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Right linear map preserving the left spectrum of 2×2 quaternion matrices

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Abstract. In this paper, the form of a right linear map preserving the left spectrum of quaternion matrices of order 2 is characterized. The obtained conclusion is different from the classical results of the linear map preserving eigenvalues of complex matrices.

Key words: linear preserving map, quaternion matrix, left spectrum.

1. INTRODUCTION

Let $\mathbb{R}$ and $\mathbb{C}$ be the fields of the real and complex numbers, respectively. The quaternion division ring over $\mathbb{R}$, denoted by $\mathbb{H}$, is the set of all elements with the form $a_0+a_1i+a_2j+a_3k$, where a_0,a_1,a_2, and $a_4\in\mathbb{R}$; moreover,

$$i^2=j^2=k^2=ijk=-1;$$

$$ij=-ji=k, jk=-kj=i, ki=-ik=j.$$

If $a=a_0+a_1i+a_2j+a_3k$, let

$$\overline{a}=a_0-a_1i-a_2j-a_3k,\ |a|=\sqrt{a_0^2+a_1^2+a_2^2+a_3^2},\ \text{and } \mathrm{Re}\,a=(a+\overline{a})/2$$

be the conjugate, modulus, and real part of a, respectively. It is clear that $\mathbb{R}\subset\mathbb{C}\subset\mathbb{H}$, and the multiplication operation of quaternions is noncommutative.

Let $M_n(\mathbb{R})$, $M_n(\mathbb{C})$, and $M_n(\mathbb{H})$ denote the set of $n\times n$ matrices over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, respectively. Clearly, $M_n(\mathbb{R})\subset M_n(\mathbb{C})\subset M_n(\mathbb{H})$. Let $E\in M_n(\mathbb{C})$ denote the identity matrix, $E_{ij}\in M_n(\mathbb{C})$ the matrix whose (i,j)th entry is 1 and the other entries are zero. If $A\in M_n(\mathbb{C})$, we write $\sigma_p(A)$ as the set of distinct complex eigenvalues of A and $tr_{\mathbb{C}}(A)$ as the trace of A. In addition, for $A=[a_{st}]\in M_n(\mathbb{H})$, let $A^T=[a_{ts}]$ be the

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transpose of A, observing that there exist unique $A_1, A_2 \in M_n(\mathbb{C})$ such that $A = A_1 + A_2 j$, thus we have $A^T = A_1^T + A_2^T j$.

Due to the noncommutativity of quaternions, there are two types of eigenvalues and linear maps: left and right eigenvalues of quaternion matrices and left and right quaternion linear maps. This paper only concerns left eigenvalues of a quaternion matrix and right quaternion linear maps, so their definitions are given below, but the introductions to right eigenvalues and left quaternion linear maps are omitted. Readers can refer to [1], [9], and [14] for more information about eigenvalues and linear maps related to the quaternion matrix.

Definition 1.1 ([9,14]). *Let $A \in M_n(\mathbb{H})$, $\lambda \in \mathbb{H}$ is called a left eigenvalue of A if $Ax = \lambda x$ for some nonzero $x \in \mathbb{H}^n$, where $\mathbb{H}^n$ is the set of vectors of n components over $\mathbb{H}$. The set of distinct left eigenvalues is called the left spectrum of A, denoted $\sigma_l(A)$.*

Definition 1.2 ([1,9]). *A map $\Phi : M_n(\mathbb{H}) \to M_n(\mathbb{H})$ is said to be a right quaternion linear map if Φ satisfies*

$$\Phi(A+B) = \Phi(A) + \Phi(B) \text{ and } \Phi(Aq) = \Phi(A)q$$

for all $A, B \in M_n(\mathbb{H})$ and $q \in \mathbb{H}$.

As for the studies of the left spectrum of a quaternion matrix, in 1985, Wood [13] used a topological method to show that the left eigenvalue always exists and demonstrated that left eigenvalues of a 2×2 quaternion matrix can be found by solving a quaternionic quadratic equation; in 2001, Huang and So [2] computed the left spectrum of a 2×2 quaternion matrix by solving quaternionic polynomials of degree 2; in 2005, So [10] also showed that the left spectrum of a 3×3 quaternion matrix can be found by this algebraic approach. So far it is still an open problem whether such algebraic approach works for general $n \times n$ quaternion matrices for $n \geq 4$. For other properties and applications about quaternions and quaternion matrices, readers can refer to [9,11,14] and references therein.

Linear preserver problems are the questions about characterizing linear maps on rings or algebras that preserve certain properties, which are a very old and active research area in matrix and operator theory. There has been a great deal of research in this area, especially on spaces of complex matrices. Here, we omit the detailed introduction to linear preserver problems. For some surveys related to the linear preserver problems, readers can consult [3–6,8,11].

From [9,12,14], we can see that there is a 2×2 complex matrix whose left spectrum is an infinite set, and the left spectrum of a quaternion matrix is not a similarity invariant in general, thus the classical result ([7, Theorem 3]) about a linear map preserving eigenvalues of complex matrices is not valid for the left spectrum of a quaternion matrix. So we will characterize the form of the linear map preserving the left spectrum of quaternion matrices in this paper.

Considering that the left spectrum of a 2×2 quaternion matrix can be found by the explicit formulas introduced in [2], and finding the left spectrum of a quaternion matrix is very difficult to deal with in general, actually so far there is no algorithm for computing the left spectrum of an $n \times n$ quaternion matrix for $n \geq 3$, so we have decided to work only with 2×2 quaternion matrices.

2. PRELIMINARIES

In order to prove the following Theorem 3.2, which characterizes the form of the linear map preserving the left spectrum of 2×2 quaternion matrices, we need the following lemmas. For convenience, some known results are also listed as Lemmas 2.1, 2.2, and 2.3.

Lemma 2.1 ([10, Lemma 3.1]). *Let $A \in M_n(\mathbb{H})$, and $X \in M_n(\mathbb{R})$ be invertible, then $\sigma_l(A) = \sigma_l(XAX^{-1})$.*

Lemma 2.2 ([12, Lemma II.5.1.1 (ii)]). *Let $A \in M_n(\mathbb{C})$, then $\sigma_l(A) \cap \mathbb{C} = \sigma_p(A)$.*

Lemma 2.3 ([2, Theorem 2.3, Corollary 3.7, Theorem 3.10])**.** *Let $A \in M_2(\mathbb{H})$ and* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(1) *If $bc = 0$, then $\sigma_l(A) = \{a, d\}$.*
(2) *If $bc \neq 0$, then $\sigma_l(A) = \{a + b\lambda : \lambda^2 + b^{-1}(a-d)\lambda - b^{-1}c = 0\}$.*
(3) *$\sigma_l(A)$ is infinite but with a unique modulus and real part if and only if $a, b, c, d \in \mathbb{R}$ such that $(d-a)^2 + 4bc < 0$.*
(4) *Furthermore, if $A \in M_2(\mathbb{C})$, then $\sigma_l(A)$ is finite if and only if $\sigma_l(A) = \sigma_p(A)$.*

Remark. The above Lemma 2.3 characterizes the left spectrum of a 2×2 quaternion matrix, and will be used many times in the proof of Lemma 2.4. In particular, Lemma 2.3 (3) shows that $\sigma_l(A)$ is an infinite set and all elements in $\sigma_l(A)$ are of the same modulus and real part if and only if $a, b, c, d \in \mathbb{R}$ and $(d-a)^2 + 4bc < 0$. Thus Lemma 2.3 (3) also gives a method for determining whether a 2×2 quaternion matrix is a real matrix.

Lemma 2.4. *Let Φ be a right quaternion linear map from $M_2(\mathbb{H})$ into itself. If $\sigma_l(A) = \sigma_l(\Phi(A))$ for all $A \in M_2(\mathbb{H})$, then $\Phi(A) \in M_2(\mathbb{C})$ for every $A \in M_2(\mathbb{C})$.*

Proof. Let

$$\Phi(E_{12}) = B_{12} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Since $\sigma_l(E_{12}) = \{0\}$, by the assumptions of Φ, one has $\sigma_l(E_{12}) = \sigma_l(B_{12}) = \{0\}$.

When $bc \neq 0$, note that $\sigma_l(B_{12}) = \{0\}$, then $a + b\lambda = 0$. If $a = 0$, then $b\lambda = 0$. Since $bc \neq 0$, we have $\lambda = 0$. By Lemma 2.3 (2), then $\lambda = 0$ satisfies the following equation:

$$\lambda^2 + b^{-1}(a-d)\lambda - b^{-1}c = 0.$$

Consequently, $c = 0$; this contradicts to $bc \neq 0$. Hence $a \neq 0$. By $a + b\lambda = 0$, then $\lambda = -b^{-1}a$. Again using Lemma 2.3 (2), then

$$(-b^{-1}a)^2 + b^{-1}(a-d)(-b^{-1}a) - b^{-1}c = 0.$$

Note that $a \neq 0$, by simple computation, we imply that $db^{-1} = ca^{-1}$ from the above equality.

Write $db^{-1} = t$, then

$$B_{12} = \begin{bmatrix} a & b \\ ta & tb \end{bmatrix}.$$

When $bc = 0$, since $\sigma_l(B_{12}) = \{0\}$, by Lemma 2.3 (1), we have

$$B_{12} = bE_{12} \text{ or } B_{12} = cE_{21}.$$

According to the above discussions, then B_{12} has three types of matrix representations. That is,

$$B_{12} = \begin{bmatrix} a & b \\ ta & tb \end{bmatrix}, \; B_{12} = bE_{12}, \text{ or } B_{12} = cE_{21}.$$

Let

$$\Phi(E_{21}) = B_{21} = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}.$$

Similar to the arguments of B_{12}, then B_{21} also has matrix representations

$$B_{21} = \begin{bmatrix} a' & b' \\ t'a' & t'b' \end{bmatrix}, \; B_{21} = b'E_{12}, \text{ or } B_{21} = c'E_{21}.$$

In the following, we give the proofs of B_{12} and $B_{21} \in M_2(\mathbb{R})$.

By Lemma 2.3 (3), we imply that $\sigma_l(E_{21}-nE_{12})$ is an infinite set and its elements are of the same modulus and real part for $n=1,2$. Note that

$$\sigma_l(E_{21}-nE_{12})=\sigma_l(\Phi(E_{21}-nE_{12}))=\sigma_l(\Phi(E_{21})-n\Phi(E_{12}))=\sigma_l(B_{21}-nB_{12}).$$

Thus $\sigma_l(B_{21}-nB_{12})$ is also an infinite set and its elements are of the same modulus and real part for $n=1,2$.

If

$$B_{12}=\begin{bmatrix} a & b \\ ta & tb \end{bmatrix},\ B_{21}=\begin{bmatrix} a' & b' \\ t'a' & t'b' \end{bmatrix},$$

apply Lemma 2.3 (3) to $B_{21}-B_{12}$ and $B_{21}-2B_{12}$, then

$$a-a',\ b-b',\ ta-t'a',\ tb-t'b'\in\mathbb{R}.$$

$$2a-a',\ 2b-b',\ 2ta-t'a',\ 2tb-t'b'\in\mathbb{R}.$$

By simple computation, then B_{12} and $B_{21}\in M_2(\mathbb{R})$.

If

$$B_{12}=\begin{bmatrix} a & b \\ ta & tb \end{bmatrix} \text{ and } B_{21}=b'E_{12},\ \text{or } B_{12}=\begin{bmatrix} a & b \\ ta & tb \end{bmatrix} \text{ and } B_{21}=c'E_{21},$$

also apply Lemma 2.3 (3) to $B_{21}-B_{12}$, we can obtain B_{12} and $B_{21}\in M_2(\mathbb{R})$.

If $B_{12}=bE_{12}$ and

$$B_{21}=\begin{bmatrix} a' & b' \\ t'a' & t'b' \end{bmatrix},$$

applying Lemma 2.3 (3) to $B_{21}-B_{12}$, then B_{12} and $B_{21}\in M_2(\mathbb{R})$.

If $B_{12}=bE_{12}$ and $B_{21}=b'E_{12}$, then $\sigma_l(B_{21}-B_{12})=\{0\}$. This gives a contradiction because $\sigma_l(B_{21}-B_{12})$ is an infinite set. Hence such case is impossible to arise.

If $B_{12}=bE_{12}$ and $B_{21}=c'E_{21}$, apply Lemma 2.3 (3) to $B_{21}-B_{12}$, then $b,c'\in\mathbb{R}$. Thus B_{12} and $B_{21}\in M_2(\mathbb{R})$.

For $B_{12}=cE_{21}$ and $B_{21}=\begin{bmatrix} a' & b' \\ t'a' & t'b' \end{bmatrix}$, $B_{12}=cE_{21}$ and $B_{21}=b'E_{12}$, as well as $B_{12}=cE_{21}$ and $B_{21}=c'E_{21}$, similar to the above arguments, we can show B_{12} and $B_{21}\in M_2(\mathbb{R})$.

Consequently, we conclude that $\Phi(E_{12})$ and $\Phi(E_{21})\in M_2(\mathbb{R})$ from the above proofs.

In the following, we show that $\Phi(E_{11})$ and $\Phi(E_{22})$ are also real matrices.

Let

$$\Phi(E_{11})=B_{11}=\begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix}.$$

Note that B_{12} and $B_{21}\in M_2(\mathbb{R})$, so we can write $B_{11}-B_{12}+B_{21}$ as

$$\begin{bmatrix} a''+x_{11} & b''+x_{12} \\ c''+x_{21} & d''+x_{22} \end{bmatrix},$$

where x_{11}, x_{12}, x_{21}, and $x_{22}\in\mathbb{R}$.

Apply Lemma 2.3 (3) to $E_{11}-E_{12}+E_{21}$, then $\sigma_l(E_{11}-E_{12}+E_{21})$ is an infinite set and its elements are of the same modulus and real part. Since $\Phi(E_{11}-E_{12}+E_{21})=B_{11}-B_{12}+B_{21}$ and

$$\sigma_l(E_{11}-E_{12}+E_{21}))=\sigma_l(B_{11}-B_{12}+B_{21}),$$

we also have that $\sigma_l(B_{11}-B_{12}+B_{21})$ is an infinite set and its elements are of the same modulus and real part. Again apply Lemma 2.3 (3) to $B_{11}-B_{12}+B_{21}$, then a'', b'', c'', and $d''\in\mathbb{R}$. Thus $\Phi(E_{11})\in M_2(\mathbb{R})$.

Similar to the proof of $\Phi(E_{11})\in M_2(\mathbb{R})$, we can show $\Phi(E_{22})\in M_2(\mathbb{R})$.

Let $A = (a_{ij}) \in M_2(\mathbb{C})$, note that Φ is a right quaternion linear map and $\Phi(E_{ij}) \in M_2(\mathbb{R})$ for $i, j = 1, 2$, then $\Phi(A) \in M_2(\mathbb{C})$. The proof is complete. $\square$

By Lemma 2.2 and 2.4, the following lemma 2.5 is valid.

Lemma 2.5. *Let Φ be a right quaternion linear map from $M_2(\mathbb{H})$ into itself. If $\sigma_l(A) = \sigma_l(\Phi(A))$ for all $A \in M_2(\mathbb{H})$, then $\sigma_p(A) = \sigma_p(\Phi(A))$ for all $A \in M_2(\mathbb{C})$.*

Lemma 2.6. *Let $X, Y \in M_2(\mathbb{C})$ with matrix representations*

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}, \; Y = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}.$$

If $\sigma_l(A) = \sigma_l(XAY)$ for all $A = tE_{12} + sE_{21} \in M_2(\mathbb{C})$, then

$$x_{11}y_{21},\ x_{11}y_{22},\ x_{21}y_{21},\ x_{21}y_{22},\ x_{12}y_{11},\ x_{12}y_{12},\ x_{22}y_{11},\ \textit{and}\ x_{22}y_{21} \in \mathbb{R}.$$

Proof. By simple computation, then

$$XAY = \begin{bmatrix} x_{12}sy_{11} + x_{11}ty_{21} & x_{12}sy_{12} + x_{11}ty_{22} \\ x_{22}sy_{11} + x_{21}ty_{21} & x_{22}sy_{12} + x_{21}ty_{22} \end{bmatrix}.$$

Take $s = 1$ and $t = -1$, then $A = E_{21} - E_{12}$. By Lemma 2.3 (3), then $\sigma_l(A)$ is an infinite set and its elements are of the same modulus and real part. Note that the assumption $\sigma_l(A) = \sigma_l(XAY)$, and again apply Lemma 2.3 (3) to $\sigma_l(XAY)$, then

$$\begin{aligned} &x_{12}y_{11} - x_{11}y_{21} \in \mathbb{R}, \quad x_{12}y_{12} - x_{11}y_{22} \in \mathbb{R}, \\ &x_{22}y_{11} - x_{21}y_{21} \in \mathbb{R}, \quad x_{22}y_{12} - x_{21}y_{22} \in \mathbb{R}. \end{aligned} \tag{1}$$

Take $s = 1$ and $t = -2$, similar to the above arguments, we can show

$$\begin{aligned} &x_{12}y_{11} - 2x_{11}y_{21} \in \mathbb{R}, \quad x_{12}y_{12} - 2x_{11}y_{22} \in \mathbb{R}, \\ &x_{22}y_{11} - 2x_{21}y_{21} \in \mathbb{R}, \quad x_{22}y_{12} - 2x_{21}y_{22} \in \mathbb{R}. \end{aligned} \tag{2}$$

With equalities (1) and (2), by simple computation, we can imply that Lemma 2.6 holds. $\square$

Lemma 2.7. *Let $X \in M_2(\mathbb{C})$ be invertible. If $\sigma_l(A) = \sigma_l(XAX^{-1})$ for all $A = tE_{12} + sE_{21} \in M_2(\mathbb{C})$ and $A = diag(s,t) \in M_2(\mathbb{C})$, then there exist $\theta \in [0, 2\pi)$ and an invertible matrix $B \in M_2(\mathbb{R})$ such that $X = e^{i\theta}B$.*

Proof. Note that $X \in M_2(\mathbb{C})$ is invertible, let

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}, \; X^{-1} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}.$$

Since $A = diag(s,t) \in M_2(\mathbb{C})$, by simple computation, one has

$$XAX^{-1} = \begin{bmatrix} x_{11}sy_{11} + x_{12}ty_{21} & x_{11}sy_{12} + x_{12}ty_{22} \\ x_{21}sy_{11} + x_{22}ty_{21} & x_{21}sy_{12} + x_{22}ty_{22} \end{bmatrix}.$$

Take $s = 1, t = 0$, by Lemma 2.3 (1), then $\sigma_l(A) = \{0, 1\}$. Since

$$Xdiag(1,0)X^{-1} \in M_2(\mathbb{C}) \text{ and } \sigma_l(A) = \sigma_l(XAX^{-1}),$$

by Lemma 2.3 (4), we have $\sigma_p(XAX^{-1}) = \{0, 1\}$. Hence, $tr_{\mathbb{C}}(XAX^{-1}) = 1$, that is

$$x_{11}y_{11} + x_{21}y_{12} = 1. \tag{3}$$

Take $s=0, t=1$, similar to the above arguments, we have

$$x_{12}y_{21}+x_{22}y_{22}=1. \tag{4}$$

Since $XX^{-1}=E$, by simple computation, one has

$$x_{11}y_{11}+x_{12}y_{21}=1 \text{ and } x_{21}y_{12}+x_{22}y_{22}=1. \tag{5}$$

By equalities (3), (4), and (5), then

$$x_{12}y_{21}=x_{21}y_{12} \text{ and } x_{11}y_{11}=x_{22}y_{22}. \tag{6}$$

Note that $X\in M_2(\mathbb{C})$, then there exist $\theta_{ij}\in[0,2\pi)$, $i,\ j=1,2$, such that

$$\begin{aligned} x_{11}&=|x_{11}|e^{i\theta_{11}}, \quad x_{12}=|x_{12}|e^{i\theta_{12}},\\ x_{21}&=|x_{21}|e^{i\theta_{21}}, \quad x_{22}=|x_{22}|e^{i\theta_{22}}. \end{aligned} \tag{7}$$

Case 1. $x_{11}x_{12}x_{21}x_{22}\neq 0$.

Since $x_{11}x_{12}x_{21}x_{22}\neq 0$, note that $X\in M_2(\mathbb{C})$ and the assumption $\sigma_l(A)=\sigma_l(XAX^{-1})$ for every $A=tE_{12}+sE_{21}\in M_2(\mathbb{C})$, by Lemma 2.6 and equality (7), we can imply that there exist a_{21}, a'_{21}, a_{22}, a'_{22}, b_{11}, b'_{11}, b_{12}, and $b'_{12}\in\mathbb{R}$ such that

$$\begin{aligned} y_{11}&=b_{11}e^{-i\theta_{12}}=b'_{11}e^{-i\theta_{22}},\\ y_{12}&=b_{12}e^{-i\theta_{12}}=b'_{12}e^{-i\theta_{22}}. \end{aligned} \tag{8}$$

$$\begin{aligned} y_{21}&=a_{21}e^{-i\theta_{11}}=a'_{21}e^{-i\theta_{21}},\\ y_{22}&=a_{22}e^{-i\theta_{11}}=a'_{22}e^{-i\theta_{21}}. \end{aligned} \tag{9}$$

By equalities (3) and (4), then $y_{11}\neq 0$ or $y_{12}\neq 0$; moreover, $y_{21}\neq 0$ or $y_{22}\neq 0$. Using equalities (9) and (8), one has

$$\theta_{11}=\theta_{21} \text{ or } \theta_{11}\pm\pi=\theta_{21}, \tag{10}$$

$$\theta_{12}=\theta_{22} \text{ or } \theta_{12}\pm\pi=\theta_{22}. \tag{11}$$

In addition, by equalities (4), (7), and (9), then

$$|x_{12}|a_{21}e^{i(\theta_{12}-\theta_{11})}+|x_{22}|a_{22}e^{i(\theta_{22}-\theta_{11})}=1.$$

Combining the above equality with equality (11), we obtain that

$$\theta_{11}=\theta_{12} \text{ or } \theta_{11}\pm\pi=\theta_{12}. \tag{12}$$

Let $\theta=\theta_{11}$, by equalities (10), (11), and (12), then X is one of the following matrices

$$e^{i\theta}\begin{bmatrix} |x_{11}| & |x_{12}|\\ |x_{21}| & |x_{22}| \end{bmatrix},\quad e^{i\theta}\begin{bmatrix} |x_{11}| & -|x_{12}|\\ |x_{21}| & -|x_{22}| \end{bmatrix},$$

$$e^{i\theta}\begin{bmatrix} |x_{11}| & |x_{12}|\\ |x_{21}| & -|x_{22}| \end{bmatrix},\quad e^{i\theta}\begin{bmatrix} |x_{11}| & -|x_{12}|\\ |x_{21}| & |x_{22}| \end{bmatrix},$$

$$e^{i\theta}\begin{bmatrix} |x_{11}| & |x_{12}|\\ -|x_{21}| & |x_{22}| \end{bmatrix},\quad e^{i\theta}\begin{bmatrix} |x_{11}| & -|x_{12}|\\ -|x_{21}| & -|x_{22}| \end{bmatrix},$$

$$e^{i\theta}\begin{bmatrix} |x_{11}| & |x_{12}|\\ -|x_{21}| & -|x_{22}| \end{bmatrix},\quad e^{i\theta}\begin{bmatrix} |x_{11}| & -|x_{12}|\\ -|x_{21}| & |x_{22}| \end{bmatrix}.$$

Consequently, there are invertible matrices $B\in M_2(\mathbb{R})$ and $\theta\in[0,2\pi)$ such that $X=e^{i\theta}B$.

Case 2. $x_{11}x_{12}x_{21}x_{22} = 0$.

(a) When $x_{11} = 0$. By equalities (3) and (5), then

$$x_{21}y_{12} = x_{12}y_{21} = 1. \tag{13}$$

By equalities (13) and (7), then there exist c_{12} and $c_{21} \in \mathbb{R}$ such that

$$y_{12} = c_{12}e^{-i\theta_{21}} \text{ and } y_{21} = c_{21}e^{-i\theta_{12}}.$$

By equality (13), then $x_{12} \neq 0$, $y_{12} \neq 0$. In terms of Lemma 2.6, we have $x_{12}y_{12} \in \mathbb{R}$. So there exists $c'_{12} \in \mathbb{R}$ such that $y_{12} = c'_{12}e^{-i\theta_{12}}$. Note that $y_{12} = c_{12}e^{-i\theta_{21}}$ and $y_{12} \neq 0$, then

$$\theta_{12} = \theta_{21} \text{ or } \theta_{12} \pm \pi = \theta_{21}. \tag{14}$$

Since $x_{11} = 0$, by equality (6), one has $x_{22}y_{22} = 0$. By Lemma 2.6, we know $x_{22}y_{21} \in \mathbb{R}$. If $x_{22} \neq 0$, then there exists $c'_{21} \in \mathbb{R}$ such that $y_{21} = c'_{21}e^{-i\theta_{22}}$. Since $y_{21} = c_{21}e^{-i\theta_{12}}$ and $y_{21} \neq 0$, we have

$$\theta_{12} = \theta_{22} \text{ or } \theta_{12} \pm \pi = \theta_{22}. \tag{15}$$

If $x_{22} = 0$, write $x_{22} = 0\,e^{i\theta_{12}}$. Consequently, let $\theta = \theta_{12}$, by equalities (7), (14), and (15), then X is one of the following matrices

$$e^{i\theta}\begin{bmatrix} 0 & |x_{12}| \\ |x_{21}| & |x_{22}| \end{bmatrix}, \quad e^{i\theta}\begin{bmatrix} 0 & |x_{12}| \\ |x_{21}| & -|x_{22}| \end{bmatrix},$$

$$e^{i\theta}\begin{bmatrix} 0 & |x_{12}| \\ -|x_{21}| & -|x_{22}| \end{bmatrix}, \quad e^{i\theta}\begin{bmatrix} 0 & |x_{12}| \\ -|x_{21}| & |x_{22}| \end{bmatrix}.$$

Hence, there are invertible matrices $B \in M_2(\mathbb{R})$ and $\theta \in [0, 2\pi)$ such that $X = e^{i\theta}B$.

(b) When $x_{22} = 0$. Similar to the proof of Case (a), we can show that there are invertible matrices $B \in M_2(\mathbb{R})$ and $\theta \in [0, 2\pi)$ such that $X = e^{i\theta}B$.

(c) When $x_{12} = 0$. By equalities (4), (6), and (7), then $x_{11}y_{11} = x_{22}y_{22} = 1$. Thus there exist d_{11} and $d_{21} \in \mathbb{R}$ such that $y_{11} = d_{11}e^{-i\theta_{11}}$ and $y_{22} = d_{22}e^{-i\theta_{22}}$. By Lemma 2.6, then $x_{11}y_{22}$ and $x_{21}y_{22} \in \mathbb{R}$. Hence there exists $d'_{11} \in \mathbb{R}$ such that $y_{22} = d'_{11}e^{-i\theta_{11}}$. Since $y_{22} \neq 0$, one has

$$\theta_{11} = \theta_{22} \textit{ or } \theta_{11} \pm \pi = \theta_{22}. \tag{16}$$

Note that $x_{21}y_{22} \in \mathbb{R}$, if $x_{21} \neq 0$, then there exists $d_{21} \in \mathbb{R}$ such that $y_{22} = d_{21}e^{-i\theta_{21}}$. Again using $y_{22} = d_{22}e^{-i\theta_{22}}$ and $y_{22} \neq 0$, then

$$\theta_{21} = \theta_{22} \textit{ or } \theta_{21} \pm \pi = \theta_{22}, \tag{17}$$

if $x_{21} = 0$, write $x_{21} = 0\,e^{i\theta_{11}}$. Consequently, let $\theta = \theta_{11}$, by equalities (7), (16), and (17), then X is of one of the following forms:

$$e^{i\theta}\begin{bmatrix} |x_{11}| & 0 \\ |x_{21}| & |x_{22}| \end{bmatrix}, \quad e^{i\theta}\begin{bmatrix} |x_{11}| & 0 \\ -|x_{21}| & |x_{22}| \end{bmatrix},$$

$$e^{i\theta}\begin{bmatrix} |x_{11}| & 0 \\ -|x_{21}| & -|x_{22}| \end{bmatrix}, \quad e^{i\theta}\begin{bmatrix} |x_{11}| & 0 \\ |x_{21}| & -|x_{22}| \end{bmatrix}.$$

Hence, there are invertible matrices $B \in M_2(\mathbb{R})$ and $\theta \in [0, 2\pi)$ such that $X = e^{i\theta}B$.

(d) When $x_{21} = 0$. Analogous to the proof of Case (c), we can also show that there are invertible matrices $B \in M_2(\mathbb{R})$ and $\theta \in [0, 2\pi)$ such that $X = e^{i\theta}B$.

In conclusion, by the above arguments, the proof is completed. $\square$

Lemma 2.8. *There exists $A \in M_2(\mathbb{H})$ such that $\sigma_l(A) \neq \sigma_l(A^T)$.*

Proof. Let A be the same as that of [14, Example 7.3], that is

$$A = \begin{bmatrix} 1 & i \\ j & k \end{bmatrix}.$$

If $Ax = 0$, where $x = (x_1, x_2)^T$, by simple computation, then $x = 0$. Consequently, $0 \notin \sigma_l(A)$. Take $y = (-i,\ k)^T$, then $A^T y = 0$, we have $0 \in \sigma_l(A^T)$. Hence $\sigma_l(A) \neq \sigma_l(A^T)$. □

3. RIGHT LINEAR MAP PRESERVING THE LEFT SPECTRUM

With the preparations in Section 2, we prove the following Theorem 3.2. For convenience, we also list [7, Theorem 3] as Lemma 3.1 in the following.

Lemma 3.1 ([7, Theorem 3])**.** *Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map, then $\sigma_p(A) = \sigma_p(\Phi(A))$ for all $A \in M_n(\mathbb{C})$ if and only if there exists an invertible matrix $X \in M_n(\mathbb{C})$ such that*

$$\Phi(A) = XAX^{-1} \text{ or } \Phi(A) = XA^TX^{-1}.$$

Theorem 3.2. *Let Φ be a right quaternion linear map from $M_2(\mathbb{H})$ into itself. Then $\sigma_l(A) = \sigma_l(\Phi(A))$ for all $A \in M_2(\mathbb{H})$ if and only if there exists an invertible matrix $B \in M_2(\mathbb{R})$ such that $\Phi(A) = BAB^{-1}$.*

Proof. The sufficiency follows from Lemma 2.1. In the following, we prove the necessity of Theorem 3.2.

If $\sigma_l(A) = \sigma_l(\Phi(A))$ for all $A \in M_2(\mathbb{H})$, by Lemma 2.5, we imply that

$$\sigma_p(A) = \sigma_p(\Phi(A))$$

for all $A \in M_2(\mathbb{C})$. By Lemma 3.1, then there exists an invertible matrix $X \in M_2(\mathbb{C})$ such that

$$\Phi(A) = XAX^{-1} \text{ or } \Phi(A) = XA^TX^{-1}$$

for all $A \in M_2(\mathbb{C})$.

Case 1. If $\Phi(A) = XAX^{-1}$ for all $A \in M_2(\mathbb{C})$.

Since $X \in M_2(\mathbb{C})$ and $\sigma_l(A) = \sigma_l(XAX^{-1})$ for all $A \in M_2(\mathbb{H})$, by Lemma 2.7, we imply that there exist invertible matrices $B \in M_2(\mathbb{R})$ and $\theta \in [0, 2\pi)$ such that $X = Be^{i\theta}$. Note that Φ is a right quaternion linear map; moreover, $A \in M_2(\mathbb{H})$ can be uniquely expressed as $A = A_1 + A_2 j$, where A_1 and $A_2 \in M_2(\mathbb{C})$, then

$$\Phi(A) = \Phi(A_1) + \Phi(A_2)j = XA_1X^{-1} + XA_2X^{-1}j. \tag{18}$$

Note that B is a real matrix, then $B^{-1}j = jB^{-1}$. Since $X = e^{i\theta}B$, $A_1, A_1 \in M_2(\mathbb{C})$, by equality (18), we have

$$\Phi(A) = BA_1B^{-1} + BA_2B^{-1}j = B(A_1 + A_2 j)B^{-1} = BAB^{-1}.$$

Case 2. If $\Phi(A) = XA^TX^{-1}$ for all $A \in M_2(\mathbb{C})$.

Let $X^{-1} = Y$, then $\sigma_l(A) = \sigma_l(\Phi(A)) = \sigma_l(XA^TY)$ for all $A \in M_2(\mathbb{C})$. Hence, $\sigma_l(A) = \sigma_l(XA^TY)$ for all $A = tE_{12} + sE_{21} \in M_2(\mathbb{C})$ and $A = diag(s,t) \in M_2(\mathbb{C})$. Since $tE_{12} + sE_{21} = (tE_{12} + sE_{21})^T$ and $diag(s,t) = diag(s,t)^T$, by Lemma 2.6 and 2.7, we obtain that there exist invertible matrices $B \in M_2(\mathbb{R})$ and $\theta \in [0, 2\pi)$ such that $X = e^{i\theta}B$. Similar to Case 1, we can show that

$$\Phi(A) = BA_1^TB^{-1} + BA_2^TB^{-1}j = B(A_1^T + A_2^T j)B^{-1} = BA^TB^{-1} \tag{19}$$

for all $A = A_1 + A_2 j$, where A_1 and $A_2 \in M_2(\mathbb{C})$. Since B is an invertible real matrix, moreover, $\sigma_l(A) = \sigma_l(\Phi(A))$ for all $A \in M_2(\mathbb{H})$, by equality (19) and Lemma 2.1, we have

$$\sigma_l(A) = \sigma_l(A^T)$$

for every $A \in M_2(\mathbb{H})$. By Lemma 2.8, a contradiction is yielded. Hence Case 2 is impossible to arise.

By the above arguments, the proof is completed. □

4. CONCLUSIONS

We have studied the problem of a right linear map preserving the left spectrum of 2×2 quaternion matrices and characterized the form of such map. By Lemma 3.1 and Theorem 3.2, it is easy to see that the form of a right linear map preserving the left spectrum of 2×2 quaternion matrices is not analogous to the classical results of the linear map preserving eigenvalues of $n\times n$ complex matrices for $n\geq 2$. By Lemma 2.1 and Theorem 3.2, we conjecture that, for $n>2$, the right quaternion linear map Φ preserves the left spectrum of $n\times n$ quaternion matrices if and only if there exists an invertible $n\times n$ real matrix X such that $\Phi(A)=XAX^{-1}$ for every $n\times n$ quaternion matrix A. This will be dealt with in future works.

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Kvaternioonide (2×2)-maatriksite vasakspektrit säilitavad parempoolsed lineaarsed kujutused

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On kirjeldatud kvaternioonide (2×2)-maatriksite ruumi parempoolsed lineaarsed kujutused, mis säilitavad kõigi kvaternioonide (2×2)-maatriksite vasakspektrid. Saadud kirjeldus erineb klassikalisest tulemusest samalaadsele probleemile komplekssete maatriksite jaoks.