\}. \tag{37}$$

(b) Because the relative degree of $X_{i_1,k+1},\ldots,X_{i_k,2}$ is $k^\star - k$, one can express them in terms of invariants $I_{k^\star-k-1} = \varphi_{k^\star-k-1}(x)$ of $\Delta_{k^\star-k-1}$. That is, there exist the functions $\psi_{k^\star-k,i_1},\ldots,\psi_{k^\star-k,i_k}$ such that $X_{i_1,k+1} = \psi_{k^\star-k,i_1}(I_{k^\star-k-1}),\ldots,X_{i_k,2} = \psi_{k^\star-k,i_k}(I_{k^\star-k-1})$. In order to compute $X_{i_1,k+1},\ldots,X_{i_k,2}$, one has to find these functions. For this purpose shift the last relations back by one step:

$$X_{i_1,k} = \psi_{k^\star-k,i_1}(I^{\langle -1\rangle}_{k^\star-k-1}), \ldots, \quad X_{i_k,1} = \psi_{k^\star-k,i_k}(I^{\langle -1\rangle}_{k^\star-k-1}), \tag{38}$$

and compute the backward shifts of $I_{k^\star-k-1} = \varphi_{k^\star-k-1}(x)$ using (2):

$$I^{\langle -1\rangle}_{k^\star-k-1} = \varphi_{k^\star-k-1}(\Lambda(x, z^{\langle -1\rangle})). \tag{39}$$

Now, in order to express $X_{i_1,k+1},\ldots,X_{i_k,2}$ in terms of $I^{\langle -1\rangle}_{k^\star-k-1}$, eliminate from the system of equations (33)–(36), and (39) the variables x and $z^{\langle -1\rangle}$, resulting in $n_{k^\star-1} + \ldots + n_{k^\star-k}$ independent implicit functions

$$F_k(X_{i_1,k+1}, \ldots, X_{i_k,2}, I^{\langle -1\rangle}_{k^\star-k-1}) = 0. \tag{40}$$

After solving (40) with respect to $X_{i_1,k+1},\ldots,X_{i_k,2}$ we get the functions in (38) in the explicit form. Next shift the relations (38) forward by one step, writing $X_{i_1,k+1} = \psi_{k^\star-k,i_1}(I_{k^\star-k-1}),\ldots,X_{i_k,2} = \psi_{k^\star-k,i_k}(I_{k^\star-k-1})$. To express $X_{i_1,k+1},\ldots,X_{i_k,2}$ in terms of x, substitute $I_{k^\star-k-1} = \varphi_{k^\star-k-1}(x)$ into the last relations:

$$\begin{aligned} X_{i_1,k+1} &= \psi_{k^\star-k,i_1}(\varphi_{k^\star-k-1}(x)), \\ &\vdots \\ X_{i_k,2} &= \psi_{k^\star-k,i_k}(\varphi_{k^\star-k-1}(x)). \end{aligned} \tag{41}$$

The algorithm stops when $\dim_{\mathscr{K}} X = n$. As a result we obtain the coordinate transformation

$$X = \Psi(x). \tag{42}$$

4.4. Static state feedback

In this subsection we will show how to find the static state feedback of the form (8), which together with the state transformation (42) allows us to represent the state equations in the Brunovsky form (9). Find the inverse of (42)

$$x = \Psi^{-1}(X). \tag{43}$$

Shift the relation (42) back according to the system dynamics (14):

$$X^{\langle -1\rangle} = \Psi(\Lambda(x, u^{\langle -1\rangle})) \tag{44}$$

and replace in (44) the variables x via X using (43). This results in the backward shift equations in the new coordinates X:

$$X^{\langle -1\rangle} = \overline{\Psi}(X, u^{\langle -1\rangle}) := \Psi(\Lambda(\Psi^{-1}(X), u^{\langle -1\rangle})), \tag{45}$$

which have, due to the definition of X, the following structure:

$$X_{i,1}^{\langle -1\rangle} = \overline{\Psi}_{i,1}(X, u^{\langle -1\rangle}), \quad i = 1, \ldots, m, \quad X_{i,l}^{\langle -1\rangle} = X_{i,l-1}, \quad l = 2, \ldots, r_i. \tag{46}$$

Shift (46) "formally" forward, obtaining

$$X_{i,1} = \overline{\Psi}_{i,1}(X^{\langle 1\rangle}, u), \quad i = 1, \ldots, m, \tag{47}$$

$$X_{i,l} = X_{i,l-1}^{\langle 1\rangle}, \qquad l = 2, \ldots, r_i. \tag{48}$$

Observe that the last $r_i - 1$ equations of each chain above can be simply rewritten in the form

$$X_{i,l}^{\langle 1\rangle} = X_{i,l+1}, \quad i = 1, \ldots, m, \quad l = 1, \ldots, r_i - 1 \tag{49}$$

by changing the order of equations (47) and (48). In order to get the last equations $X_{i,r_i}^{\langle 1\rangle} = v_i$ of each chain in (9), one has to apply the feedback. To find the feedback, we replace in (47), according to (49), all $X_{i,l}^{\langle 1\rangle}$, $l = 1, \ldots, r_i - 1$, by $X_{i,l+1}$, and all $X_{i,r_i}^{\langle 1\rangle}$ by v_i. Solving the system of algebraic equations obtained that way with respect to u results in the feedback.

4.5. The independence of coordinate transformation and feedback on the choice of z

In this subsection we will prove that the coordinate transformation $X = \Psi(x)$ and feedback (8) do not depend on the choice of z. The proof consists in two parts.

Lemma 10. *The coordinates $X_{i,1} = \Psi_{i,1}(x)$, $i = 1, \ldots, m$, do not depend on the choice of z.*

Proof. Using Algorithms 1 and 2, one finds $X_{i,1}$, $i = 1, \ldots, m$, as the appropriately chosen invariants of (involutive) distributions Δ_k, where $k+1$ is the relative degree of respective $X_{i,1}$. We will show below that if Δ_k is involutive, then its invariants I_k do not depend on the choice of z.

Suppose that the invariants I_{k-1} do not depend on the choice of z (for $k = 1$ this is true, because $I_0 = x$), and show that if Δ_k is involutive, then also I_k does not depend on the choice of z. According to the definition of the projection of a vector field (6), one can rewrite the distribution Δ_k in (19) as

$$\Delta_k = \operatorname{span}_{\mathscr{K}} \left\{ \Xi_{k-1}^{\pi}, \frac{\partial}{\partial z^{\langle -1\rangle}} \right\}, \tag{50}$$

where Ξ_{k-1} are the basis vector fields of $\mathrm{Ker}\, T\phi_{k-1}$, see (18). Therefore,

$$\Delta_k^\pi = \mathrm{span}_{\mathscr{K}}\{\Xi_{k-1}^\pi\}. \tag{51}$$

If Δ_k is involutive (being necessary for the existence of the coordinate transformation $X = \Psi(x)$), due to Definition 4 and (50) the following conditions must be satisfied: 1) $[\Xi_{k-1,1}^\pi, \Xi_{k-1,2}^\pi] \in \Delta_k$, 2) $[\partial/\partial z^{\langle -1\rangle}, \Xi_{k-1,2}^\pi] \in \Delta_k$ for all $\Xi_{k-1,1}^\pi, \Xi_{k-1,2}^\pi$ and $\partial/\partial z^{\langle -1\rangle}$. As by definition $\Xi_{k-1,1}^\pi, \Xi_{k-1,2}^\pi \in \mathrm{span}_{\mathscr{K}}\{\partial/\partial x\}$, it follows from 1) and 2) that 3) $[\Xi_{k-1,1}^\pi, \Xi_{k-1,2}^\pi] \in \mathrm{span}_{\mathscr{K}}\{\partial/\partial x\}$, 4) $[\partial/\partial z^{\langle -1\rangle}, \Xi_{k-1,2}^\pi] \in \mathrm{span}_{\mathscr{K}}\{\partial/\partial x\}$. Due to 1) and 3), $[\Xi_{k-1,1}^\pi, \Xi_{k-1,2}^\pi] \in \Delta_k^\pi$. Consequently, if Δ_k is involutive, then Δ_k^π is also involutive and I_k can be defined as the invariants of an involutive distribution Δ_k^π. This means that they can be defined as a complete set of independent functions $I_k = \varphi_k(x)$ such that

$$\langle \mathrm{d}\varphi_k, \tilde{\Xi}\rangle \equiv 0, \quad \forall \tilde{\Xi} \in \Delta_k^\pi. \tag{52}$$

On the other hand, from 2) and 4) it follows that $[\partial/\partial z^{\langle -1\rangle}, \Xi_{k-1}^\pi] \in \Delta_k^\pi$ for $\partial/\partial z^{\langle -1\rangle}$ and Ξ_{k-1}^π, i.e., the Lie derivative with respect to $\partial/\partial z^{\langle -1\rangle}$ does not affect Δ_k^π. Therefore one can define the basis $\Delta_k^\pi = \mathrm{span}_{\mathscr{K}}\{\tilde{\Xi}\}$ such that the components of $\tilde{\Xi}$ do not depend on $z^{\langle -1\rangle}$. Because the single criterion for the choice of I_k is (52), I_k (and also $X_{i,1}$) do not depend on the choice of z. □

Corollary 11. *The coordinates $X_{i,2},\ldots,X_{i,r_i}$ and the feedback (8) do not depend on the choice of $z^{\langle -1\rangle}$.*

Proof. Take into account that $X_{i,2},\ldots,X_{i,r_i}$ and v_i are obtained via forward shifting of $X_{i,1}$. According to Lemma 10, the choice of $X_{i,1}$ does not depend on the choice of z and also the forward shift equations $x^{\langle 1\rangle} = \bar{\Phi}(x,u)$ do not contain z; then also $X_{i,2},\ldots,X_{i,r_i}$ and v_i do not depend on the choice of z. Then the feedback (8) does not depend on z either. □

5. EXAMPLE

Consider the state equations of a hydraulic press with the vertical cylinder, described in [6] as

$$\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= (S(x_3 - x_4) - Mg - \mu x_2)/M,\\
\dot{x}_3 &= \frac{\beta(u_1 - x_2)}{l_0 + x_1},\\
\dot{x}_4 &= \frac{\beta(x_2 - u_2)}{l - l_0 - x_1},
\end{aligned} \tag{53}$$

where x_1 and x_2 are, respectively, the position and velocity of the piston, and x_3 and x_4 are the pressures under and above the piston, respectively. The system constants have the following meaning: M is the mass of the piston, S is the effective piston area, μ is the damping coefficient, l_0 is the height of the chamber under the piston, l is the total length of the cylinder, and β is the isothermal bulk modulus of the oil. The inputs u_i, $i = 1, 2$, are defined as

$$u_i = \begin{cases} \frac{K_1}{S}\sqrt{p_s - x_{i+2}}|U_i| & \text{if} \quad U_i \geq 0 \\ \frac{K_2}{S}\sqrt{x_{i+2} - p_t}|U_i| & \text{if} \quad U_i < 0 \end{cases},$$

where $K_1, K_2 \in \mathbb{R}^+$, p_s and p_t are the supply and tank pressures, respectively, and U_1 and U_2 are the positions of valves at the supply and the tank, respectively. The implicit Euler discretization of (53) has the form

$$\begin{aligned} x_1^{\langle 1\rangle} &= x_1 + x_2^{\langle 1\rangle} T, \\ x_2^{\langle 1\rangle} &= x_2 + \frac{(S(x_3^{\langle 1\rangle} - x_4^{\langle 1\rangle}) - Mg - \mu x_2^{\langle 1\rangle})T}{M}, \\ x_3^{\langle 1\rangle} &= x_3 + \frac{\beta(u_1 - x_2^{\langle 1\rangle})T}{l_0 + x_1^{\langle 1\rangle}}, \\ x_4^{\langle 1\rangle} &= x_4 + \frac{\beta(x_2^{\langle 1\rangle} - u_2)T}{l - l_0 - x_1^{\langle 1\rangle}}, \end{aligned} \tag{54}$$

where T is the sampling time. To find the system description (1) requires solving equations (54) with respect to $x^{\langle 1\rangle}$. This leads to extremely complicated equations [6]. However, one can easily convert equations (54) into the system description (2):

$$\begin{aligned} x_1^{\langle -1\rangle} &= x_1 - x_2 T, \\ x_2^{\langle -1\rangle} &= x_2 - (S(x_3 - x_4) - Mg - \mu x_2)T/M, \\ x_3^{\langle -1\rangle} &= x_3 - \frac{\beta(z^{\langle -1\rangle}{}_1 - x_2)T}{l_0 + x_1}, \\ x_4^{\langle -1\rangle} &= x_4 - \frac{\beta(x_2 - z^{\langle -1\rangle}{}_2)T}{l - l_0 - x_1}, \end{aligned} \tag{55}$$

taking $z_1 = u_1$, $z_2 = u_2$. Equations (55) define the map $\phi_0 : \mathbf{X} \times \mathbf{Z} \to \mathbf{X}$, whose Jacobi matrix reads

$$T\phi_0 = \begin{pmatrix} 1 & -T & 0 & 0 & 0 & 0 \\ 0 & 1 + \dfrac{\mu T}{M} & -\dfrac{ST}{M} & \dfrac{ST}{M} & 0 & 0 \\ \dfrac{(-x_2 + z_1^{\langle -1\rangle})\beta T}{(l_0 + x_1)^2} & -\dfrac{\beta T}{l_0 + x_1} & 1 & 0 & \dfrac{\beta T}{l_0 + x_1} & 0 \\ \dfrac{(x_2 - z_2^{\langle -1\rangle})\beta T}{(l_0 - l - x_1)^2} & -\dfrac{\beta T}{l_0 - l - x_1} & 0 & 1 & 0 & \dfrac{\beta T}{l_0 - l - x_1} \end{pmatrix}. \tag{56}$$

In order to check the linearizability property of the discrete-time model (55), find the distributions Δ_k, $k = 1, \ldots, k^\star$, with the help of Algorithm 1.

Step 0. Take $I_0 = x$, $n_0 = 4$; the map ϕ_0 is given by equations (55).

Step 1. Find Δ_1 as described in (18) and (19). After simplification one gets

$$\Delta_1 = \operatorname{span}_{\mathscr{K}} \left\{ T\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} + \frac{M - \mu T}{ST}\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}, \frac{\partial}{\partial z_1^{\langle -1\rangle}}, \frac{\partial}{\partial z_2^{\langle -1\rangle}} \right\},$$

which is an involutive distribution. Because $\dim_{\mathscr{K}} \Delta_1 = 4 = n + m - 2$, the number of its independent invariants is 2. One can easily check that these invariants are

$$I_{1,1} = (x_3 - x_4)ST - x_2(M - \mu T), \quad I_{1,2} = (x_1 - x_2 T)ST. \tag{57}$$

According to Lemma 7, their relative degree is at least 2.

Step 2. Find Δ_2. Shift the invariants (57) back as in (16), to obtain the map ϕ_1:

$$
\begin{aligned}
I_{1,1}^{\langle -1\rangle} = \phi_{1,1} &:= -(\mu T - M)\left(x_2 - \frac{((x_3-x_4)S - Mg - \mu x_2)\,T}{M}\right) + \left(x_3 + \frac{(x_2 - z_1^{\langle -1\rangle})\beta T}{l_0+x_1}\right)ST \\
&\quad - \left(x_4 - \frac{(x_2 - z_2^{\langle -1\rangle})\beta T}{l-l_0-x_1}\right)ST, \\
I_{1,2}^{\langle -1\rangle} = \phi_{1,2} &:= (x_1 - x_2T)ST - \left(x_2 - \frac{((x_3-x_4)S - Mg - \mu x_2)\,T}{M}\right)ST^2.
\end{aligned}
\tag{58}
$$

The Jacobi matrix of ϕ_1 reads

$$
T\phi_1 = \begin{pmatrix}
-\frac{\left(x_2 - z_1^{\langle -1\rangle}\right)\beta ST^2}{(l_0+x_1)^2} - \frac{\left(z_2^{\langle -1\rangle} - x_2\right)\beta ST^2}{(l-l_0-x_1)^2} & \frac{\beta ST^2}{l_0+x_1} + \frac{\beta ST^2}{l-l_0-x_1} - (\mu T + M)\left(1 + \frac{\mu T}{M}\right) & \frac{(\mu T+2M)ST}{M} & -\frac{(\mu T+2M)ST}{M} & -\frac{\beta ST^2}{l_0+x_1} & -\frac{\beta ST^2}{l-l_0-x_1} \\
ST & -\left(2+\frac{\mu T}{M}\right)ST^2 & \frac{S^2T^3}{M} & -\frac{S^2T^3}{M} & 0 & 0
\end{pmatrix}.
$$

Compute Δ_2, using (18) and (19):

$$
\Delta_2 = \operatorname{span}_{\mathscr{K}}\left\{\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}, ST\frac{\partial}{\partial x_2} + (\mu T + 2M)\frac{\partial}{\partial x_3}, ST^2\frac{\partial}{\partial x_1} - M\frac{\partial}{\partial x_3}, \frac{\partial}{\partial z_1^{\langle -1\rangle}}, \frac{\partial}{\partial z_2^{\langle -1\rangle}}\right\}, \tag{59}
$$

which is again involutive. Note that Δ_2 has one independent invariant

$$
I_{2,1} = x_1 - 2x_2T + \frac{((x_3-x_4)S - Mg - \mu x_2)T^2}{M}
$$

as the single independent function with relative degree 3.

Step 3. Finally, one gets

$$
\Delta_3 = \operatorname{span}_{\mathscr{K}}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial z^{\langle -1\rangle}}\right\}.
$$

Because $\dim_{\mathscr{K}} \Delta_3 = 6 = n + m$, the algorithm stops here. Due to involutivity of Δ_1, Δ_2, and Δ_3, system (31) is, according to Theorem 9, static state feedback linearizable.

Find next, with the help of Algorithm 2, the state transformation $X = \Psi(x)$. We start with Δ_2. Note that Δ_2 is the largest distribution that has non-zero invariants. Therefore, the relative degree of its single invariant $I_{2,1}$ is equal to 3 (and not at least 3). So, $i_2 = 1$. Recall that $n_2 = 1$ is the number of independent invariants of Δ_2, and $i_2 = 1$ is the number of the Brunovsky chains, whose first elements are the invariants of Δ_2.

Step 1. (a) Define, according to (22),

$$
X_{1,1} = I_{2,1} = x_1 - 2x_2T + \frac{((x_3-x_4)S - Mg - \mu x_2)T^2}{M}. \tag{60}
$$

(b) Compute, by (26), $X_{1,2} = I_{2,1}^{\langle 1\rangle}$. Since the relative degree of $X_{1,2}$ is 2, one can express it in terms of the invariants I_1 only. So there must exist a function $\psi_{2,1}$ such that $X_{1,1} = \psi_{2,1}(I_1^{\langle -1\rangle})$, see (23). To find this function, combine a system of equations from (58) and (60) and eliminate from these $z^{\langle -1\rangle}$ and x to get $X_{1,1} = I_{1,2}^{\langle -1\rangle}/(ST)$. Shifting this relation forward gives $X_{1,2} = I_{1,2}/(ST)$. Taking $I_{1,2}$ from (57), we obtain, according to (26),

$$X_{1,2} = I_{2,1}^{\langle 1\rangle} = x_1 - x_2 T.$$

Step 2. (a) Because Δ_1 is involutive, the number of its independent invariants is $n + m - \dim_{\mathscr{K}} \Delta_2 = 2$. In step 1 we obtained already two independent invariants $X_{1,1}$ and $X_{1,2}$ and therefore, $n_1 = 0$.

(b) Find the forward shift $X_{1,3}$ of $X_{1,2}$. From (55) one can easily see that $X_{1,2} = x_1^{\langle -1\rangle}$ and therefore $X_{1,3} = x_1$.

Step 3. At this step we define a complete set of functions with relative degree at least 1, denoted by I_0. In previous steps we got already 3 invariants, $X_{1,1}$, $X_{1,2}$, and $X_{1,3}$. In order to complete the set, define the 4th one, for instance one can take $X_{2,1} = I_{0,4} = x_4$. Define the state transformation (where $X_{1,1}$, $X_{1,2}$, and $X_{1,3}$ mean the positions of the piston, and $X_{2,1}$ means the pressure above the piston):

$$\begin{aligned} X_{1,1} &= x_1 - 2x_2T - gT^2 + \frac{((x_3 - x_4)S - \mu x_2)T^2}{M}, \\ X_{1,2} &= x_1 - x_2T, \quad X_{1,3} = x_1, \quad X_{2,1} = x_4, \end{aligned} \tag{61}$$

and find its inverse

$$\begin{aligned} x_1 &= X_{1,3}, \quad x_2 = \frac{X_{1,3} - X_{1,2}}{T}, \\ x_3 &= \frac{1}{ST^2}\left[M(X_{1,3} - 2X_{1,2} + X_{1,1}) + \mu T(X_{1,3} - X_{1,2}) + (X_{2,1}S + Mg)T^2\right], \\ x_4 &= X_{2,1}. \end{aligned} \tag{62}$$

Shifting (61) back with the help of (55) and substituting in the obtained equations x by X using (62), we get the backward shift equations (46) in the new coordinates:

$$\begin{aligned} X_{1,1}^{\langle -1\rangle} &= \frac{u_2^{\langle -1\rangle}\beta ST^3 + (X_{1,2} - X_{1,3})\beta ST^2}{M(l_0 - l - X_{1,3})} - \frac{u_1^{\langle -1\rangle}\beta ST^3 + (X_{1,2} - X_{1,3})\beta ST^2}{M(l_0 + X_{1,3})} \\ &\quad + \frac{\mu T}{M}(X_{1,3} - 2X_{1,2} + X_{1,1}) + (X_{1,3} - 3X_{1,2} + 3X_{1,1}), \\ X_{1,2}^{\langle -1\rangle} &= X_{1,1}, \quad X_{1,3}^{\langle -1\rangle} = X_{1,2}, \\ X_{2,1}^{\langle -1\rangle} &= X_{2,1} + \frac{(X_{1,3} - X_{1,2})\beta - u_2^{\langle -1\rangle}\beta T}{l_0 - l - X_{1,3}}. \end{aligned} \tag{63}$$

To convert the above equations into the Brunovsky form

$$X_{1,1}^{\langle 1\rangle} = X_{1,2}, \quad X_{1,2}^{\langle 1\rangle} = X_{1,3}, \quad X_{1,3}^{\langle 1\rangle} = v_1, \quad X_{2,1}^{\langle 1\rangle} = v_2, \tag{64}$$

shift first equations (63) "formally" forward, substituting $X^{\langle -1\rangle}$, X, and $u^{\langle -1\rangle}$ simply by X, $X^{\langle 1\rangle}$, and u, respectively, to get

$$
\begin{aligned}
X_{1,1} &= \frac{u_2\beta ST^3 + \left(X_{1,2}^{\langle 1\rangle} - X_{1,3}^{\langle 1\rangle}\right)\beta ST^2}{M\left(l_0 - l - X_{1,3}^{\langle 1\rangle}\right)} - \frac{u_1\beta ST^3 + \left(X_{1,2}^{\langle 1\rangle} - X_{1,3}^{\langle 1\rangle}\right)\beta ST^2}{M\left(l_0 + X_{1,3}^{\langle 1\rangle}\right)} \\
&\quad + \frac{\mu T}{M}\left(X_{1,3}^{\langle 1\rangle} - 2X_{1,2}^{\langle 1\rangle} + X_{1,1}^{\langle 1\rangle}\right) + \left(X_{1,3}^{\langle 1\rangle} - 3X_{1,2}^{\langle 1\rangle} + 3X_{1,1}^{\langle 1\rangle}\right), \\
X_{1,2} &= X_{1,1}^{\langle 1\rangle}, \quad X_{1,3} = X_{1,2}^{\langle 1\rangle}, \\
X_{2,1} &= X_{2,1}^{\langle 1\rangle} + \frac{\left(X_{1,3}^{\langle 1\rangle} - X_{1,2}^{\langle 1\rangle}\right)\beta - u_2\beta T}{l_0 - l - X_{1,3}^{\langle 1\rangle}}.
\end{aligned} \tag{65}
$$

Observe that the second and the third equations of (65) give already the first and the second equations in (64), but to convert the first and the fourth equations of (65) into the Brunovsky form, one needs to use the feedback of the form (8). To find the feedback, (1) replace $X_{1,1}^{\langle 1\rangle}$ by $X_{1,2}$ and $X_{1,2}^{\langle 1\rangle}$ by $X_{1,3}$ in the first and the fourth equations of (65), and (2) because our aim is to get the third and fourth equations of (64), replace $X_{1,3}^{\langle 1\rangle}$ by v_1, and $X_{2,1}^{\langle 1\rangle}$ by v_2:

$$
\begin{aligned}
X_{1,1} &= \frac{u_2\beta ST^3 + (X_{1,3} - v_1)\beta ST^2}{M(l_0 - l - v_1)} - \frac{u_1\beta ST^3 + (X_{1,3} - v_1)\beta ST^2}{M(l_0 + v_1)} \\
&\quad + \frac{\mu T}{M}(v_1 - 2X_{1,3} + X_{1,2}) + (v_1 - 3X_{1,3} + 3X_{1,2}), \\
X_{2,1} &= v_2 + \frac{(v_1 - X_{1,3})\beta - u_2\beta T}{l_0 - l - v_1}.
\end{aligned}
$$

Solving the above equations with respect to u gives the static state feedback

$$
\begin{aligned}
u_1 &= \frac{1}{\beta T}\left[(v_2 - X_{2,1})(l_0 + v_1) + (v_1 - X_{1,3})\beta\right] + \frac{\mu}{\beta ST^2}(v_1 - 2X_{1,3} + X_{1,2})(l_0 + v_1) \\
&\quad + \frac{M(l_0 + v_1)}{\beta ST^3}(v_1 - 3X_{1,3} + 3X_{1,2} - X_{1,1}), \\
u_2 &= \frac{1}{\beta T}\left[(v_2 - X_{2,1})(l_0 - l - v_1) + \beta(v_1 - X_{1,3})\right].
\end{aligned}
$$

6. CONCLUSIONS

The paper provides an alternative computational method for solving the static state feedback linearization problem for a discrete-time control system. The proposed method is built upon the results of [5], based on the vector fields. However, instead of distributions in [5], another but related sequence of distributions is suggested, which can easily be computed with the help of backward shift equations only, while the computation of the related distributions in [5] requires both forward and backward shift equations. This fact makes the method especially useful for a certain subclass of discrete-time systems, obtained from the implicit Euler discretization of continuous-time systems. The idea that allows us to avoid forward shifts is to use the concepts of distribution invariants and their relative degrees in computations. As an example, the implicit Euler discretization of hydraulic press equations is considered.

Similar results can be obtained for the case when only the forward shift equations are available. Moreover, the approach can also be used to solve the partial feedback linearization problem, either using only backward shifts or only forward shifts, and possibly to solve the problem of realization of the input-output equations in the state space form. Observe that some preliminary results in this direction have been obtained in [15] regarding the realization problem. That is, paper [15] constructs an alternative sequence of vector spaces of differential forms, tightly related to the sequence from paper [3] and, unlike those from [3], needs only forward shifts in their constructions. However, compared to the approach of this paper, it allows much less. Namely, although the subspaces of one-forms that can be computed based on the forward shifts only are useful to check the solvability of the problem, they are of no use for constructing the state coordinates, that is, for providing a full solution.

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Diskreetsete mittelineaarsete juhtimissüsteemide tagasisidega lineariseerimine: arvutuslikud aspektid

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On esitatud alternatiivne meetod, mis võimaldab kontrollida diskreetaja mittelineaarse juhtimissüsteemi staatilise olekutagasisidega lineariseeritavust ja leida ka lineariseerimiseks vajalikku olekuteisendust. Meetodi eelis varasematega võrreldes seisneb selle lihtsuses: meetod vajab ainult tagasinihkeoperaatori rakendamist, samas kui varasemad nõuavad nii tagasinihke- kui ka edasinihkeoperaatori kasutamist. Meetod on eriti sobiv diskreetaja süsteemide alamklassi korral, mis on saadud ilmutamata Euleri diskretiseerimismeetodi rakendamisel pideva ajaga süsteemidele. Lineariseeritavuse tarvilike ja piisavate tingimuste kontroll põhineb rekurrentse algoritmi abil arvutatud juhtimissüsteemiga seotud vektorväljade jaotuste involutiivsusel. Seni kasutatud jaotuste asemel tuuakse sisse uued jaotused, kusjuures sarnaselt varasematele sisaldub iga järgnev jaotus eelmises. Uued jaotused langevad kokku varasematega, kui jaotused on involutiivsed. Et uute olekukoordinaatide arvutamisel vältida nendele edasinihkeoperaatori rakendamist, defineeritakse nende edasinihked (kasutades jaotuste invariantide ja nende hilinemisjärkude mõisteid) järgmiselt. Juba olemasolevad olekukoordinaadid on alati ühesse eelkirjeldatud järjestusse kuuluva jaotuse sõltumatud invariandid, nende edasinihked arvutatakse kui sellele jaotusele eelneva jaotuse invariandid.

Näitena on käsitletud meetodi rakendamist hüdraulilise pressi võrrandite diskretiseerimisel saadud mudeli tagasisidega lineariseerimisel.