Proceedings of the Estonian Academy of Sciences,
2019, **68**, 3, 324–332
https://doi.org/10.3176/proc.2019.3.10
Available online at www.eap.ee/proceedings

TOPOLOGICAL
ALGEBRAS

On non-unital locally pseudoconvex Q-algebras

Reyna María Pérez Tiscareño

Institute of Mathematics and Statistics, University of Tartu, J. Liivi 2, 50409 Tartu, Estonia; reyna@ut.ee

Received 6 February 2019, accepted 21 May 2019, available online 13 August 2019

Abstract. Some equivalent conditions for a topological algebra to be a Q-algebra have been studied by several researchers. They have studied Q-algebras, mainly for unital topological algebras. In this paper some equivalent conditions are studied to be a Q-algebra for non-unital locally pseudoconvex algebras, locally A-pseudoconvex algebras and locally m-pseudoconvex algebras.

Key words: topological algebra, locally pseudoconvex algebras, locally m-pseudoconvex algebras, locally A-pseudoconvex algebras, quasi-invertibility, Q-algebras.

1. INTRODUCTION

An algebra, E, over the field of complex numbers $\mathbb{C}$ is called topological algebra if E is equipped with a topology such that E is a topological linear space with a separately continuous multiplication (it is, for each $a \in E$, the maps $l_a, r_a : E \to E, l_a(x) = ax, r_a(x) = xa$ are continuous).

We define a map $\circ : E \times E \to E$ such that $\circ(x,y) := x \circ y = x + y - xy$. An element $x \in E$ is called *quasi-invertible*, if there exists $y \in E$ such that $x \circ y = y \circ x = \theta$, where θ is the zero element of E.

A topological algebra E is called a *Q-algebra*, when the set $Qinv(E)$ of quasi-invertible elements of E is open. A unital algebra E is called a Q-algebra, when the set $Inv(E)$ of invertible elements of E is open.

We notice that, if the topological algebra E has a unit e, then we can consider the sets $Qinv(E)$ and $Inv(E)$. By [6],

$$x \circ y = 0 \Leftrightarrow (e-x)(e-y) = e \text{ for every } x, y \in E.$$

Namely, x is (right) quasi-invertible if and only if $e - x$ is (right) invertible. The same holds for “left”. So,

$$Inv(E) \text{ is open if and only if } QinvE \text{ is open.}$$

So, we could also say that a topological algebra with a unit is Q-algebra if $Qinv(E)$ is open.

It is well known that Banach algebras are Q-algebras, but they are not the only ones. Several researchers have studied these kind of algebras [1,4,5,9], some of them have given equivalent conditions for a topological algebra to be a Q-algebra.

In [7], some equivalent conditions for unital complex normed algebras were given in order to be Q-algebras. Later, in [5] and [8], analogous conditions for non-unital locally m-convex and unital locally m-pseudoconvex algebras, respectively, were given, so that to be Q-algebras. Notice, that in all these papers the norms, seminorms or pseudoseminorms, that give the topology to the algebra, have the submultiplicativity condition.

Later, in [10], some results that characterize Q-algebras in the context of unital locally pseudoconvex Q-algebras were studied, in these algebras the submultiplicativity condition for the pseudoseminorms is not needed. But the proofs used strongly the fact that the algebra has a unit. In this paper, we give a generalization of these results to the case that the algebra does not have a unit. We also give some results that characterize Q-algebras in the context of non-unital locally A-pseudoconvex and locally m-pseudoconvex algebras. These results, assuming that the algebra has a unit, were studied in [11] and [8], respectively.

2. PRELIMINARIES

A subset S of a linear space X over a field $\mathbb{K}$ ($\mathbb{K}$ denotes $\mathbb{C}$ or $\mathbb{R}$) is called ρ*-convex*, with $0<\rho\leqslant 1$, if $\alpha x+\beta y\in S$ for any $x,y\in S$ and any $\alpha,\beta\in\mathbb{R}$ such that $\alpha,\beta\geqslant 0$ and $\alpha^\rho+\beta^\rho=1$.

Notice that, when $\rho=1$, then the notions of ρ-convexity and convexity coincide. A subset of X is called *pseudoconvex*, if it is ρ-convex for some ρ $(0<\rho\leqslant 1)$. A topological linear space is *locally pseudoconvex*, if it has a basis of pseudoconvex neighbourhoods of zero

$$\{U_\alpha:\alpha\in\Lambda\},$$

where each U_α is ρ_α-convex $(0<\rho_\alpha\leqslant 1)$. If $\rho_\alpha=\rho$, for every $\alpha\in\Lambda$, then X is called *locally* ρ*-convex* and *locally convex*, if $\rho=1$.

Let ρ be a real number such that $0<\rho\leqslant 1$. A *non-homogeneous seminorm* (called ρ-seminorm [4, p. 110] or pseudoseminorm in [4, p. 189]), is a real-valued function $p=p(x)$ on X which satisfies:
(i) $p(x+y)\leqslant p(x)+p(y)$ for all $x,y\in X$;
(ii) $p(\lambda x)=|\lambda|^\rho p(x)$ for all $x\in X,\ \lambda\in\mathbb{K}$.
The real number ρ is called the homogeneity index of p. A ρ-seminorm p is called ρ-norm if $p(x)=0\Rightarrow x=0$.

Let $\{p_\alpha:\alpha\in\Lambda\}$ be a family of non-homogeneous seminorms on a linear space X and $O_{\alpha,r}=\{x\in X:p_\alpha(x)<r\}$, where $r\in\mathbb{R}, r>0$. Then the family of finite intersections of sets $O_{\alpha,r}$ (varying α and r) gives a base of pseudoconvex neighbourhoods of zero for X. Conversely, if X is a locally pseudoconvex space then it has a base of balanced pseudoconvex neighbourhoods of zero $\mathscr{U}=\{U_\alpha:\alpha\in\Lambda\}$, then the Minkowski functionals $p_{U_\alpha}(x)=\inf_\beta\{\beta>0:x\in\beta^{\frac{1}{\rho_\alpha}}U_\alpha\}$, associated to each $U_\alpha\in\mathscr{U}$, determine a family $\{p_\alpha:=p_{U_\alpha}:\alpha\in\Lambda\}$ of ρ_α-seminorms (see [4, p.179, Proposition 4.1.10]). If the underlying topological linear space of a topological algebra E is locally pseudoconvex, then E is called a *locally pseudoconvex algebra* and its topology can be defined by a family $\mathscr{P}=\{p_\alpha:\alpha\in\Lambda\}$ of pseudoseminorms. When every p_α in the family $\mathscr{P}$ satisfies the submultiplicativity condition

$$p_\alpha(xy)\leqslant p_\alpha(x)p_\alpha(y)\ \text{ for each }\ \alpha\in\Lambda, \tag{2.1}$$

then E is called a *locally m-pseudoconvex algebra*.

When each $p_\alpha\in\mathscr{P}$ is a seminorm (the homogeneity index of every $p_\alpha\in\mathscr{P}$ is equal to 1), then E is a *locally convex algebra* and, if also every element of the family $\mathscr{P}$ satisfies the submultiplicativity condition (2.1), then E is a *locally m-convex algebra*.

A locally pseudoconvex algebra is called *locally A-pseudoconvex*, if for every $x\in E$ and every $\alpha\in\Lambda$ there exist $M=M(x,\alpha)>0$ and $N=N(x,\alpha)>0$ (which depend on x and α) such that,

$$p_\alpha(xy)\leqslant M^{\rho_\alpha}p_\alpha(y)\ \text{ and }\ p_\alpha(yx)\leqslant N^{\rho_\alpha}p_\alpha(y),\ \text{ for every }\ y\in E.$$

If $\rho_\alpha=\rho$ for every $\alpha\in\Lambda$, then the algebra is called *locally A-*(ρ*-convex*).

If E is an algebra over $\mathbb{C}$, the *spectrum* $\sigma(x)$ of x is given by

$$\sigma(x) = \{\lambda \in \mathbb{C} : \lambda^{-1}x \notin Qinv(E)\}$$

with zero added unless E has a unit and x is invertible (when E is a unital algebra, then by

$$\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda e \notin Inv(E)\})$$

and the *spectral radius* of x is defined by

$$r(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}.$$

Remember that if E is a topological algebra, a net $(x_\lambda)_{\lambda\in\Lambda}$ is called *advertibly convergent*, if there exists $x \in E$ such that $(x \circ x_\lambda)_{\lambda\in\Lambda}$ and $(x_\lambda \circ x)_{\lambda\in\Lambda}$ converge to θ. A topological algebra is *advertive* if every advertibly convergent net is convergent.

3. SOME RESULTS FOR NON-UNITAL LOCALLY PSEUDOCONVEX Q-ALGEBRAS

From here on, we assume that the family of pseudoseminorms, that induces the topology in any locally pseudoconvex algebra, is saturated. Otherwise, we can saturate the family (see [4, p. 191]).

It is known that in the case of unital locally pseudoconvex algebras, the following theorem holds (see [10, Theorem 3.1]).

Theorem 1. *Let* $(E, \{p_\alpha : \alpha \in \Lambda\})$ *be a* (*complex*) *locally pseudoconvex algebra with unit e. Consider the following conditions:*
(1) *E is a Q-algebra*;
(2) $\exists\ \alpha \in \Lambda$ *and* ε *with* $0 < \varepsilon < 1$ *such that*

$$p_\alpha(e - x) < \varepsilon \Rightarrow x \in Inv(E);$$

(3) $\exists\ \alpha \in \Lambda$ *and* ε *with* $0 < \varepsilon < 1$ *such that* $p_\alpha(x) < \varepsilon \Rightarrow \sum_{n=0}^{\infty} x^n$ *converges in* E;
(4) $\exists\ \alpha \in \Lambda$ *such that* $\sup\limits_{p_\alpha(x)\leqslant 1} r(x) < \infty$;
(5) $\exists\ \alpha \in \Lambda$ *and* ε *with* $0 < \varepsilon \leqslant 1$ *such that* $r(x)^{\rho_\alpha} \leqslant \frac{1}{\varepsilon} p_\alpha(x)$ *for all* $x \in E$.
Then, (3) *implies* (2), *and* (1), (2), (4) *and* (5) *are equivalent.*

In this section, we give a generalization of this theorem for non-unital locally pseudoconvex algebras. Notice that $\sum_{n=0}^{\infty} x^n$ converges in E, if and only if $-\sum_{n=1}^{\infty} x^n$ converges in E.

Theorem 2. *Let* $(E, \{p_\alpha : \alpha \in \Lambda\})$ *be a* (*complex*) *locally pseudoconvex algebra. Consider the following conditions:*
(1) *E is a Q-algebra*;
(2) $\exists\ \alpha \in \Lambda$ *and* ε *with* $0 < \varepsilon < 1$ *such that*

$$p_\alpha(x) < \varepsilon \Rightarrow x \in Qinv(E);$$

(3) $\exists\ \alpha \in \Lambda$ *and* ε *with* $0 < \varepsilon < 1$ *such that* $p_\alpha(x) < \varepsilon \Rightarrow -\sum_{n=1}^{\infty} x^n$ *converges in* E;
(4) $\exists\ \alpha \in \Lambda$ *such that* $\sup\limits_{p_\alpha(x)\leqslant 1} r(x) < \infty$;
(5) $\exists\ \alpha \in \Lambda$ *and* ε *with* $0 < \varepsilon < 1$ *such that* $r(x)^{\rho_\alpha} \leqslant \frac{1}{\varepsilon} p_\alpha(x)$ *for all* $x \in E$.
Then, (3) *implies* (2), *and* (1), (2), (4) *and* (5) *are equivalent.*

Proof. $(3) \Rightarrow (2)$ Let $\alpha \in \Lambda$ and $0 < \varepsilon < 1$ be as in (3) and $x \in E$ be such that $p_\alpha(x) < \varepsilon$. Then, by (3)

$$-\sum_{n=1}^{\infty} x^n$$

converges in E. Hence,

$$x \circ \left(-\sum_{n=1}^{\infty} x^n\right) = x - \sum_{n=1}^{\infty} x^n + \sum_{n=1}^{\infty} x^{n+1} = \theta.$$

Analogously, $(-\sum_{n=1}^{\infty} x^n) \circ x = \theta$. So, $x \in Qinv(E)$. $(2) \Rightarrow (1)$ Let $\alpha \in \Lambda$ and $0 < \varepsilon < 1$ be as in (2). Then we have that

$$\{x : p_\alpha(x) < \varepsilon\} \subset Qinv(E).$$

This is, there exists a neighbourhood of θ, contained in $Qinv(E)$, so E is a Q-algebra (see [6, p. 43, Lemma 6.4]).

$(1) \Rightarrow (5)$ Since $\mathscr{P} = \{p_\alpha : \alpha \in \Lambda\}$ is saturated, then

$$\mathscr{B} = \{O_{\alpha,\varepsilon} : \alpha \in \Lambda, \varepsilon > 0\},$$

where $O_{\alpha,\varepsilon} = \{x \in E : p_\alpha(x) < \varepsilon\}$ is a base of neighbourhoods of zero in E (see [4, p. 191, Lemma 4.3.8]) and every $O_{\alpha,\varepsilon} \in \mathscr{B}$ is absorbing, ρ_α-convex and balanced by Proposition 4.1.13, Remark 4.1.14 and Lemma 4.1.4 in [4]. As, by (1), E is a Q-algebra, then $Qinv(E)$ is open. Thus, there exists

$$O = O_{\alpha,\varepsilon} = \{x \in E : p_\alpha(x) < \varepsilon\} \in \mathscr{B}$$

such that $O \subseteq Qinv(E)$. We can assume that $\varepsilon < 1$.

Now, we consider any $x \in E$. Since O is absorbing, then there exists $\lambda_x > 0$ such that $\kappa x \in O$ for every κ with $0 < |\kappa| \leqslant \lambda_x$.

If $\mu \geqslant \frac{1}{\lambda_x^{\rho_\alpha}}$, then $0 < \frac{1}{\mu^{\frac{1}{\rho_\alpha}}} \leqslant \lambda_x$. Thus,

$$\frac{1}{\mu^{\frac{1}{\rho_\alpha}}} x \in O.$$

So, there exists $\mu > 0$ such that $x \in \mu^{\frac{1}{\rho_\alpha}} O$. Consider any $\varphi > 0$ such that $x \in \varphi^{\frac{1}{\rho_\alpha}} O$, then

$$\frac{1}{\varphi^{\frac{1}{\rho_\alpha}}} x \in O \subseteq Qinv(E).$$

Hence, $\varphi^{\frac{1}{\rho_\alpha}} \notin \sigma(x)$. Moreover, if $|\lambda| > \varphi^{\frac{1}{\rho_\alpha}}$, then

$$x \in \varphi^{\frac{1}{\rho_\alpha}} O \subset \lambda O$$

because O is balanced. Then $\frac{x}{\lambda} \in O \subset Qinv(E)$, it means that $\lambda \notin \sigma(x)$. So, $r(x) \leqslant \varphi^{\frac{1}{\rho_\alpha}}$ for any $\varphi > 0$ such that $x \in \varphi^{\frac{1}{\rho_\alpha}} O$.

Thus,

$$r(x)^{\rho_\alpha} \leqslant \inf_{\varphi}\{\varphi > 0 : x \in \varphi^{\frac{1}{\rho_\alpha}} O\} = p_O(x). \tag{3.1}$$

By the definition of O, we have that

$$p_O(x) = \inf_{\varphi}\{\varphi > 0 : p_\alpha(x) < \varphi\varepsilon\} = \inf_{\varphi}\left\{\varphi > 0 : p_\alpha\left(\frac{1}{\varepsilon^{\frac{1}{\rho_\alpha}}} x\right) < \varphi\right\}. \tag{3.2}$$

Since $\{\varphi > 0 : p_\alpha\left(\frac{1}{\varepsilon^{\frac{1}{\rho_\alpha}}}x\right) < \varphi\}$ is the interval $\left(p_\alpha\left(\frac{1}{\varepsilon^{\frac{1}{\rho_\alpha}}}x\right), \infty\right)$ it is clear that

$$\inf_{\varphi}\{\varphi > 0 : p_\alpha\left(\frac{1}{\varepsilon^{\frac{1}{\rho_\alpha}}}x\right) < \varphi\} = p_\alpha\left(\frac{1}{\varepsilon^{\frac{1}{\rho_\alpha}}}x\right). \tag{3.3}$$

Now, by $(3.1), (3.2)$ and (3.3), we can conclude that

$$r(x)^{\rho_\alpha} \leqslant p_\alpha\left(\frac{1}{\varepsilon^{\frac{1}{\rho_\alpha}}}x\right) = \frac{1}{\varepsilon}p_\alpha(x).$$

$(5) \Rightarrow (2)$ We consider $\alpha \in \Lambda$ and ε as in (5). Let $x \in E$ be such that $p_\alpha(x) < \varepsilon$. Then $r(x)^{\rho_\alpha} \leqslant \frac{1}{\varepsilon}p_\alpha(x) < 1$. Thus, we have that $r(x) < 1$. Hence, $1 \notin \sigma(x)$ or, equivalently, $x \in Qinv(E)$.

In [10, Theorem 3.3], it is shown that (5) implies (4) for unital locally pseudoconvex algebras. However, the proof of these facts do not need the algebra to be unital.

$(4) \Rightarrow (5)$ Let $\alpha \in \Lambda$ be as in (4) and

$$0 \leqslant M_\alpha = \sup_{p_\alpha(x) \leqslant 1} r(x) < \infty.$$

If $p_\alpha(x) = 0$, then $0 = p_\alpha(mx)$ for every $m > 0$. By [9, Proposition 2.2.1 (a)], $mr(x) = r(mx)$ and, since $mr(x) = r(mx) \leqslant M_\alpha$, then $r(x) \leqslant \frac{M_\alpha}{m}$ for every $m > 0$. This means that $r(x) = 0$ and (5) holds.

If $p_\alpha(x) \neq 0$, then

$$p_\alpha\left(\frac{x}{p_\alpha(x)^{\frac{1}{\rho_\alpha}}}\right) = 1$$

implies that

$$\frac{1}{p_\alpha(x)}r(x)^{\rho_\alpha} = r\left(\frac{x}{p_\alpha(x)^{\frac{1}{\rho_\alpha}}}\right)^{\rho_\alpha} \leqslant M_\alpha^{\rho_\alpha}.$$

So, $r(x)^{\rho_\alpha} \leqslant M_\alpha^{\rho_\alpha}p_\alpha(x)$. Take $N = \max(M_\alpha^{\rho_\alpha}, 2)$ and $\varepsilon = \frac{1}{N}$. Then $\varepsilon < 1$ and $r(x)^{\rho_\alpha} \leqslant \frac{1}{\varepsilon}p_\alpha(x)$. □

For a unital locally m-pseudoconvex algebra $(E, \{p_\alpha : \alpha \in \Lambda\})$, it is known (see [8, Theorem 3.1]) that E is a Q-algebra if and only if there exists $\alpha \in \Lambda$ such that

$$r(x)^{\rho_\alpha} = \lim_n \ p_\alpha(x^n)^{\frac{1}{n}} = \inf_n \ p_\alpha(x^n)^{\frac{1}{n}}.$$

Moreover, as a consequence of [4, Lemma 3.3.6] we have that for each $\alpha \in \Lambda$, there exists $\lim_n \ p_\alpha(x^n)^{\frac{1}{n}}$ and

$$\lim_n \ p_\alpha(x^n)^{\frac{1}{n}} = \inf_n \ p_\alpha(x^n)^{\frac{1}{n}}. \tag{3.4}$$

But, when we consider locally pseudoconvex algebras, we cannot be sure that $\lim_n \ p_\alpha(x^n)^{\frac{1}{n}}$ exists and even if it exists it is not always true, the equality (3.4), for every $x \in E$. In [3, Example 2.7], is given a locally A-convex algebra, E, where $\lim_n \ p_\alpha(x^n)^{\frac{1}{n}}$ exists and there exists $x \in E$ such that $\lim_n \ p_\alpha(x^n)^{\frac{1}{n}} > \inf_n \ p_\alpha(x^n)^{\frac{1}{n}}$.

Nevertheless, if we consider locally A-pseudoconvex algebras, then there exists $\limsup_{n\to\infty} \ p_\alpha(x_n)^{\frac{1}{n}}$, i.e. it is finite.

In the next section, we will consider non-unital locally A-pseudoconvex and locally m-pseudoconvex algebras. We will study some characterizations of these algebras to be Q-algebras, one of these characterizations will involve the limits mentioned above.

4. SOME RESULTS FOR NON-UNITAL LOCALLY A-PSEUDOCONVEX AND LOCALLY m-PSEUDOCONVEX ALGEBRAS

Let $(E,\{p_\alpha : \alpha \in \Lambda\})$ be a locally A-pseudoconvex algebra. Then, for every $x \in E$ and $\alpha \in \Lambda$, there exists, $M = M(x,\alpha) > 0$ (which depends on x and α) such that

$$p_\alpha(x^n)^{\frac{1}{n}} \leqslant M^{\frac{(n-1)\rho_\alpha}{n}} p_\alpha(x)^{\frac{1}{n}}, \quad \text{for every} \quad \alpha \in \Lambda.$$

Hence,

$$\limsup_{n\to\infty} \ p_\alpha(x^n)^{\frac{1}{n}} \leqslant \limsup_{n\to\infty} \ M^{\frac{(n-1)\rho_\alpha}{n}} p_\alpha(x)^{\frac{1}{n}} = \lim_{n\to\infty} \ M^{\frac{(n-1)\rho_\alpha}{n}} p_\alpha(x)^{\frac{1}{n}} = M^{\rho_\alpha}$$

for every $\alpha \in \Lambda$. So, there exists $\limsup_{n\to\infty} \ p_\alpha(x^n)^{\frac{1}{n}}$ for every $\alpha \in \Lambda$. According to the next theorem, if E is a Q-algebra, then $\exists \ \alpha \in \Lambda$ such that $r(x)^{\rho_\alpha} = \limsup_{n\to\infty} \ p_\alpha(x^n)^{\frac{1}{n}}$ for every $x \in E$.

Theorem 3. *Let* $(E,\{p_\alpha : \alpha \in \Lambda\})$ *be a locally A-pseudoconvex algebra. Consider* (1)–(5) *as in Theorem* 2 *and*
(6) $\exists \ \alpha \in \Lambda$ *such that* $r(x)^{\rho_\alpha} = \limsup_{n\to\infty} \ p_\alpha(x^n)^{\frac{1}{n}}$ *for every* $x \in E$.
Then, (3) *implies* (2), (5) *implies* (6) *and* (1), (2), (4) *and* (5) *are equivalent.*

Proof. Since every locally A-pseudoconvex algebra is a locally pseudoconvex algebra, then, by Theorem 2, we need only to prove that (5) implies (6). This implication was proved in [11, Theorem 1] for unital locally A-pseudoconvex algebras but the proof does not need the algebra to be unital. □

In the previous theorem we could not show that (6) implies (5). But, as a remark, we can say that if a locally A-pseudoconvex algebra satisfies (6), then there exists ε_x (ε_x depends on x), $0 < \varepsilon_x < 1$ such that

$$r(x)^{\rho_\alpha} \leqslant \frac{1}{\varepsilon_x} p_\alpha(x).$$

Indeed, if $p_\alpha(x) = 0$, then $p_\alpha(x^n) = 0$ and $r(x) = 0$ (by property 6)). If $p_\alpha(x) \neq 0$, take $M_x = \max\left(\frac{r(x)^{\rho_\alpha}}{p_\alpha(x)}, 1\right)$ and $\varepsilon_x = \frac{1}{M_x}$.

In the next theorem we prove that adding a condition to the hyphotesis of the previous theorem, then (6) implies (5). For it, we need the following definition.

Definition. *Let* $(E,\{p_\alpha : \alpha \in \Lambda\})$ *be a* (complex) *locally A-pseudoconvex algebra. By [2, Theorem 3], for every* $\alpha \in \Lambda$*, there exists a submultiplicative* ρ_α*-seminorm* q_α *on E such that* $p_\alpha(x) \leqslant q_\alpha(x)$*, for every* $x \in E$*, we will say, from now on, that such* q_α *is the submultiplicative* ρ_α*-seminorm associated to* p_α.

Theorem 4. *Let* $(E,\{p_\alpha : \alpha \in \Lambda\})$ *be a* (*complex*) *locally A-pseudoconvex algebra. Property* (6) *in Theorem* 3*, implies property* (5) *of Theorem* 2 *if*

$$\delta = \inf_{y\in E} \left\{ \frac{p_\alpha(y)}{q_\alpha(y)} : p_\alpha(y) \neq 0 \ \ and \ \ q_\alpha(y) \neq 0 \right\} > 0, \tag{4.1}$$

where q_α *is the submultiplicative* ρ_α*-seminorm associated to* p_α.

Proof. Let α be as in property 6). Then, there exists α such that $r(x)^{\rho_\alpha} = \limsup_{n\to\infty} \ p_\alpha(x^n)^{\frac{1}{n}}$ for every $x \in E$. Let $q_\alpha(x)$ be the submultiplicative ρ_α-seminorm associated to p_α. Then

$$\limsup_{n\to\infty} \ p_\alpha(x^n)^{\frac{1}{n}} \leqslant \limsup_{n\to\infty} \ q_\alpha(x^n)^{\frac{1}{n}} = \lim_{n\to\infty} \ q_\alpha(x^n)^{\frac{1}{n}} \leqslant q_\alpha(x).$$

So, $r(x)^{\rho_\alpha} \leqslant q_\alpha(x)$ for every $x \in E$.

If $p_\alpha(x)=0$, then $r(x)^{\rho_\alpha}=0$. If $p_\alpha(x)\neq 0$ and $q_\alpha(x)=0$, then $r(x)^{\rho_\alpha}\leqslant q_\alpha(x)=0$. So, $r(x)^{\rho_\alpha}\leqslant p_\alpha(x)$. If $q_\alpha(x)\neq 0$ and $\varepsilon=\frac{\delta}{2}$, where δ is defined as in (4.1), then $0<\varepsilon<1$ and

$$q_\alpha(x)\leqslant \frac{1}{\varepsilon}p_\alpha(x), \text{ for every } x\in E \text{ such that } p_\alpha(x)\neq 0 \text{ and } q_\alpha(x)\neq 0.$$

Thus, $r(x)^{\rho_\alpha}\leqslant \frac{1}{\varepsilon}p_\alpha(x)$ for every $x\in E$ such that $p(x)\neq 0$. So, $r(x)^{\rho_\alpha}\leqslant \frac{1}{\varepsilon}p_\alpha(x)$ for every $x\in E$. □

Corollary 1. *Let $(E,\{p_\alpha:\alpha\in\Lambda\})$ be a (complex) locally A-pseudoconvex algebra. Consider (1), (2), (4), and (5) as in Theorem 2; (6) as in Theorem 3 and assume that E satisfies (4.1) in the Theorem 4 with α as in (6). Then, (1), (2), (4), (5), and (6) are equivalent.*

Proof. Immediate from Theorems 3 and 4. □

Let $(E,\{p_\alpha:\alpha\in\Lambda\})$ be a (complex) locally m-pseudoconvex algebra. Then, (as an immediate consequence of [4, Lemma 3.3.6]) for each $\alpha\in\Lambda$, there exists $\lim_n\ p_\alpha(x^n)^{\frac{1}{n}}$ and

$$\lim_n\ p_\alpha(x^n)^{\frac{1}{n}}=\inf_n\ p_\alpha(x^n)^{\frac{1}{n}} \tag{4.2}$$

for every $x\in E$.

A locally m-pseudoconvex algebra with same homogeneity indexes ρ for all pseudoseminorms that give their topology is called m-(ρ-convex) algebra.

Theorem 5. *Let $(E,\{p_\alpha:\alpha\in\Lambda\})$ be a (complex) locally m-pseudoconvex algebra. We consider (1)–(5) as in Theorem 2 and*
(6) $\exists\ \alpha\in\Lambda$ such that $r(x)^{\rho_\alpha}=\lim_n\ p_\alpha(x^n)^{\frac{1}{n}}\left(=\inf_n\ p_\alpha(x^n)^{\frac{1}{n}}\right)$ for every $x\in E$;
(7) $\exists\ \alpha\in\Lambda$ such that $r(x)^{\rho_\alpha}\leqslant p_\alpha(x)$ for all $x\in E$.
Then (1), (2), (4), (5), (6), and (7) are equivalent and (3) implies (2).
Moreover, if E is an advertive m-(ρ-convex) algebra, then we have that (6) implies (3). In this case (1), (2), (3), (4), (5), (6), and (7) are equivalent.

Proof. Since any locally m-pseudoconvex algebra is a locally pseudoconvex algebra, then by Theorem 2, we have that (3) implies (2) and (1), (2), (4), and (5) are equivalent. Moreover, as every locally m-pseudoconvex algebra is locally A-pseudoconvex algebra, we conclude by Theorem 3 that (5) implies (6). Since p_α is submultiplicative for any $\alpha\in\Lambda$, then (6) implies (5). Indeed, by (6), there exists $\alpha\in\Lambda$ such that

$$r(x)^{\rho_\alpha}=\lim_n\ p_\alpha(x^n)^{\frac{1}{n}}\leqslant p_\alpha(x) \text{ for all } x\in E.$$

We have that (5) is equivalent to (7) because (7) ⇒ (5) ⇒ (6) ⇒ (7).

(6) ⇒ (3) We will proceed analogously as in the proof of (6) ⇒ (3) in [8, Theorem 3.1]. Let $\alpha\in\Lambda$ be as in (6) and $x\in E$ be such that $p_\alpha(x)<\varepsilon$ for some $0<\varepsilon<1$. For every $\gamma\in\Lambda$, we consider $E_\gamma=E/\ker p_\gamma$ and define on E_γ the ρ_γ-norm $p'_\gamma([x]_\gamma)=p_\gamma(x)$, where $[x]_\gamma=x+\ker p_\gamma$ and denote by $\pi_\gamma:E\to E_\gamma$, the quotient map and by $\overline{E_\gamma}$ the completion of the algebra E_γ (a ρ_γ-Banach algebra) and by $r_{\overline{E_\gamma}}$ the spectral radius function in $\overline{E_\gamma}$.

Since[1]

$$r(x)^\rho\geqslant r_{E_\gamma}(\pi_\gamma(x))^\rho\geqslant r_{\overline{E_\gamma}}(\pi_\gamma(x))^\rho=\lim_n\ p'_\gamma(\pi_\gamma(x)^n)^{\frac{1}{n}}=\lim_n\ p_\gamma(x^n)^{\frac{1}{n}}$$

[1] Since $\overline{E_\gamma}$ is a ρ-Banach algebra, by [4, Theorem 7.4.6], this equality holds.

for every $\gamma \in \Lambda$, then $\sup_{\beta} \lim_{n} p_{\beta}(x^n)^{\frac{1}{n}} \leqslant r(x)^{\rho}$. Now, since p_{α} is submultiplicative, then by (6), we have that $r(x)^{\rho} \leqslant p_{\alpha}(x)$, and by hypothesis, $p_{\alpha}(x) < \varepsilon$, where $0 < \varepsilon < 1$. Thus, $\sup_{\beta} \lim_{n} p_{\beta}(x^n)^{\frac{1}{n}} < 1$. Therefore, there exists $m < 1$ such that

$$\lim_{n} p_{\beta}(x^n)^{\frac{1}{n}} < m < 1$$

for every $\beta \in \Lambda$. Since $\lim_{n} p_{\beta}(x^n) < m^n$ with $m < 1$ for each β, then $x^n \to \theta$.

Let $s_n = -\sum_{k=1}^{n} x^k$, then $(s_n \circ x)_n$ and $(x \circ s_n)_n$ converge to θ, indeed,

$$s_n \circ x = -\sum_{k=1}^{n} x^k + x + \sum_{k=2}^{n+1} x^k = x^{n+1}.$$

Analogously, $x \circ s_n = x^{n+1}$. So, $(s_n)_n$ is advertibly convergent and, since E is advertive, the net $(s_n)_n$ converges. Namely, the series $-\sum_{n=1}^{\infty} x^n$ converges in E. □

Remark 1. We can generalize [8, Theorem 3.1] for non-unital algebras. Namely, if $(E, \{p_{\alpha} : \alpha \in \Lambda\})$ is a (complex) locally m-pseudoconvex algebra, (1)–(5) as in Theorem 2 with $\varepsilon = 1$ and (6) as in Theorem 5. Then the conditions (1), (2), (4), (5), and (6) are equivalent and (3) implies (2). Moreover, if E is advertive m-(ρ-convex), then we have that (6) implies (3). In this case the conditions (1), (2), (3), (4), (5), and (6) are equivalent.

By the equivalence between (1), (2) and (6) in [8, Theorem 2.5] and [11, Remark 1], we have that (2), (5) and (6) are equivalent and (5) is equivalent for (E, p_{α}) to be Q-algebra, but as the topology induced by p_{α} on E is contained in the topology induced by the family of pseudoseminorms, $\{p_{\alpha} : \alpha \in \Lambda\}$, then we have that $(E, \{p_{\alpha} : \alpha \in \Lambda\})$ is a Q-algebra. Namely, (5) implies (1).

(1) $\Rightarrow$ (5) By [4, p.195, Lemma 4.4.2], we have that a local subbase of the origin is given by the sets

$$\{x \in E : p_{\alpha}(x) < \delta\}, \alpha \in \Lambda, \delta > 0.$$

Now, as $\theta \in Qinv(E)$ and the set $Qinv(E)$ is open, then there exists $\alpha \in \Lambda$ and $0 < \delta < 1$ such that, if $p_{\alpha}(x) < \delta$, then $x \in Qinv(E)$. We consider $\lambda \in \mathbb{C}$ such that $|\lambda| > \left(\frac{p_{\alpha}(x)}{\delta}\right)^{\frac{1}{\rho_{\alpha}}}$. Then,

$$p_{\alpha}\left(\frac{x}{\lambda}\right) = \frac{1}{|\lambda|^{\rho_{\alpha}}} p_{\alpha}(x) < \delta.$$

This implies that $\frac{x}{\lambda} \in Qinv(E)$, so we conclude that $r(x) \leqslant \left(\frac{p_{\alpha}(x)}{\delta}\right)^{\frac{1}{\rho_{\alpha}}}$ and by the equivalence between (1) and (5) in [8, Theorem 2.5] we have that $r(x)^{\rho_{\alpha}} \leqslant p_{\alpha}(x)$, for every $x \in E$.

We notice that (4) implies (5) because in the proof of [8, Theorem 3.1] (5) $\Rightarrow$ (6)) it is not needed the algebra to be unital, (5) implies (4) is obvious. We can consider $\varepsilon = 1$ in the proof of (3) implies (2) given in Theorem 2. Notice that in the proof of (6) implies (3) in the Theorem 5, we could also consider that $\varepsilon = 1$.

Theorem 6. *Let $(E, \{p_{\alpha} : \alpha \in \Lambda\})$ be a (complex) advertive A-(ρ-convex) algebra. Suppose that property* (4.1) *with $\rho_{\alpha} = \rho$ for every $\alpha \in \Lambda$, in Theorem* 4 *holds. Then* (6) *in Theorem* 3, *with $\rho_{\alpha} = \rho$, implies* (3) *in Theorem* 2.

Proof. Analogous to [11, Theorem 2]. □

Corollary 2. *Let $(E, \{p_{\alpha} : \alpha \in \Lambda\})$ be a (complex) advertive A-(ρ-convex) algebra,* (1)–(6) *as in Theorem* 3 *and E satisfies* (4.1) *in the Theorem* 4 *with α as in* (6). *Then,* (1), (2), (3), (4), (5), *and* (6) *are equivalent.*

Proof. Since an advertive A-(ρ-convex) algebra is a special case of a locally A-pseudoconvex algebra, then from Theorem 3 we have that (3) implies (2), (5) implies (6) and (1), (2), (4), and (5) are equivalent. By Theorem 4, we have that (6) implies (5). Moreover, by Theorem 6 we have that (6) implies (3). So, (1), (2), (3), (4), (5), and (6) are equivalent. □

5. CONCLUSIONS

In the present paper, we have showed, for non-unital locally pseudoconvex algebras, some conditions that are equivalent to be a Q-algebra. We have also given equivalent conditions to be a Q-algebra for non-unital A-pseudoconvex algebras and m-pseudoconvex algebras.

ACKNOWLEDGEMENTS

The research was supported by the institutional research funding IUT20-57 of the Estonian Ministry of Education and Research. The author expresses her gratitude to the groups of topological algebras in the Institute of Mathematics at the Universidad Nacional Autonoma de México (México) and in the Institute of Mathematics and Statistics at the University of Tartu (Estonia) for their comments that helped to improve this paper. The author thanks the referees for their remarks. The publication costs of this article were covered by the Estonian Academy of Sciences.

REFERENCES

1. Abel, M. and Żelazko, W. Properties of TQ-algebras. *Proc. Est. Acad. Sci.*, 2011, **60**(3), 141–148.
2. Abel, M. and Arhippainen, J. Locally m-pseudoconvex topologies on locally A-pseudoconvex algebras. *Czech. Math. J.*, 2004, **54**(3), 675–680.
3. Arizmendi-Peimbert, H., Perez-Tiscareño, R. M., and Roa-Fajardo, J. On the spectral radii in $(C_b(x), \beta)$ and the $M(\beta)$ topology. In *Proceedings of the International Conference on Topological Algebras and their Applications (ICTAA 2008), Tartu, Estonia, January 24–27, 2008*, (Abel, M., ed.). Math. Studies, 4, Est. Math. Soc., Tartu, 2008, 29–33.
4. Balachandran, V. K. *Topological Algebras*. North-Holland Math. Studies, 185, North-Holland Publ. Co., Amsterdam, 2000.
5. Fragoulopoulou, M. *Topological Algebras With Involution*. North-Holland Math. Studies 200, Elsevier Science, Amsterdam, 2005.
6. Mallios, A. *Topological Algebras. Selected Topics*. North-Holland Publ. Co., Amsterdam, 1986.
7. Mascioni, V. Some characterizations of complex normed Q-algebras. *El. Math.*, 1984, **42**, 10–14.
8. Palacios, L., Pérez-Tiscareño, R. M., and Signoret, C. On Q-algebras and spectral algebras. *Poincare J. Anal. Appl.*, 2016, **I**, 21–28.
9. Palmer, T. W. *Banach algebras and the general theory of *-algebras, Vol. 1, Encyclopedia of math. and its appl. 49*, Cambridge Univ. Press, Cambridge, 1994.
10. Pérez-Tiscareño, R. M. On some characterizations of Q-algebras for unital locally pseudoconvex algebras. *Poincare J. Anal. Appl.*, 2019, 1, 23–30.
11. Pérez-Tiscareño, R. M. When is a unital locally A-pseudoconvex algebra a Q-algebra? In *Proceeding of ICTAA 2018, Tallinn, Estonia, January 25–28, 2018 (Abel, M., ed.).* Math. Stud., 7, Est. Math. Soc., Tartu, 2018, 112–120.

Ühikuta pseudokumeratest Q-algebratest

Reyna María Pérez Tiscareño

Mõned teadlased on uurinud topoloogiliste algebrate jaoks ekvivalentseid tingimusi Q-algebraks olekuga. Eelkõige on nad uurinud, mis juhtub, kui topoloogiline algebra on ühikuga. Käesolevas artiklis on Q-algebraks olekuga ekvivalentsed tingimused saadud juhul, kui topoloogiline algebra ei ole ühikuga ja kuulub ühte järgmistest klassidest: lokaalselt pseudokumerad algebrad, lokaalselt A-pseudokumerad algebrad või lokaalselt m-pseudokumerad algebrad.