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MATHEMATICS

Novel results on a fixed function and their application based on the best approximation of the treatment plan for tumour patients getting intensity modulated radiation therapy (IMRT)

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Abstract. In the present paper, we extend the concept of contraction in a new manner by introducing $\mathfrak{D}$-contraction defined on a family $\mathfrak{F}$ of bounded functions. We also introduce a new notion of a fixed function on a metric space. Some fixed function theorems along with illustrative examples and application are also given to verify the effectiveness of our results.

Key words: fixed function, complete metric space, $\mathfrak{D}$-contraction, $\alpha-\psi$ contractive mapping.

1. INTRODUCTION AND PRELIMINARIES

The concept of *Contraction* in the field of fixed point theory was first introduced by Banach [2] in 1922. His principle, known as *Banach Contraction*, ensures that the application of a continuous self mapping on two points of a complete metric space contracts the distance between these points. After that, many authors gave extensions to this result by presenting more robust contractive conditions. For more details, references [1,3,5–8,10,12] can be cited.

This paper deals with a unique approach in the field of contraction mappings introduced with a family of bounded functions. The contents of this paper are divided into four sections. Section 1 is concerned with some basic definitions and results related to this paper. In section 2, main results are presented with some illustrative examples whereas section 3 deals with an application to medical science. The last section of this paper presents the conclusion.

In order to prove the main results, we need some basic concepts, definitions, and results from the literature.

Definition 1.1. [2] *For a metric space (X,d), a mapping $T: X \to X$ is called a contraction mapping on X if for any real number λ with $0 \leq \lambda < 1$, the following inequality holds:*

$$d(Tx,Ty) \leq \lambda d(x,y) \quad for \ \ all \ \ x,y \in X.$$

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Remark 1.2. It can be easily seen that the distance between the images of any two points of a given set is contracting by a uniform factor $\lambda < 1$.

Example 1.3. [2] Let $X = \mathbb{R}^2$ be a set equipped with standard metric d

$$d((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \text{ for all } x_1,x_2,y_1,y_2 \in X$$

and $T:\mathbb{R}^2 \to \mathbb{R}^2$ be the mapping defined as $Tx = \frac{3}{8}x$ for all $x \in \mathbb{R}^2$ where $x = (x_1,x_2)$. Then T is a contraction on X as $d(Tx,Ty) = \frac{3}{8}\sqrt{(x_1-y_1)^2+(x_2-y_2)^2} = \frac{3}{8}d(x,y)$.

Theorem 1.4. [2] *Let (X,d) be a complete metric space and T be the contraction mapping defined on X. Then T possesses a unique fixed point x in X, i.e. $Tx = x$.*

After this well-known result, Reich [9] presented the following theorem:

Theorem 1.5. [9] *Let (X,d) be a complete metric space and T be the self mapping defined on X which satisfy the condition*

$$d(Tx,Ty) \leq \alpha d(x,Tx) + \beta d(y,Ty) + \gamma d(x,y)$$

for all $x,y \in X$ and α,β,γ nonnegative with $\alpha+\beta+\gamma < 1$. Then T admits a unique fixed point in X.

In 2012, Samet et al. [11] obtained some fixed point results by defining $\alpha-\psi$ contractive mapping as follows:

Definition 1.6. [11] *Let Ψ be the family of all functions $\psi:[0,+\infty)\to[0,+\infty)$ satisfying the following properties:*
(1) $\sum_{n=1}^{+\infty}\psi^n(t) < +\infty$ for every $t > 0$, where ψ^n is the n^{th} iterate of ψ;
(2) ψ is nondecreasing.

Definition 1.7. [11] *Let (X,d) be a metric space and T be a self mapping defined on X. The mapping T is said to be an $\alpha-\psi$ contractive mapping if there exist two functions $\alpha: X\times X \to [0,+\infty)$ and $\psi \in \Psi$ satisfying*

$$\alpha(x,y)d(Tx,Ty) \leq \psi(d(x,y)) \;\; for \;\; all \;\; x,y \in X.$$

Definition 1.8. [11] *Let $T: X\to X$ and $\alpha: X\times X \to [0,+\infty)$. The mapping T is known as α-admissible mapping if*

$$\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1 \; for \;\; every \;\; x,y \in X.$$

2. MAIN RESULTS

This section presents some fixed function theorems using the notions of a fixed function and $\mathfrak{D}$-contraction.

Definition 2.1. ***Fixed function***: *Let $\mathfrak{D}$ be any self mapping defined on a family of functions $\mathfrak{F}$, then $f \in \mathfrak{F}$ is said to be a fixed function of $\mathfrak{D}$ if $\mathfrak{D}f = f$.*

Example 2.2. Let $U = [1,2]$ and the mapping $\mathfrak{D}$ be defined as $\mathfrak{D}f(u) = f^2(u) - 2f(u) + 2$ for all $f \in \mathfrak{F}$ and $u \in U$. Then $f(u) = 2$ for all $u \in U$ and $f(u) = 1$ for all $u \in U$ are two fixed functions of $\mathfrak{D}$.

Example 2.3. Let $U = \mathbb{R}^+$ and $\mathfrak{D}$ be the self mapping on $\mathfrak{F}$. Let $f \in \mathfrak{F}$ be a function defined on U as

$$f(u) = \begin{cases} -1 & 0 \le u \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then f^3 is a fixed function of $\mathfrak{D}$.

Definition 2.4. *Let $(U, \hat{d})$ be a complete metric space and let $\mathfrak{F}$ be the collection of all bounded functions defined on U. Let $\mathfrak{D}$ be any self mapping on $\mathfrak{F}$. Then the given mapping is called $\mathfrak{D}$-contraction mapping on $\mathfrak{F}$ if for any real number $\lambda \in [0,1)$, we have*

$$d^*(\mathfrak{D}f, \mathfrak{D}g) \le \lambda d^*(f,g) \;\; for \;\; all \;\; f,g \in \mathfrak{F},$$

where

$$d^*(f,g) = \sup\{\hat{d}(f(u), g(v)) |\, u,v \in U\} = \sup\{|f(u) - g(v)|\; |u,v \in U\}. \tag{2.1}$$

Remark 2.5. Clearly, d^* is a metric on $\mathfrak{F}$ as $d^*(f,g) = 0 \Leftrightarrow f \sim g$ for all $f,g \in \mathfrak{F}$. Also, for all $u,v \in U$ and $f,g,h \in \mathfrak{F}$,

$$\begin{aligned} |f(u) - g(v)| &\le |f(u) - h(w)| + |h(w) - g(v)| \\ &\le \sup\{|f(u) - h(w)|\; |u,w \in U\} \\ &\quad + \sup\{|h(w) - g(v)|\; |w,v \in U\} \\ \Rightarrow \sup\{|f(u) - g(v)|\; |u,v \in U\} &\le \sup\{|f(u) - h(w)|\; |u,w \in U\} \\ &\quad + \sup\{|h(w) - g(v)|\; |w,v \in U\} \\ \Rightarrow d^*(f,g) &\le d^*(f,h) + d^*(h,g). \end{aligned}$$

Theorem 2.6. *Let $(U, \hat{d})$ be a complete metric space with metric $\hat{d}$ defined as $\hat{d}(u,v) = |u - v|$ for all $u,v \in U$. Let $\mathfrak{F}$ be the collection of all bounded functions f defined on U with metric d^*(as defined in (2.1)). Also, let $\mathfrak{D}$ be the $\mathfrak{D}$-contraction mapping defined on $\mathfrak{F}$. Then there exists a unique fixed function $f \in \mathfrak{F}$, i.e. there exists some $f \in \mathfrak{F}$ such that $\mathfrak{D}f = f$.*

Proof. Let f,g be any two functions from the family $\mathfrak{F}$. Since $\mathfrak{D}$ is the $\mathfrak{D}$-contraction mapping on $\mathfrak{F}$, there exists a real number $\lambda \in [0,1)$ such that

$$d^*(\mathfrak{D}f, \mathfrak{D}g) \le \lambda d^*(f,g) \text{ for all } f,g \in \mathfrak{F},$$

where

$$d^*(f,g) = \sup\{\hat{d}(f(u), g(v)) |\, u,v \in U\}.$$

This further implies that

$$\begin{aligned} d^*(\mathfrak{D}^2 f, \mathfrak{D}^2 g) &\le \lambda d^*(\mathfrak{D}f, \mathfrak{D}g) \\ &\le \lambda^2 d^*(f,g) \text{ for all } f,g \in \mathfrak{F}. \end{aligned}$$

Continuing in the same manner, we get

$$d^*(\mathfrak{D}^n f, \mathfrak{D}^n g) \le \lambda^n d^*(f,g) \text{ for all } f,g \in \mathfrak{F}. \tag{2.2}$$

***Step I**: We will show that $\{f_n\}_{(n\in\mathbb{N})}$ is a Cauchy sequence.*
Let f_0 be any function in $\mathfrak{F}$. Let us define the sequence $\{f_n\}_{(n\in\mathbb{N})}$ by setting

$$f_1 = \mathfrak{D}(f_0),$$

$$f_2 = \mathfrak{D}(f_1) = \mathfrak{D}^2(f_0),$$

$$\cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot$$

$$f_n = \mathfrak{D}(f_{n-1}) = \mathfrak{D}^2(f_{n-2}) = ... = \mathfrak{D}^n(f_0).$$

Let $p,q \in N$ be some positive integers with $p > q$. Let $p = q+t$, where $t \geq 1$. Now,

$$\begin{aligned}
d^*(f_q,f_p) &= d^*(f_q,f_{q+t}) \\
&\leq d^*(f_q,f_{q+1}) + d^*(f_{q+1},f_{q+2}) + ... + d^*(f_{q+t-1},f_{q+t}) \\
&= d^*(\mathfrak{D}^q f_0,\mathfrak{D}^q f_1) + d^*(\mathfrak{D}^{q+1} f_0,\mathfrak{D}^{q+1} f_1) + ... + d^*(\mathfrak{D}^{q+t-1} f_0,\mathfrak{D}^{q+t-1} f_1) \\
&\leq \lambda^q d^*(f_0,f_1) + \lambda^{q+1} d^*(f_0,f_1) + ... + \lambda^{q+t-1} d^*(f_0,f_1) \qquad \text{(using (2.2))} \\
&= \lambda^q d^*(f_0,f_1).[1+\lambda+\lambda^2+...+\lambda^{t-1}] \\
&\leq \frac{\lambda^q}{1-\lambda} d^*(f_0,f_1), \text{ where } \lambda < 1.
\end{aligned}$$

Since $\mathfrak{F}$ is a family of bounded functions, $d^*(f_q,f_p) \to 0$ *as* p , $q \to \infty$. Hence, $\{f_n\}_{(n\in\mathbb{N})}$ is a Cauchy sequence in $\mathfrak{F}$.

***Step II**: Existence of a fixed function.*
As $\mathfrak{F}$ is the family of bounded functions defined on the complete metric space $(U,\hat{d})$, $(\mathfrak{F},d^*)$ is a complete metric space and thus the sequence $\{f_n\}_{(n\in\mathbb{N})}$ is convergent in $\mathfrak{F}$.

Let $f \in \mathfrak{F}$ be the limit of $\{f_n\}_{(n\in\mathbb{N})}$, i.e. $\lim_{n\to\infty} f_n = f$. By the continuity of $\mathfrak{D}$, we get

$$\lim_{n\to\infty} \mathfrak{D} f_n = \mathfrak{D} f.$$

Also, $\mathfrak{D} f_n = f_{n+1} \to f$ as $n \to \infty$.

Thus, the uniqueness of the limit implies that $\mathfrak{D} f = f$. This shows that f is a fixed function of $\mathfrak{D}$.

***Step III**: Uniqueness of a fixed function.*
Let g be another fixed function of $\mathfrak{D}$, i.e. $\mathfrak{D} g = g$ and $f \nsim g$. Now,

$$\begin{aligned}
0 \leq d^*(f,g) &= d^*(\mathfrak{D} f,\mathfrak{D} g) \\
&\leq \lambda d^*(f,g) \\
&< d^*(f,g).
\end{aligned}$$

Thus, we arrive at a contradiction. Hence, f is a unique fixed function of $\mathfrak{D}$. □

Example 2.7. Let $U = \mathbb{R}$ and $\hat{d}$ be the metric defined on $\mathbb{R}$. Clearly, $(U,\hat{d})$ is a complete metric space. Let $\mathfrak{F}$ be the family of bounded functions defined on U and d^* be the metric on $\mathfrak{F}$ defined as

$$d^*(f,g) = \sup\{\hat{d}(f(u),g(v)) | \, u,v \in U\} = \sup\{|f(u)-g(v)| \, |u,v \in U\}.$$

It can be easily seen that $(\mathfrak{F}, d^*)$ is a complete metric space, being the family of bounded functions defined on the complete metric space $(U, \hat{d})$. Let

$$f(u) = \begin{cases} 1 & u \text{ is rational} \\ 0 & u \text{ is irrational} \end{cases}$$

and

$$g(u) = \begin{cases} -1 & u \text{ is rational} \\ 0 & u \text{ is irrational.} \end{cases}$$

Let the mapping $\mathfrak{D}$ be defined as $\mathfrak{D}f = f^2$ for all $f \in \mathfrak{F}$. Then, we only need to show that the mapping $\mathfrak{D}$ is a $\mathfrak{D}$-contraction mapping. For this, we have

$$\begin{aligned} d^*(\mathfrak{D}f, \mathfrak{D}g) &= \sup\{\, \hat{d}(\mathfrak{D}f(u), \mathfrak{D}g(v)) \,|u, v \in U\} \\ &= \sup\{|f^2(u) - g^2(v)| \,|u, v \in U\} \\ &\leq \lambda \sup\{|f(u) - g(v)| \,|u, v \in U\} \text{ where } 0 \leq \lambda < 1 \\ \Rightarrow d^*(\mathfrak{D}f, \mathfrak{D}g) &\leq \lambda d^*(f, g). \end{aligned}$$

Since all the conditions required for Theorem 2.6 are fulfilled, there exists a unique fixed function of $\mathfrak{D}$. In this example, f^2, f^4, f^6 etc. yield the same fixed function of $\mathfrak{D}$.

Example 2.8. Let $U = [0, 1]$ and $\hat{d}$ be the metric defined on U. Let $\mathfrak{F} = C[0, 1]$ (i.e. the set of all real-valued continuous functions defined on [0,1]) and the mapping $\mathfrak{D} : \mathfrak{F} \to \mathfrak{F}$ be defined as

$$\mathfrak{D}f(u) = \frac{2}{3} f(u) \text{ for all } f \in \mathfrak{F} \text{ and } u \in [0, 1].$$

Here, $(U, \hat{d})$ is a complete metric space and $\mathfrak{F} = C[0, 1]$ is the collection of all real-valued continuous (and hence bounded) functions defined on $U = [0, 1]$. Let $f_n(u) = \frac{u^n}{n}$ for all $u \in [0, 1]$. Then $\{f_n(u)\}_{(u \in [0,1])}$ is a uniformly convergent sequence in $\mathfrak{F}$ and therefore is a Cauchy sequence. Also, the given mapping is a $\mathfrak{D}$-contraction mapping as

$$\begin{aligned} d^*(\mathfrak{D}f, \mathfrak{D}g) &= d^*\left(\frac{2}{3} f, \frac{2}{3} g\right) \\ &= \sup\left\{\hat{d}\left(\frac{2}{3} f(u), \frac{2}{3} g(v)\right) \,|u, v \in U\right\} \\ &= \frac{2}{3} \sup\{\, \hat{d}(f(u), g(v)) \,|u, v \in U\} \\ &< \lambda d^*(f, g) \text{ for } \frac{2}{3} < \lambda < 1. \end{aligned}$$

Since all the conditions required for Theorem 2.6 are fulfilled, there exists a unique fixed function of $\mathfrak{D}$. In this example, null function is a unique fixed function.

Theorem 2.9. *Let $(U, \hat{d})$ be a complete metric space (where $\hat{d}$ is the metric as defined earlier) and $\mathfrak{F}$ be the collection of all bounded functions f defined on U with metric d^*(as defined in (2.1)). Also, let $\mathfrak{D}$ be the modified $\mathfrak{D}$-contraction mapping on $\mathfrak{F}$ satisfying*

$$d^*(\mathfrak{D}f, \mathfrak{D}g) \leq \alpha d^*(f, \mathfrak{D}f) + \beta d^*(g, \mathfrak{D}g) + \gamma d^*(f, g)$$

for all $f, g \in \mathfrak{F}$; α, β, γ nonnegative with $\alpha + \beta + \gamma < 1$. Then $\mathfrak{D}$ has a unique fixed function.

Proof. Let us define a sequence $\{f_n\}_{(n\in\mathbb{N})}$ of functions of $\mathfrak{F}$ in the following way.

Let $f_0 \in \mathfrak{F}$ be any arbitrary function and $f_n = \mathfrak{D} f_{n-1} = \mathfrak{D}^n f_0$.

Step I*: It is shown that $\{f_n\}_{(n\in\mathbb{N})}$ is a Cauchy sequence in $\mathfrak{F}$. For this, consider*

$$\begin{aligned}
d^*(f_1,f_2) &= d^*(\mathfrak{D}f_0,\mathfrak{D}f_1)\\
&\leq \alpha d^*(f_0,\mathfrak{D}f_0)+\beta d^*(f_1,\mathfrak{D}f_1)+\gamma d^*(f_0,f_1)\\
&= \alpha d^*(f_0,f_1)+\beta d^*(f_1,f_2)+\gamma d^*(f_0,f_1)\\
&= (\alpha+\gamma)d^*(f_0,f_1)+\beta d^*(f_1,f_2)\\
\Rightarrow (1-\beta)d^*(f_1,f_2) &\leq (\alpha+\gamma)d^*(f_0,f_1)\\
\Rightarrow d^*(f_1,f_2) &\leq \left(\frac{\alpha+\gamma}{1-\beta}\right)d^*(f_0,f_1) \qquad \text{(where} \quad \beta<1).
\end{aligned}$$

Similarly

$$\begin{aligned}
d^*(f_2,f_3) &\leq \left(\frac{\alpha+\gamma}{1-\beta}\right)d^*(f_1,f_2)\\
&\leq \left(\frac{\alpha+\gamma}{1-\beta}\right)^2 d^*(f_0,f_1)
\end{aligned}$$

and so on.

As $\left(\frac{\alpha+\gamma}{1-\beta}\right)<1$ and $f_0, f_1 \in \mathfrak{F}$ are bounded, $\{f_n\}_{(n\in\mathbb{N})}$ is a Cauchy sequence in $\mathfrak{F}$. Since $\mathfrak{F}$ is complete, being the family of bounded functions defined on a complete metric space $(U,\hat{d})$, the sequence $\{f_n\}_{(n\in\mathbb{N})}$ is convergent in $\mathfrak{F}$ (say it converges to $f \in \mathfrak{F}$).

Step II*: Existence of a fixed function.*

Now it will be shown that f is a fixed function of $\mathfrak{D}$. Let s be any arbitrary positive integer. Now,

$$\begin{aligned}
d^*(f,\mathfrak{D}f) &\leq d^*(f,f_s)+d^*(f_s,\mathfrak{D}f)\\
&= d^*(f,f_s)+d^*(\mathfrak{D}f_{s-1},\mathfrak{D}f)\\
&= d^*(f,f_s)+d^*(\mathfrak{D}f,\mathfrak{D}f_{s-1})\\
\Rightarrow \quad & d^*(f,\mathfrak{D}f) \leq d^*(f,f_s)+\alpha d^*(f,\mathfrak{D}f)+\beta d^*(f_{s-1},\mathfrak{D}f_{s-1})+\gamma d^*(f,f_{s-1})\\
\Rightarrow \quad & (1-\alpha)d^*(f,\mathfrak{D}f) \leq d^*(f,f_s)+\beta d^*(f_{s-1},\mathfrak{D}f_{s-1})+\gamma d^*(f,f_{s-1}).
\end{aligned}$$

The right side expression can be made arbitrarily small enough by taking s sufficiently large. Thus $0 \leq d^*(f,\mathfrak{D}f) < \varepsilon$. This implies that $d^*(f,\mathfrak{D}f)=0$, i.e. f is a fixed function of $\mathfrak{D}$.

Step III*: Uniqueness of a fixed function.*

Suppose $g \in \mathfrak{F}$ is another fixed function of $\mathfrak{D}$, i.e. $\mathfrak{D}g = g$ and $g \nsim f$. Then,

$$\begin{aligned}
d^*(f,g) &= d^*(\mathfrak{D}f,\mathfrak{D}g)\\
&\leq \alpha d^*(f,\mathfrak{D}f)+\beta d^*(g,\mathfrak{D}g)+\gamma d^*(f,g)\\
\Rightarrow (1-\gamma)d^*(f,g) &\leq 0 \qquad \text{(where } \gamma<1)\\
\Rightarrow d^*(f,g) &\leq 0,
\end{aligned}$$

which is a contradiction to our assumption. This implies that f is unique. □

In this paper, we have extended the concept of $\alpha-\psi$ contractive mapping in the following manner.

Definition 2.10. *The mapping $\mathfrak{D}:\mathfrak{F}\to\mathfrak{F}$ is said to be an $\alpha-\psi$ contractive mapping if there exist two functions $\alpha:U\times U\to[0,+\infty)$ and $\psi\in\Psi$ satisfying*

$$\alpha(f(u),g(v))d^*(\mathfrak{D}f,\mathfrak{D}g)\le\psi(d^*(f,g)) \tag{2.3}$$

for all $f,g\in\mathfrak{F}$ and $u,v\in U$.

Definition 2.11. *Let $\mathfrak{D}:\mathfrak{F}\to\mathfrak{F}$ and $\alpha:U\times U\to[0,+\infty)$. The mapping $\mathfrak{D}$ is called an α-admissible mapping if*

$$\alpha(f(u),g(v))\ge 1\Rightarrow\alpha(\mathfrak{D}f(u),\mathfrak{D}g(v))\ge 1$$

for every $f,g\in\mathfrak{F}$ and $u,v\in U$.

Theorem 2.12. *Let $(U,\hat{d})$ be a complete metric space and $\mathfrak{F}$ be the collection of all bounded functions f (defined on U) with metric d^* (as defined in (2.1)). Let $\mathfrak{D}:\mathfrak{F}\to\mathfrak{F}$ be an $\alpha-\psi$ contractive mapping. Also, suppose that*
(i) *$\mathfrak{D}$ is α-admissible,*
(ii) *there is some $f_0\in\mathfrak{F}$ for which $\alpha(f_0(u),\mathfrak{D}f_0(v))\ge 1$ for all $u,v\in U$,*
(iii) *$\mathfrak{D}$ is continuous.*
Then $\mathfrak{D}$ possesses a fixed function in $\mathfrak{F}$.

Proof. Let $f_0\in\mathfrak{F}$ be a function such that

$$\alpha(f_0(u),\mathfrak{D}f_0(v))\ge 1 \text{ for all } u,v\in U.$$

Define the sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathfrak{F}$ by $f_{n+1}=\mathfrak{D}f_n$ for every $n\in\mathbb{N}$. If $f_n=f_{n+1}$ for some $n\in\mathbb{N}$, then f_n is a fixed function of $\mathfrak{D}$. Let us assume that $f_n\neq f_{n+1}$ for every $n\in\mathbb{N}$.

As by condition (i) $\mathfrak{D}$ is α-admissible, for all $u,v\in U$ we have

$$\begin{aligned}\alpha(f_0(u),f_1(v)) &= \alpha(f_0(u),\mathfrak{D}f_0(v))\ge 1\\ \Rightarrow\alpha(\mathfrak{D}f_0(u),\mathfrak{D}f_1(v)) &= \alpha(f_1(u),f_2(v))\ge 1.\end{aligned}$$

By mathematical induction we get

$$\alpha(f_n(u),f_{n+1}(v))\ge 1 \text{ for all } \quad n\in\mathbb{N} \text{ and } u,v\in U. \tag{2.4}$$

Using (2.3) and (2.4),

$$\begin{aligned}d^*(f_n,f_{n+1}) &= d^*(\mathfrak{D}f_{n-1},\mathfrak{D}f_n)\\ &\le \alpha(f_{n-1}(u),f_n(v))d^*(\mathfrak{D}f_{n-1},\mathfrak{D}f_n)\\ &\le \psi(d^*(f_{n-1},f_n)).\end{aligned}$$

Repetition of the above process implies

$$d^*(f_n,f_{n+1})\le\psi^n(d^*(f_0,f_1)) \text{ for all } n\in\mathbb{N}.$$

Let $n>m\ge N$ for $N\in\mathbb{N}$. Using triangular inequality, we have

$$\begin{aligned}d^*(f_m,f_n) &\le d^*(f_m,f_{m+1})+d^*(f_{m+1},f_{m+2})+d^*(f_{m+2},f_{m+3})+\ldots+d^*(f_{n-1},f_n)\\ &\le \psi^m(d^*(f_0,f_1))+\psi^{m+1}(d^*(f_0,f_1))+\ldots+\psi^{n-1}(d^*(f_0,f_1))\\ &= \sum_{k=m}^{n-1}\psi^k(d^*(f_0,f_1)).\end{aligned}$$

As $\sum_{n=1}^{+\infty}\psi^n(u) < +\infty$ for each $u > 0$, $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathfrak{F}$, with $\mathfrak{F}$ being the collection of bounded functions defined on complete metric space $(U,\hat{d})$; $(\mathfrak{F},d^*)$ is itself a complete metric space. Therefore, there exists a function $f \in \mathfrak{F}$ such that $f_n \to f$ as $n \to +\infty$.

As $\mathfrak{D}$ is a continuous mapping, we have

$$\mathfrak{D}f_n \to \mathfrak{D}f \text{ as } n \to +\infty \quad \Rightarrow f_{n+1} \to \mathfrak{D}f \text{ as } n \to +\infty.$$

Since the limit of a convergent sequence is always unique, we have $f = \mathfrak{D}f$, i.e. f is a fixed function of $\mathfrak{D}$. This completes the proof. □

Example 2.13. Let $U = [0,2]$ and $\hat{d}(u,v) = |u-v|$. Let $\mathfrak{F}$ be the family of bounded functions on $[0,2]$ and $\mathfrak{D} : \mathfrak{F} \to \mathfrak{F}$ be defined as $\mathfrak{D}f = f^2$ and d^* be the metric defined on $\mathfrak{F}$ as

$$d^*(f,g) = \int_0^2 |f(u)-g(u)|du.$$

Let

$$f(u) = \begin{cases} 1 & u \in [0,1] \\ 0 & \text{otherwise,} \end{cases}$$

$$g(u) = \begin{cases} -1 & u \in [0,1] \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\alpha(f(u),g(v)) = \begin{cases} 2 & u \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $(U,\hat{d})$ is a complete metric space and $\mathfrak{D}$ is a continuous mapping. Moreover, f and g are bounded functions and $\mathfrak{D}$ is α-admissible as

$$\alpha(f(u),g(v)) \geq 1 \quad \Rightarrow \quad u \in [0,1]$$

and for $u \in [0,1]$, we have

$$\alpha(\mathfrak{D}f(u),\mathfrak{D}g(v)) = \alpha(f^2(u),g^2(v)) \geq 1.$$

Now we show that $\mathfrak{D}$ is an $\alpha - \psi$ contractive mapping. To prove this, let $\psi \in \Psi$ be a function defined as $\psi(u) = \frac{u}{2}$.

Case I: When $u \in [0,1]$, then

$$\begin{aligned} \alpha(f(u),g(v))d^*(\mathfrak{D}f,\mathfrak{D}g) &= 2\int_0^2 |f^2(u)-g^2(u)|du \\ &= 2\int_0^1 |f^2(u)-g^2(u)|du + 2\int_1^2 |f^2(u)-g^2(u)|du \\ &= 2(0)+2(0) = 0 \\ &\leq u = \psi(d^*(f,g)). \end{aligned}$$

Case II: When $u \in (1,2]$, then

$$\alpha(f(u),g(v))d^*(\mathfrak{D}f,\mathfrak{D}g) = 0 \leq \psi(d^*(f,g)).$$

Thus, all the conditions needed for Theorem 2.12 are fulfilled, so, there must exist a fixed function in $\mathfrak{F}$. In this example, f is a fixed function of $\mathfrak{D}$.

Uniqueness: By considering the following hypothesis, the uniqueness of a fixed function in Theorem 2.12 will be assured.
(H): for all $f, g \in \mathfrak{F}$, there exists $h \in \mathfrak{F}$ such that

$$\alpha(f(u), h(v)) \geq 1 \text{ and } \alpha(g(u), h(v)) \geq 1.$$

Theorem 2.14. *Adding condition* (H) *to the hypothesis of Theorem 2.12, we obtain the uniqueness of a fixed function of* $\mathfrak{D}$.

Proof. Let us suppose that f^* and g^* are two fixed functions of $\mathfrak{D}$. From (H) there exists some $h^* \in \mathfrak{F}$ such that

$$\alpha(f^*(u), h^*(v)) \geq 1 \text{ and } \alpha(g^*(u), h^*(v)) \geq 1. \tag{2.5}$$

Since $\mathfrak{D}$ is α-admissible, by (2.5) we have

$$\alpha(f^*(u), \mathfrak{D}^n h^*(v)) \geq 1 \text{ and } \alpha(g^*(u), \mathfrak{D}^n h^*(v)) \geq 1 \text{ for all } \in \mathbb{N}. \tag{2.6}$$

Using (2.6) and the $\alpha - \psi$ contractive condition

$$\begin{aligned} d^*(f^*, \mathfrak{D}^n h^*) &= d^*(\mathfrak{D} f^*, \mathfrak{D}(\mathfrak{D}^{n-1} h^*)) \\ &\leq \alpha(f^*(u), \mathfrak{D}^{n-1} h^*(v)) d^*(\mathfrak{D} f^*, \mathfrak{D}(\mathfrak{D}^{n-1} h^*)) \\ &\leq \psi(d^*(f^*, \mathfrak{D}^{n-1} h^*)), \end{aligned}$$

this implies that $d^*(f^*, \mathfrak{D}^n h^*) \leq \psi^n(d^*(f^*, h^*))$ for all $n \in \mathbb{N}$.

Taking the limit $n \to +\infty$, we get

$$\mathfrak{D}^n h^* \to f^*.$$

Similarly,

$$\mathfrak{D}^n h^* \to g^*.$$

The uniqueness of the limit gives $f^* = g^*$. This proves the theorem. □

3. APPLICATION

The application in this section is based on the best approximation of the treatment plan for tumour patients getting intensity modulated radiation therapy (IMRT). In this technique, a proper dose deposition coefficient (DDC) matrix truncation has been used, which significantly improves the accuracy of results. In 2013, Tian et al. [14] presented a fluence map optimization (FMO) model for dose calculation, by splitting the DDC matrix into two components, on the basis of a threshold intensity value. Following this concept, a sequence of functions can be constructed through the presented results, which contains different dose distributions corresponding to different patients and finally converges to a fixed function. The fixed function obtained in this application represents the suitable doses of a number of tumour patients at the same time. Bortfeld [4] and Shepard et al. [13] also presented some useful techniques to develop algorithms for the problems encountered in tomotherapy.

In these techniques, a DDC matrix is often computed to approximate dose distribution to each voxel in the required volume of interest from every beamlet with unit intensity. But we usually get a large set of data during calculation that requires a huge computer memory and computational efficiency. As a result, small values from the DDC matrix are usually truncated, which affects the quality of the treatment plan.

The fixed point iteration method is a very efficient and effective technique to solve this problem. In this technique, a proper DDC matrix truncation is used, which significantly improves the accuracy of results. In the FMO model of Tian et al. [14] the DDC matrix was divided into two components $\mathfrak{D}_1$ and $\mathfrak{D}_2$ on the basis of a threshold value. The matrix $\mathfrak{D}_1$ (major component) consists of those values of the DDC matrix which are higher than the threshold, whereas the minor component $\mathfrak{D}_2$ consists of the remaining values. In fact, $\mathfrak{D}_1$ represents those doses which correspond to tumour area voxels (specifically), while $\mathfrak{D}_2$ represents scatter doses passing at large distances. The problem can be interpreted as:

$$x^{(k+1)} = \text{argmin}_x |\mathfrak{D}_1 x + \delta^{(k)} - T|, \tag{3.1}$$

$$\delta^{(k+1)} = \mathfrak{D}_2 x^{(k+1)}. \tag{3.2}$$

Equation (3.1) represents an inner loop that can be solved by using an iterative algorithm for value $\delta^{(k)}$, which is the dose value corresponding to $\mathfrak{D}_2$. The matrix $\mathfrak{D}_1$ contains a much reduced number of non-zero elements as compared to the DDC full matrix. So, the inner loop will converge more quickly than the original matrix. The outer loop, represented by equation (3.2), updates the values of $\delta^{(k+1)}$ using the minor matrix $\mathfrak{D}_2$. The symbol T represents the prescription dose for planned target volume voxels and the threshold dose for organs at risk voxels. This mapping gives rise to a sequence $x^{(0)}, x^{(1)}, x^{(2)}, \ldots$ containing different dose distributions corresponding to a patient. If the matrix $\mathfrak{D}_2$ contains very small (or negligible) values and there exists such a λ for which the contraction condition is satisfied, then the suitable dose exists for a patient at a time. Following this concept, the treatment plan for more than a patient at a time is presented through our results in a more effective way. The results proposed in this paper provide a very efficient and easy technique for the estimation of a suitable treatment plan.

In the present case, two tumour patients were considered with different tumour levels. Let U denote the set of all threshold intensity values (with unit Gy) to be given on particular days and in particular sessions. A patient is getting the therapy two times a day. Days and sessions are denoted by D and S, respectively:

$$U = \begin{cases} (1, D_1S_1), (\frac{1}{2}, D_1S_2), (1, D_2S_1), (\frac{1}{2}, D_2S_2) & \text{Patient-I ,} \\ (1, D_1S_1), (2, D_1S_2), (1, D_2S_1), (2, D_2S_2) & \text{Patient-II .} \end{cases}$$

Note that U is complete, being a closed and bounded subset of $\mathbb{R}^2$. Let $\mathfrak{F} = \{f_1, f_2\}$ be the family of dose functions and each function represent different dose distributions (to tumour locations) of different tumour patients during IMRT:

$$f_1(u) = \begin{cases} 2u & \text{Patient-I,} \\ u & \text{Patient-II.} \end{cases} \text{ and } f_2(u) = \begin{cases} \frac{u}{3} & \text{Patient-I,} \\ \frac{2u}{3} & \text{Patient-II.} \end{cases}$$

It is to be noted that $\mathfrak{F}$ is the family of bounded functions. Let $\mathfrak{D} : \mathfrak{F} \to \mathfrak{F}$ be the mapping defined as $\mathfrak{D}f = f^2 - 2f + 2 \;\; \forall \;\; f \in \mathfrak{F}$. It is required to prove that $\mathfrak{D}$ is a $\mathfrak{D}$-contraction mapping. For $u, v \in U$, we have the following cases:

For Patient-I

Case I. When $u = v = 1$, then
$|\mathfrak{D}f_1 - \mathfrak{D}f_2| = \frac{5}{9}$ and $|f_1 - f_2| = \frac{5}{3}$.

Case II. When $u = v = \frac{1}{2}$, then
$|\mathfrak{D}f_1 - \mathfrak{D}f_2| = \frac{25}{36}$ and $|f_1 - f_2| = \frac{5}{6}$.

Case III. When $u = 1, \;\; v = \frac{1}{2}$, then
$|\mathfrak{D}f_1 - \mathfrak{D}f_2| = \frac{11}{36}$ and $|f_1 - f_2| = \frac{11}{6}$.

Case IV. When $u = \frac{1}{2}, \;\; v = 1$, then
$|\mathfrak{D}f_1 - \mathfrak{D}f_2| = \frac{4}{9}$ and $|f_1 - f_2| = \frac{2}{3}$.

For Patient-II

Case I. When $u = v = 1$, then
$|\mathfrak{D}f_1 - \mathfrak{D}f_2| = \frac{1}{9}$ and $|f_1 - f_2| = \frac{1}{3}$.

Case II. When $u = v = 2$, then
$|\mathfrak{D}f_1 - \mathfrak{D}f_2| = \frac{8}{9}$ and $|f_1 - f_2| = \frac{2}{3}$.

Case III. When $u = 1, \;\; v = 2$, then
$|\mathfrak{D}f_1 - \mathfrak{D}f_2| = \frac{1}{9}$ and $|f_1 - f_2| = \frac{1}{3}$.

Case IV. When $u = 2, \;\; v = 1$, then
$|\mathfrak{D}f_1 - \mathfrak{D}f_2| = \frac{8}{9}$ and $|f_1 - f_2| = \frac{4}{3}$.

From all above cases, for Patient-I

$$\begin{aligned} d^*(\mathfrak{D}f_1, \mathfrak{D}f_2) &= \sup\{|\mathfrak{D}f_1 - \mathfrak{D}f_2| \; u, v \in U\} \\ &= \frac{25}{36} \leq \frac{2}{3} \times \frac{11}{6} \\ &= \lambda d^*(f_1, f_2), \end{aligned}$$

and for Patient-II

$$\begin{aligned} d^*(\mathfrak{D}f_1, \mathfrak{D}f_2) &= \sup\{|\mathfrak{D}f_1 - \mathfrak{D}f_2| \; u, v \in U\} \\ &= \frac{8}{9} \leq \frac{2}{3} \times \frac{4}{3} \\ &= \lambda d^*(f_1, f_2), \end{aligned}$$

where $\lambda = \frac{2}{3} < 1$.

Thus, all the conditions required for Theorem 2.6 are fulfilled. Therefore, there exists a unique fixed function f_1 of $\mathfrak{D}$ that yields suitable doses for two patients at the same time.

4. CONCLUSIONS

Till now, fixed point results have a wide range of applications in various fields such as engineering, functional analysis, optimization theory, etc. However, the concept of a fixed function is yet not defined. In future, this new extended concept can be applied in various forms to prove the existence and uniqueness of fixed functions, i.e. by changing the nature of mappings, using different contractive conditions, by changing the space or by using different topological structures. The effectiveness of this result is directly given by the application which helps us to diagnose a number of patients at the same time.

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