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ALGEBRA

About the density property in the space of continuous maps vanishing at infinity

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Abstract. The conditions when $C_0(X) \otimes Y$ is dense in $C_0(X,Y)$ in the compact-open topology on $C_0(X,Y)$ are given. This result is used for describing the properties of topological Segal algebras.

Key words: Segal algebra, density property, approximation property, algebra of continuous functions vanishing at infinity.

1. INTRODUCTION

Let $\mathbb{K}$ denote either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers, X a topological space and Y a topological linear space over $\mathbb{K}$ (shortly, a topological linear space), $C(X,Y)$ the set of all continuous maps from X to Y, and $C_0(X,Y)$ the subset of all such $f \in C(X,Y)$ that vanish at infinity. In case we want to specify the topology of a topological space X, we write instead of X a pair (X, τ_X), where τ_X denotes the topology of X.

In [1], some results about Segal algebras were obtained, where one of the conditions that had to be fulfilled was that the set $C_0(X,\mathbb{K}) \otimes B$ (the full definition of this set will be given further in this paper) had to be dense in $C_0(X,B)$ (in the compact-open topology) for a topological algebra B. In [2], a result (Theorem 1 on page 27) is given describing the density of a subset of $C(X,\mathbb{K}) \otimes Y$ in $C(X,Y)$ for a Tikhonov space X and topological linear Hausdorff space Y over $\mathbb{K}$ (again in the compact-open topology). In [10], some similar results (Theorem 1 on page 98, Corollaries 1 and 2 on page 99) are presented for a compact Hausdorff space X and a topological linear space Y. It appeares that some of the ideas of [2] were such that they could be modified in order to obtain the density needed in our case. The present paper gives some sufficient conditions on a topological space X and a topological linear space Y under which the set $C_0(X,\mathbb{K}) \otimes Y$ is dense in $C_0(X,Y)$ in the compact-open topology. The obtained results will be applied to the results of [1] at the end of the paper.

2. PRELIMINARY DEFINITIONS AND RESULTS

Let Y be a topological linear space and K a subset of Y.

Definition 2.1. *A map $L : K \to Y$ is said to be* **finite-dimensional**[1] *if there exist a positive integer n and an n-dimensional subspace Z of Y such that $L(K) \subseteq Z$. Moreover, a finite-dimensional map $L : K \to Y$, which can be represented in a form $L(y) = \lambda_1(y)e_1 + \cdots + \lambda_n(y)e_n$ for every $y \in K$, where $\{e_1, \ldots, e_n\}$ is a basis of Z, is said to have* **continuous coordinate functions** *if the maps $\lambda_i : K \to \mathbb{K}$ are continuous for every $i \in \{1, \ldots, n\}$. A topological space Y is said to have* **continuous coordinate functions** *if every continuous finite-dimensional map $L : Y \to Y$ has continuous coordinate functions.*

Definition 2.2. *It is said that a topological linear space Y is* **Klee admissible** *if for every compact set $K \subseteq Y$ and for every neighbourhood O of zero in Y there exists a continuous finite-dimensional map $L : K \to Y$ such that $L(y) - y \in O$ for every $y \in K$.*

Let us recall some definitions of approximation properties.

Definition 2.3. *A topological linear space Y has*
(a) *the* **approximation property** *if for every compact set $K \subseteq Y$ and for every neighbourhood O of zero in Y there exists a continuous finite-dimensional linear map $L : Y \to Y$ such that $L(y) - y \in O$ for every $y \in K$;*
(b) *the* **nonlinear approximation property** *if for every compact set $K \subseteq Y$ and for every neighbourhood O of zero in Y there exists a continuous finite-dimensional map $L : Y \to Y$ such that $L(y) - y \in O$ for every $y \in K$.*

Remark 2.4. The term 'nonlinear approximation property' was suggested for that class of topological linear spaces already by Waelbroeck in [12] in 1972.

It is easy to see that every topological linear space that has the approximation property has also the nonlinear approximation property, and that every topological space that has the nonlinear approximation property is Klee admissible.

In [5], p. 826, the authors claim that every locally convex space is Klee admissible and pose an open problem to find out whether every topological linear space is Klee admissible.

In the proofs of the present paper we need some 'stronger' versions of the Klee admissibility and nonlinear approximation property.

Definition 2.5. *We will say that a topological linear space Y*
(a) *is* **strongly Klee admissible** *if for every compact set $K \subseteq Y$ and for every neighbourhood O of zero in Y there exists a continuous finite-dimensional map $L : K \cup \{\theta_Y\} \to Y$ such that $L(\theta_Y) = \theta_Y$ and $L(y) - y \in O$ for every $y \in K$;*
(b) *has the* **strong nonlinear approximation property** *if for every compact set $K \subseteq Y$ and for every neighbourhood O of zero in Y there exists a continuous finite-dimensional map $L : Y \to Y$ such that $L(\theta_Y) = \theta_Y$ and $L(y) - y \in O$ for every $y \in K$.*

Note that the condition $L(\theta_Y) = \theta_Y$ gives for a finite-dimensional map $L : Y \to Y$, which can be written as $L(y) = \lambda_1(y)e_1 + \cdots + \lambda_n(y)e_n$, that $\lambda_1(\theta_Y) = \cdots = \lambda_n(y) = \theta_Y$.

Definition 2.6. *Let X, Y be topological spaces. It is said that a map $f : X \to Y$* **vanishes at infinity** *if for every neighbourhood U of zero in Y there exists a compact set $K \subset X$ such that $f(x) \in U$ for every $x \in X \setminus K$.*

The following lemmas and corollary will be used in the proofs later.

[1] In some books (see, for example, [8]) it is assumed that a finite-dimensional map has to be also continuous.

Lemma 2.7. *Let X be a topological space and Y a topological linear space. If $f \in C_0(X,Y)$ and $\lambda \in C(Y,\mathbb{K})$ is such that $\lambda(\theta_Y) = 0$, then $\lambda \circ f \in C_0(X,\mathbb{K})$.*

Proof. It is clear that $\lambda \circ f \in C(X,\mathbb{K})$. Take any neighbourhood O of zero in $\mathbb{K}$. As $\lambda(\theta_Y) = 0$ and λ is continuous, there exists a neighbourhood U of zero in Y such that $\lambda(U) \subseteq O$. Since $f \in C_0(X,Y)$, there exists a compact set $K \subset X$ such that $f(x) \in U$ for every $x \in X \setminus K$. Hence, there exists a compact set $K \subset X$ such that $(\lambda \circ f)(x) = \lambda(f(x)) \in \lambda(U) \subseteq O$ for every $x \in X \setminus K$. Therefore, $\lambda \circ f \in C_0(X,\mathbb{K})$. □

Now let us discuss the property of continuous coordinate functions.

Lemma 2.8. *Let Y be a Hausdorff topological linear space, n a positive integer, and $L : Y \to Y$ a continuous n-dimensional map. Hence, there exists an n-dimensional subspace Z of Y (equipped with the subspace topology) with basis $\{e_1,\dots,e_n\}$, satisfying $L(Y) \subseteq Z$ such that $L(y) = \lambda_1(y)e_1 + \dots + \lambda_n(y)e_n$ for every $y \in Y$. Then the maps $\lambda_i : Y \to \mathbb{K}$, where $i \in \{1,\dots,n\}$, are continuous, i.e. L has continuous coordinate functions.*

Proof. By Theorem 1 from [6], p. 141, we know that Z is isomorphic to $\mathbb{K}^n$ via the homeomorphism $F : Z \mapsto \mathbb{K}^n$, where

$$F(\lambda_1(y)e_1 + \dots + \lambda_n(y)e_n) = (\lambda_1(y),\dots,\lambda_n(y))$$

for all $y \in Y$. Note that the maps $p_i : \mathbb{K}^n \to \mathbb{K}$, defined by $p_i((\lambda_1,\dots,\lambda_n)) = \lambda_i$ for every $i \in \{1,\dots,n\}$, are projections, hence, continuous maps.

Using the notations defined above, we see that $\lambda_i = p_i \circ F \circ L$ is a continuous map for every $i \in \{1,\dots,n\}$ because it is a composition of three continuous maps. Hence, L has continuous coordinate functions. □

Corollary 2.9. *Every Hausdorff topological linear space has continuous coordinate functions.*

Proof. Let Y be a Hausdorff topological linear space and $L : Y \to Y$ an arbitrary continuous finite-dimensional map. Then there exist a positive integer n, an n-dimensional subspace Z of Y, and basis $\{e_1,\dots,e_n\}$ of Z such that $L(y) = \lambda_1(y)e_1 + \dots + \lambda_n(y)e_n$ for every $y \in Y$. But then, by Lemma 2.8, L has continuous coordinate functions. Since $L : Y \to Y$ was an arbitrary continuous finite-dimensional map, all continuous finite-dimensional maps $L : Y \to Y$ have continuous coordinate functions. Therefore, Y has continuous coordinate functions. □

Definition 2.10. *A topological space X is a* **completely regular Hausdorff space** *if for every closed subset Z of X and every $x \in X \setminus Z$ there is a continuous map $f : X \to [0,1]$ such that $f(x) = 0$ and $f(z) = 1$ for every $z \in Z$.*

Lemma 2.11. *Let Y be a topological linear Hausdorff space, K a compact subset of Y, and $f \in C(K,\mathbb{K})$. Then there exists $\overline{f} \in C(Y,\mathbb{K})$ such that $\overline{f}|_K = f$, i.e. every continuous $\mathbb{K}$-valued map on a compact subset K of Y has a continuous extension to the whole space Y.*

Proof. Every Hausdorff topological linear space is a Hausdorff topological group, which is a completely regular (Hausdorff) space by Theorem 5 in [7], p. 49. Every compact set in a completely regular (Hausdorff) space is C-embedded (which means that every continuous real-valued function on a compact subset of a completely regular space can be extended to a continuous real-valued function on the whole space) by 3.11 (c) in [4], p. 43.

Hence, every continuous real-valued map on a compact subset K of a Hausdorff topological linear space Y has a real-valued extension to the whole space Y and the case for $\mathbb{K} = \mathbb{R}$ is proved.

Let $f \in C(K,\mathbb{C})$. Then we can write $f = f_r + if_i$, where $f_r, f_i \in C(K,\mathbb{R})$ are defined as $f_r(y) = a, f_i(y) = b$ for every $y \in K$ with $f(y) = a + bi$. Now, by the first part of the proof, there exist continuous extensions $\overline{f_r}, \overline{f_i} \in C(Y,\mathbb{R})$ of f_r and f_i, respectively. Defining $\overline{f} = \overline{f_r} + i\overline{f_i}$, we see that $\overline{f} \in C(Y,\mathbb{K})$ is a continuous extension of f to the whole space Y. □

3. RESULTS CONNECTED WITH THE DENSITY PROPERTY

Let X be a locally compact Hausdorff space and (Y,τ_Y) a topological linear space. Consider the algebra $(C_0(X,Y),c_Y)$ of all continuous maps $f: X\to Y$ vanishing at infinity equippped with the compact-open topology c_Y, where the subbase of the topology c_Y on $C_0(X,Y)$ consists of all sets of the form

$$\{S(K,U): K\subset X, K \text{ is compact}, U\in\tau_Y\},$$

where $S(K,U)=\{f\in C_0(X,Y): f(K)\subseteq U\}$.

Define a map $\otimes: C_0(X,\mathbb{K})\times Y\to C_0(X,Y)$ by

$$(\otimes(\phi,y))(x)\equiv(\phi\otimes y)(x):=\phi(x)y$$

for every $\phi\in C_0(X,\mathbb{K}), y\in Y$, and $x\in X$. Let $C_0(X,\mathbb{K})\otimes Y$ be the linear span of the set $\{\phi\otimes y:\phi\in C_0(X,\mathbb{K}), y\in Y\}$.

Next, we will give three results similar to Theorem 1 (β) from [2], p. 27.

Proposition 3.1. *Let X be a locally compact Hausdorff space and Y a topological linear space that has the strong nonlinear approximation property. If Y is a Hausdorff space or has continuous coordinate functions, then $C_0(X,\mathbb{K})\otimes Y$ is dense in $C_0(X,Y)$ in the compact-open topology.*

Proof. Take any $f\in C_0(X,Y)$ and fix a neighbourhood $O(f)$ of f in $C_0(X,Y)$. Then there exist a compact subset $K\subset X$ and a neighbourhood U of zero in Y such that $f+S(K,U)\subseteq O(f)$. Now, $f(K)$ is a compact subset of Y. Since Y has the strong nonlinear approximation property, there exists a continuous finite-dimensional map $L: Y\to Y$ such that $L(\theta_Y)=\theta_Y$ and $L(y)-y\in U$ for every $y\in f(K)$. Hence, there exists a positive integer n and a subspace Z of Y with the basis $\{e_1,\ldots,e_n\}$ such that $L(Y)\subseteq Z$.

If Y is a Hausdorff space, then it has continuous coordinate functions by Corollary 2.9. Since in both cases Y has continuous coordinate functions, the map L has the form $L(y)=\lambda_1(y)e_1+\cdots+\lambda_n(y)e_n$ and the maps $\lambda_i: Y\to\mathbb{K}$ are continuous with $\lambda_i(\theta_Y)=0$ for all $i\in\{1,\ldots,n\}$.

Take $g_1=\lambda_1\circ f,\ldots,g_n=\lambda_n\circ f$. Then, by Lemma 2.7, $g_1,\ldots,g_n\in C_0(X,\mathbb{K})$.

Note that

$$\Big(\sum_{i=1}^n(g_i\otimes e_i)\Big)(x)-f(x)=\sum_{i=1}^n g_i(x)e_i-f(x)=\sum_{i=1}^n\lambda_i(f(x))e_i-f(x)=L(f(x))-f(x)\in U$$

for each $x\in K$. Hence, for every $f\in C_0(X,Y)$ and every neighbourhood $O(f)$ of f in $C_0(X,Y)$ there exist integer $n>0$, $g_1,\ldots,g_n\in C_0(X,\mathbb{K})$, and $e_1,\ldots,e_n\in Y$ such that

$$\sum_{i=1}^n(g_i\otimes e_i)\in f+S(K,U)\subseteq O(f).$$

Therefore, $C_0(X,\mathbb{K})\otimes Y$ is dense in $C_0(X,Y)$ in the compact-open topology. □

As every topological space that has the approximation property has also the strong nonlinear approximation property, we obtain the following corollary.

Corollary 3.2. *Let X be a locally compact Hausdorff space and Y a topological linear space that has the approximation property. Then $C_0(X,\mathbb{K})\otimes Y$ is dense in $C_0(X,Y)$ in the compact-open topology.*

Proof. Exactly as in the proof of Proposition 3.1, we choose any $f\in C_0(X,Y)$, fix neighbourhood $O(f)$ of f, and find a compact subset $K\subset X$ and a neighbourhood U of zero in Y. Since Y has the approximation property, the map $L: Y\to Y$ will be not only continuous and finite-dimensional, but also linear. Therefore, $L(\theta_Y)=\theta_Y$, which implies that $\lambda_i(\theta_Y)=0$ for every $i\in\{1,\ldots,n\}$. It is known (see e.g. [8], Part II, Chapter XIII, 4.5) that a continuous linear finite-dimensional map has continuous coordinate functions. Hence, the maps λ_i are continuous for every $i\in\{1,\ldots,n\}$. Therefore, we can now proceed as in the proof of Proposition 1 and see that $C_0(X,\mathbb{K})\otimes Y$ is dense in $C_0(X,Y)$ in the compact-open topology. □

Next, we shall prove a version of Proposition 3.1 for the case of strongly Klee admissible Hausdorff topological linear spaces.

Proposition 3.3. *Let X be a locally compact Hausdorff space and Y a strongly Klee admissible Hausdorff topological linear space. Then $C_0(X,\mathbb{K})\otimes Y$ is dense in $C_0(X,Y)$ in the compact-open topology.*

Proof. Note that Y, as a Hausdorff topological linear space, has continuous coordinate functions by Corollary 2.9. Take any $f\in C_0(X,Y)$ and fix a neighbourhood $O(f)$ of f in $C_0(X,Y)$. Exactly as in the proof of Proposition 3.1, we obtain that there exist a compact subset $f(K)\subset Y$, a neighbourhood U of zero in Y, a continuous finite-dimensional map $L: f(K)\cup\{\theta_Y\}\to Y$ with $L(\theta_Y)=\theta_Y$, a positive integer n, and a subspace Z of Y with the basis $\{e_1,\dots,e_n\}$ such that $f+S(K,U)\subseteq O(f)$, $L(y)-y\in U$ for every $y\in f(K)$ and $L(f(K))\subseteq Z$. Since Y has continuous coordinate functions, the maps $\lambda_1,\dots,\lambda_n\in C(f(K)\cup\{\theta_Y\},\mathbb{K})$. As $L(\theta_Y)=\theta_Y$, then $\lambda_1(\theta_Y)=\cdots=\lambda_n(\theta_Y)=0$.

Now, by Lemma 2.11, there exist the extensions $\overline{\lambda_1},\dots,\overline{\lambda_n}\in C(Y,\mathbb{K})$ of $\lambda_1,\dots,\lambda_n$ to the space Y, respectively.

Similarily as in the proof of Proposition 3.1, we define $g_i=\overline{\lambda_i}\circ f\in C_0(X,\mathbb{K})$ for $i\in\{1,\dots,n\}$ and obtain that

$$\sum_{i=1}^{n}(g_i\otimes e_i)\in f+S(K,U)\subseteq O(f).$$

Hence, $C_0(X,\mathbb{K})\otimes Y$ is dense in $C_0(X,Y)$ in the compact-open topology also when Y is a strongly Klee admissible Hausdorff topological linear space. □

Recall that for a topological space X, one writes $\dim(X)=n$ if n is the smallest nonnegative integer such that for any finite open cover of X one can choose a finite open refinement of that cover such that every $x\in X$ is contained in maximally $n+1$ elements of that refinement. If there exists a nonnegative integer n such that $\dim(X)=n$, then it is said that $\dim(X)$ (or, the *topological dimension* of X, or the *Lebesgue covering dimension* of X) is finite.

Remark 3.4. In the mathematical literature, there are actually several definitions of the Lebesgue covering dimension: in some books it is assumed that one can choose an open refinement for any open cover of X, in other books it is assumed that an open refinement should exist only for any finite open cover of X, and in others that a finite open refinement should exist for any finite open cover of X. In Remark 1 in [3], p. 165, it is claimed that the last two definitions coincide.

Let us recall that for a map $f: X\to\mathbb{K}$ from a topological space X to the field $\mathbb{K}$ of real or complex numbers, the closure of the set of elements $x\in X$ for which $f(x)\neq 0$, was called the *support* of f and was denoted by $\mathrm{supp}(f)$. In order to prove the next result, we will use another known result.

Lemma 3.5. *Let X be a locally compact Hausdorff space, K a compact subset of X, and $V_1,\dots,V_n$ open subsets of X such that $K\subseteq V_1\cup V_2\cup\cdots\cup V_n$. Then there are continuous functions $h_i: X\to[0,1]$, $i=1,2,\dots,n$, all of compact supports, such that* $\mathrm{supp}(h_i)\subseteq V_i$, *for all i, and*

$$\sum_{i=1}^{n}h_i(k)=1$$

for all $k\in K$.

Proof. See the proof of Theorem on slide 10 of [11]. The cited proof copies actually Rudin's ideas of the proof of Theorem 2.13 from [9], p. 40. One has just to notice in the proof of Rudin that the supports $\mathrm{supp}(h_i)$ of the constructed functions h_i are compact sets (a closed subset of a compact set is compact). □

The collection $\{h_1,\dots,h_n\}$ of functions h_i, given in Lemma 3.5, is also called a *partition of unity* of X.

Now we are ready to present a result similar to Theorem 1 (γ) from [2], p. 27.

Proposition 3.6. *Let X be a locally compact Hausdorff space and Y a topological linear space. If* $\dim(X)$ *is finite, then $C_0(X,\mathbb{K})\otimes Y$ is dense in $C_0(X,Y)$ in the compact-open topology.*

Proof. As in the proof of Proposition 3.1, fix any $f\in C_0(X,Y)$, its neighbourhood $O(f)$, compact subset $K\subset X$, and a neighbourhood U of zero in Y such that $f+S(K,U)\subseteq O(f)$.

If $\dim(X)$ is finite, then there exists a nonnegative integer n such that $\dim(X)=n$. Since the addition is continuous in Y, there exists an open balanced neighbourhood V of zero in Y such that

$$\underbrace{V+\cdots+V}_{n+1 \text{ summands}}\subseteq U.$$

Now, $f(x)+V$ is an open neighbourhood of $f(x)$ for every $x\in X$. Since f is continuous, then

$$O(x)=f^{-1}(f(x)+V)=\{z\in X: f(z)\in f(x)+V\}$$

is an open neighbourhood of x for every $x\in X$. Hence, the set $A=\{O(x):x\in K\}$ is an open cover of K.

As K is a compact set, there exists a finite subcover $B=\{O(z_1),\dots,O(z_l)\}$ of A, with $l<\infty$ a positive integer, which is still a cover of K. In a Hausdorff space, every compact set is closed. Hence, $X\setminus K$ is open and $C=\{X\setminus K,O(z_1),\dots,O(z_l)\}$ is a finite open cover of X.

Since $\dim(X)=n$, we can find a finite open subcover $D=\{O_1,\dots,O_m\}$ of C, which is still a cover of X and where every $x\in X$ is contained in maximally $n+1$ elements of the cover D. For every $x\in K$, let $I_x=\{i\in\{1,\dots,m\}:x\in O_i\}$. Then it is clear that the sets I_x can have at most $n+1$ elements.

As X is a locally compact Hausdorff space and $K\subset X=O_1\cup\cdots\cup O_m$, then, by Lemma 3.5, there exist a partition of unity $\alpha_1,\dots,\alpha_m\in C(X,[0,1])\subset C(X,\mathbb{K})$ and compact sets $K_1,\dots,K_m$ (supports of $\alpha_1,\dots,\alpha_m$) such that for every $i\in\{1,\dots,m\}$ and every $x\in X\setminus K_i$ hold $K_i\subseteq O_i$, $\alpha_i(x)=0$ and for every $x\in K$ holds $\alpha_1(x)+\cdots+\alpha_m(x)=1$. Hence, $\alpha_1,\dots,\alpha_m\in C_0(X,[0,1])\subset C_0(X,\mathbb{K})$.

For every $i\in\{1,\dots,m\}$, either there exist $k\in\{1,\dots,l\}$ such that $O_i\subseteq O(z_k)$ or $O_i\subseteq X\setminus K$. In the first case, take $x_i=z_j$, where $j\in\{1,\dots,l\}$ is a minimal such index that $O_i\subseteq O(z_j)$. In the second case, take $x_i=\theta_X$.

Now, for every $x\in K$ we have

$$\begin{aligned}\left(\sum_{i=1}^m(\alpha_i\otimes f(x_i))\right)(x)-f(x) &= \sum_{i=1}^m(\alpha_i(x)f(x_i))-\sum_{i=1}^m(\alpha_i(x)f(x))=\sum_{i=1}^m\alpha_i(x)(f(x_i)-f(x))\\ &= \sum_{i\in I_x}\alpha_i(x)(f(x_i)-f(x))\in\sum_{i\in I_x}\alpha_i(x)V\subseteq\underbrace{V+\cdots+V}_{n+1\text{ summands}}\subseteq U.\end{aligned}$$

Hence,

$$\sum_{i=1}^m(\alpha_i\otimes f(x_i))\in f+S(K,U)\subset O(f).$$

On the other hand, it is clear that

$$\sum_{i=1}^m(\alpha_i\otimes f(x_i))\in C_0(X,\mathbb{K})\otimes Y.$$

Hence, $C_0(X,\mathbb{K})\otimes Y$ is dense in $C_0(X,Y)$ in the compact-open topology. □

4. APPLICATIONS OF THE DENSITY RESULTS FOR THE CASE OF SEGAL ALGEBRAS

A topological algebra is a topological vector space over $\mathbb{K}$, where the multiplication is separately continuous.

Definition 4.1. *A topological algebra (A,τ_A)* **is a left (right or two-sided) topological Segal algebra** *if there exists a topological algebra (B,τ_B) and an algebra homomorphism $f : A \to B$ such that*
(1) *the image of A by f is dense in B, i.e.* $\mathrm{cl}_B(f(A)) = B$*;*
(2) $\tau_A \supseteq \{f^{-1}(U) : U \in \tau_B\}$*;*
(3) $f(A)$ *is a left (respectively, right or two-sided) ideal of B.*
Since for a topological algebra (A,τ_A) there might exist several different topological algebras (B,τ_B) and algebra homomorphisms $f : A \to B$ fulfilling the conditions of the definition, we say that "(A,τ_A) **is a left (right or two-sided) topological Segal algebra in** (B,τ_B) **via** $f : A \to B$*", when we want to specify which of the possibly many algebras (B,τ_B) and maps $f : A \to B$ we consider in the particular case.*

By Propositions 3.1, 3.3, and 3.6, we will have now some new results for topological Segal algebras.

Proposition 4.2. *Let X be a locally compact Hausdorff space, A,B topological algebras, and $\iota : A \to B$ a map. Consider the algebras $(C_0(X,A),c_A)$ and $(C_0(X,B),c_B)$ equipped with the compact-open topologies c_A and c_B and define a map $\omega : C_0(X,A) \to C_0(X,B)$ by $\omega(f) := \iota \circ f$ for every $f \in C_0(X,A)$. Suppose that the multiplication in B is jointly continuous, ι is continuous and open algebra monomorphism, and $\iota(A)$ is a dense left (right or two-sided) ideal of B. If one of the conditions*

(a) *B has the approximation property,*

(b) *B has the strong nonlinear approximation property and either B is a Hausdorff topological linear space or has continuous coordinate functions,*

(c) *B is a strongly Klee admissible Hausdorff topological algebra,*

(d) dim (X) *is finite*

is satisfied, then $C_0(X,A)$ is a left (respectively, right or two-sided) topological Segal algebra in $C_0(X,B)$ via ω.

Proof. Using Corollary 3.2 in case (a), Proposition 3.1 in case (b), Proposition 3.3 in case (c), and Proposition 3.6 in case (d), we see that in all cases $C_0(X,\mathbb{K}) \otimes B$ is dense in $C_0(X,B)$ in the compact-open topology. Hence, the result follows from Proposition 3 (f) in [1]. □

Proposition 4.3. *Let X be a locally compact Hausdorff space and A,B topological algebras such that A is a subalgebra of B. If the multiplication in B is jointly continuous, A is a left (right or two-sided) topological Segal algebra in B via the identity map 1_A, and one of the conditions*

(a) *B has the approximation property,*

(b) *B has the strong nonlinear approximation property and either B is a Hausdorff topological linear space or has continuous coordinate functions,*

(c) *B is a strongly Klee admissible Hausdorff topological algebra,*

(d) $\dim(X)$ *is finite*

is satisfied, then $C_0(X,A)$ is a left (respectively, right or two-sided) topological Segal algebra in $C_0(X,B)$ via the identity map $1_{C_0(X,A)}$.

Proof. As in the proof of Proposition 4.2, we see that in all cases $C_0(X,\mathbb{K}) \otimes B$ is dense in $C_0(X,B)$ in the compact-open topology. Hence, the result follows from Corollary 1 in [1]. □

Proposition 4.4. *Let X be a locally compact Hausdorff space, A,B topological algebras such that A is a subalgebra of B and the multiplication in B is jointly continuous. Consider the algebras $(C_0(X,A),c_A)$ and $(C_0(X,B),c_B)$ equippped with the compact-open topologies c_A and c_B, respectively. Suppose that one of the conditions*

(a) *B has the approximation property,*

(b) *B has the strong nonlinear approximation property and either B is a Hausdorff topological linear space or has continuous coordinate functions,*

(c) *B is a strongly Klee admissible Hausdorff topological algebra,*

(d) $\dim(X)$ *is finite*

is satisfied. Then the following conditions are equivalent:

(1) *A is a left (right or two-sided) topological Segal algebra in B via* 1_A*;*

(2) $C_0(X,A)$ *is a left (respectively, right or two-sided) topological Segal algebra in* $C_0(X,B)$ *via* $1_{C_0(X,A)}$.

Proof. As in the proof of Proposition 4.2, we see that in all cases $C_0(X,\mathbb{K})\otimes B$ is dense in $C_0(X,B)$ in the compact-open topology. Hence, the result follows from Corollary 2 in [1]. □

5. CONCLUSIONS

We found some sufficient conditions for a Hausdorff space X and a topological linear space Y ensuring that $C_0(X,\mathbb{K})\otimes Y$ is dense in $C_0(X,Y)$ in the compact-open topology. This allowed us to specify the class of topological algebras for which A is a topological Segal algebra in B if and only if $C_0(X,A)$ is a topological Segal algebra in $C_0(X,B)$.

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Kõikjal tiheduse omadusest pidevate lõpmatuses hääbuvate funktsioonide algebras

Mart Abel

Olgu $\mathbb{K}$ kas reaalarvude või kompleksarvude korpus ja Y topoloogiline vektorruum üle $\mathbb{K}$. Artiklis on leitud mõningad piisavad tingimused topoloogilise ruumi X ja topoloogilise algebra Y jaoks, mille korral algebra $C_0(X,\mathbb{K})\otimes Y$ on kõikjal tihe pidevate lõpmatuses hääbuvate funktsioonide algebras $C_0(X,Y)$.

Nende tulemuste rakendusena saadakse, et topoloogiline algebra A on topoloogiline Segali algebra topoloogilises algebras B siis ja ainult siis, kui pidevate lõpmatuses hääbuvate funktsioonide algebra $C_0(X,A)$ on topoloogiline Segali algebra pidevate lõpmatuses hääbuvate funktsioonide algebras $C_0(X,B)$.