



About pushouts in the category $\mathcal{S}(B)$ of Segal topological algebras

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Abstract. In this paper, we answer positively the open question, posed in [2], about the existence of pushouts in the category $\mathcal{S}(B)$ of Segal topological algebras.

Key words: mathematics, topological algebras, Segal topological algebras, category, pushout.

A *topological algebra* is throughout this paper a topological linear space over the field \mathbb{K} (where \mathbb{K} stands for either \mathbb{R} or \mathbb{C}), in which is defined a separately continuous associative multiplication.

In [1], the study of general Segal topological algebras started. We begin with recalling the definitions from [1].

A topological algebra (A, τ_A) is a left (right or two-sided) *Segal topological algebra* in a topological algebra (B, τ_B) via an algebra homomorphism $f : A \rightarrow B$, if

- (1) $\text{cl}_B(f(A)) = B$;
- (2) $\tau_A \supseteq \{f^{-1}(U) : U \in \tau_B\}$;
- (3) $f(A)$ is a left (respectively, right or two-sided) ideal of B .

In what follows, a Segal topological algebra will be denoted shortly by a triple (A, f, B) .

From now on, we will fix a topological algebra (B, τ_B) , which we will not change for this paper.

Let us remind to the readers also the definition of the category $\mathcal{S}(B)$ of Segal topological algebras, introduced in [2].

The set $\text{Ob}(\mathcal{S}(B))$ of objects of the category $\mathcal{S}(B)$ consists of all Segal topological algebras in the same topological algebra B , i.e., all Segal algebras in the form of triples $(A, f, B), (C, g, B), \dots$

The set $\text{Mor}((A, f, B), (C, g, B))$ of morphisms between Segal topological algebras (A, f, B) and (C, g, B) consists of all continuous algebra homomorphisms $\alpha : A \rightarrow C$, satisfying $g(\alpha(a)) = (1_B \circ f)(a) = f(a)$ for every $a \in A$.

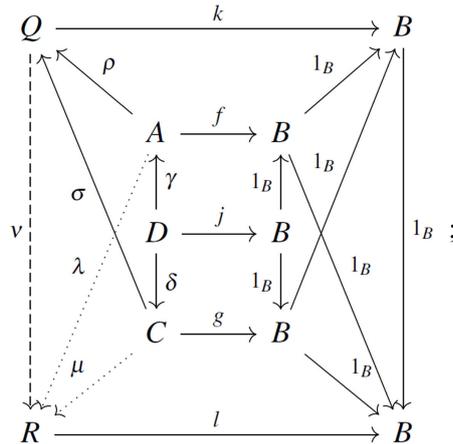
In [2] we showed that $\mathcal{S}(B)$ is really a category, which had, among other categorical constructions, also pullbacks. The existence of pushouts was an open problem posed in [2].

The present paper answers this open question positively, using some facts from category theory and results obtained in [2] and [3].

Let us start with the definition of a pushout in the context of Segal topological algebras (in the category $\mathcal{S}(B)$).

Definition 1. Let $(A, f, B), (C, g, B), (D, j, B) \in \text{Ob}(\mathcal{S}(B))$ with $\gamma \in \text{Mor}((D, j, B), (A, f, B))$ and $\delta \in \text{Mor}((D, j, B), (C, g, B))$. An object (Q, k, B) of the category $\mathcal{S}(B)$, together with morphisms $\rho \in \text{Mor}((A, f, B), (Q, k, B))$ and $\sigma \in \text{Mor}((C, g, B), (Q, k, B))$, is called the **pushout** of morphisms γ and δ , if

(1) $\rho \circ \gamma = \sigma \circ \delta$



(2) for every object (R, l, B) of the category $\mathcal{S}(B)$ and such morphisms $\lambda \in \text{Mor}((A, f, B), (R, l, B))$, $\mu \in \text{Mor}((C, g, B), (R, l, B))$ that $\lambda \circ \gamma = \mu \circ \delta$, there exists unique morphism $v \in \text{Mor}((Q, k, B), (R, l, B))$ such that $v \circ \rho = \lambda$ and $v \circ \sigma = \mu$.

Let \mathcal{C} be any category with the following two properties:

- (P1) for any pair of objects $A, B \in \text{Ob}(\mathcal{C})$, the coproduct of A and B exists in \mathcal{C} ;
- (P2) for any pair of objects $C, D \in \text{Ob}(\mathcal{C})$ and any pair of morphisms $\alpha, \beta \in \text{Mor}(C, D)$, the coequalizer of α and β exists in \mathcal{C} .

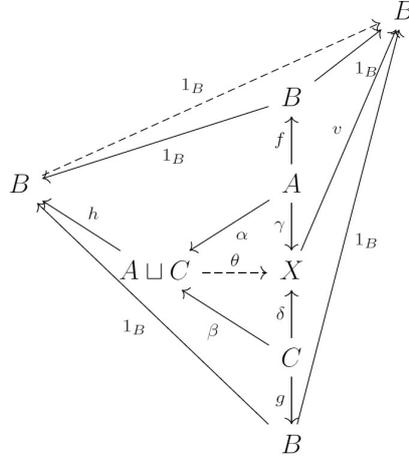
In category theory it is known¹ that, under these two conditions, for any morphisms $\gamma \in \text{Mor}(E, F)$, $\delta \in \text{Mor}(E, G)$, with $E, F, G \in \text{Ob}(\mathcal{C})$, the pushout of γ and δ exists and is constructable in the following way:

- (1) Construct the coproduct $(F \sqcup G, i_F, i_G)$ of F and G with injections $i_F : F \rightarrow F \sqcup G$ and $i_G : G \rightarrow F \sqcup G$. Then $i_F \circ \gamma, i_G \circ \delta \in \text{Mor}(E, F \sqcup G)$.
- (2) Construct the coequalizer (Q, λ) of maps $i_F \circ \gamma$ and $i_G \circ \delta$, where $\lambda \in \text{Mor}(F \sqcup G, Q)$.
- (3) The triple $(Q, \lambda \circ i_F, \lambda \circ i_G)$ is then the pushout of γ and δ .

Now we continue with the definitions and descriptions of coproduct and coequalizer in the category $\mathcal{S}(B)$. The material about coproducts comes from [3] and the material about coequalizers comes from [2].

Definition 2. The **coproduct** of $(A, f, B), (C, g, B) \in \text{Ob}(\mathcal{S}(B))$ is a triple $((A \sqcup C, h, B), \alpha, \beta)$, where $(A \sqcup C, h, B) \in \text{Ob}(\mathcal{S}(B))$, $\alpha \in \text{Mor}((A, f, B), (A \sqcup C, h, B))$, $\beta \in \text{Mor}((C, g, B), (A \sqcup C, h, B))$ such that for every $(X, j, B) \in \text{Ob}(\mathcal{S}(B))$ and every pair of morphisms $\gamma \in \text{Mor}((A, f, B), (X, j, B))$ and $\delta \in \text{Mor}((C, g, B), (X, j, B))$ there exists a unique morphism $\theta \in \text{Mor}((A \sqcup C, h, B), (X, j, B))$ such that $\theta \circ \alpha = \gamma$ and $\theta \circ \beta = \delta$

¹ It could be obtained as the dual claim of Corollary 5.8 in [4], p. 82, for example.



In the following result, we need the notion of a tensor algebra, which is explained in more details in [3]. Suppose that $(A, f, B), (C, g, B) \in \text{Ob}(\mathcal{S}(B))$, let T be the tensor algebra of A and C and define a map $h_T : T \rightarrow B$ as follows:

$$h_T(t) = \sum_{i=1}^n \sum_{j=1}^{k_i} \prod_{l=1}^{N_i} \tilde{h}_T(t_{i,j,l})$$

for every element

$$t = \bigoplus_{i=1}^n \sum_{j=1}^{k_i} t_{i,j,1} \otimes \cdots \otimes t_{i,j,N_i}$$

of T , where

$$\tilde{h}_T(t_{i,j,l}) = \begin{cases} f(t_{i,j,l}), & \text{if } t_{i,j,l} \in A \\ g(t_{i,j,l}), & \text{if } t_{i,j,l} \in C \end{cases}$$

On algebra T we consider the topology $\tau_{h_T} = \{h_T^{-1}(U) : U \in \tau_B\}$, where τ_B denotes the topology of B . Then (T, τ_{h_T}) becomes a topological algebra and h_T becomes a continuous algebra homomorphism in the topology τ_{h_T} (see [3] for details).

Lemma 1. Let $(A, f, B), (C, g, B) \in \text{Ob}(\mathcal{S}(B))$ and let T be the tensor algebra of A and C . Define the map $h_T : T \rightarrow B$ as in (0.1) and equip T with the topology τ_{h_T} . Let I be the two-sided ideal of T , generated by the set

$$\{a_1 \otimes a_2 - a_1 a_2, c_1 \otimes c_2 - c_1 c_2 : a_1, a_2 \in A, c_1, c_2 \in C\}$$

and $A \sqcup C = T/I$ be equipped with the quotient topology. Let $\kappa_I : T \rightarrow T/I$ be the quotient map. Then the triple $(A \sqcup C, h, B)$, where $h(\kappa_I(t)) = h_T(t)$ for every $t \in T$ and every $\kappa_I(t) \in A \sqcup C$, is an object of the category $\mathcal{S}(B)$.

Proof. For the proof, see the proof of Lemma 2.2 in [3]. □

The next Proposition describes the coproducts of two elements in the category $\mathcal{S}(B)$.

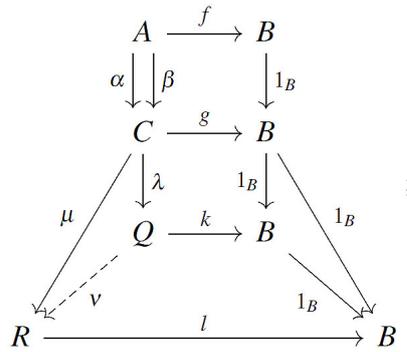
Proposition 1. For any $(A, f, B), (C, g, B) \in \text{Ob}(\mathcal{S}(B))$, their coproduct in $\mathcal{S}(B)$ exists and is the triple $((A \sqcup C, h, B), \alpha, \beta)$, where $(A \sqcup C, h, B)$ is the object of $\mathcal{S}(B)$, described in Lemma 1, $\alpha : A \rightarrow A \sqcup C$ and $\beta : C \rightarrow A \sqcup C$ are morphisms, defined by $\alpha(a) = \kappa_I(a), \beta(c) = \kappa_I(c)$ for all $a \in A$ and $c \in C$, where κ_I is the quotient map, defined in Lemma 1.

Proof. For the proof, see the proof of Proposition 3.2 in [3]. □

Now we move on to the coequalizers.

Definition 3. Let $(A, f, B), (C, g, B) \in \text{Ob}(\mathcal{S}(B))$. The **coequalizer** of morphisms $\alpha, \beta \in \text{Mor}((A, f, B), (C, g, B))$ is a pair $((Q, k, B), \lambda)$ such that

(1) $(Q, k, B) \in \text{Ob}(\mathcal{S}(B))$ and $\lambda \in \text{Mor}((C, g, B), (Q, k, B))$ with $\lambda(\alpha(a)) = \lambda(\beta(a))$ for every $a \in A$



(2) for any pair $((R, l, B), \mu)$, where $(R, l, B) \in \text{Ob}(\mathcal{S}(B))$ and $\mu \in \text{Mor}((C, g, B), (R, l, B))$ with $\mu(\alpha(a)) = \mu(\beta(a))$ for every $a \in A$, there exists unique $v \in \text{Mor}((Q, k, B), (R, l, B))$ with $v \circ \lambda = \mu$.

Next Proposition describes the coequalizers in the category $\mathcal{S}(B)$.

Proposition 2. Let $(A, f, B), (C, g, B) \in \text{Ob}(\mathcal{S}(B))$ and I be the smallest two-sided ideal of C , generated by the set

$$M = \{\alpha(a) - \beta(a) : a \in A\}.$$

Then the coequalizer of morphisms $\alpha, \beta \in \text{Mor}((A, f, B), (C, g, B))$ is the pair $((C/I, \tilde{g}, B), p)$, where $\tilde{g} : C/I \rightarrow B$ is defined by $\tilde{g}([c]) = g(c)$ for each $[c] \in C/I$, $p : C \rightarrow C/I$ is the canonical projection and C/I is equipped with the quotient topology

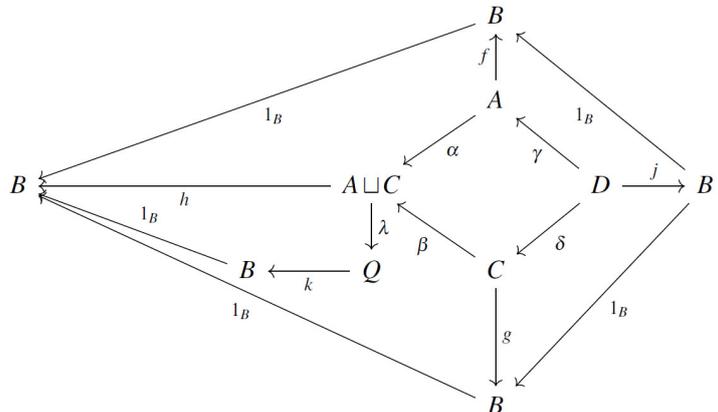
$$\tau_{C/I} = \{V \subseteq C/I : p^{-1}(V) \in \tau_C\}.$$

Proof. For the proof, see the proof of Theorem 10 in [2]. □

Thus, the conditions P1) and P2) are fulfilled for the category $\mathcal{S}(B)$. Hence, we can state the main result of this paper.

Theorem 1. The pushouts exist always in the category $\mathcal{S}(B)$.

To illustrate the situation, we give the following commutative diagram,



which describes the pushout of morphisms γ and δ , if one compares this diagram with diagrams given in the Definitions 1–3 and takes $\rho = \lambda \circ \alpha$ and $\sigma = \lambda \circ \beta$ in the diagram of Definition 1.

CONCLUSION

In this paper we showed that the pushouts always exist in the category $\mathcal{S}(B)$.

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Väljatõukajatest Segali topoloogiliste algebrate kategoorias $\mathcal{S}(B)$

Mart Abel

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