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TOPOLOGICAL
ALGEBRA

Mackey Q-algebras

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Abstract. Main properties of the topology defined by a bornology on a topological linear space and main properties of Mackey Q-algebras are presented. Relationships of Mackey Q-algebras with other classes of topological algebras are described. It is shown that every Mackey Q-algebra is an advertibly Mackey complete algebra, every strongly sequential Mackey Q-algebra is a Q-algebra, every infrasequential Mackey Q-algebra is an advertibly complete algebra, and every infrasequential advertive Hausdorff algebra is a Mackey Q-algebra.

Key words: topological algebra, Mackey Q-algebra, advertibly complete algebra, advertibly Mackey complete algebra, sequential algebra, strongly sequential algebra, netial algebra, infrasequential algebra.

1. INTRODUCTION

Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers and A a topological algebra over $\mathbb{K}$ with separately continuous multiplication (shortly, a topological algebra). Moreover, let θ_A denote the zero element in A and Qinv$A$ the set of all quasi-invertible elements in A, that is, there is an element $b \in A$ (we denote it by a_q^{-1}) such that $a \circ b = b \circ a = \theta_A$ (here $a \circ b = a + b - ab$ for each $a, b \in A$). In the case A has a unit element e_A, let Inv$A$ denote the set of all invertible elements in A. A topological algebra A is called a *Q-algebra* if the set Qinv$A$ (in the unital case the set Inv$A$) is open in A.

An element $a \in A$ is *topologically left quasi-invertible* in A if there is a net $(a_\lambda)_{\lambda\in\Lambda}$ such that $(a_\lambda \circ a)_{\lambda\in\Lambda}$ converges to θ_A. Topologically right quasi-invertible elements are defined similarly. An element $a \in A$ is *topologically quasi-invertible* if it is topologically left quasi-invertible and topologically right quasi-invertible in A. In the case where A has the unit element e_A, element $a \in A$ is *topologically left invertible* in A if there is a net $(a_\lambda)_{\lambda\in\Lambda}$ such that $(a_\lambda a)_{\lambda\in\Lambda}$ converges to e_A. Topologically right invertible elements and topologically invertible elements in A are defined similarly. We denote the set of all topologically quasi-invertible elements in A by Tqinv$A$ and the set of all topologically invertible elements in A by Tinv$A$. A topological algebra A is called an *advertive algebra* if Tqinv$A$ = Qinv$Q$. In the unital case, A is an advertive or invertive algebra if Tinv$A$ = Inv$A$ because in the unital case Tqinv$A = e_A -$ Tinv$A$ and Qinv$A = e_A -$ Inv$A$. In addition, a topological algebra A is called *locally k-convex* (*locally convex*) if it has a local base consisting of absolutely k-convex (respectively, absolutely convex) neighbourhoods of zero, and A is *locally m-convex* if it has a local base consisting of idempotent and absolutely convex neighbourhoods of zero. Recall that a set U is *absolutely k-convex* if $U = \Gamma_k(U)$ and *absolutely convex* if $U = \Gamma_1(U)$, where

$$\Gamma_k(U) = \left\{ \sum_{\nu=1}^{n} \alpha_\nu u_\nu : n \in \mathbb{N}, u_1, \ldots, u_n \in U \text{ and } \alpha_1, \ldots, \alpha_n \in \mathbb{K} \text{ with } \sum_{\nu=1}^{n} |\alpha_\nu|^k \leqslant 1 \right\}.$$

Let now A be a topological algebra over $\mathbb{C}$, Hom$A$ the set of all non-zero homomorphisms from A to $\mathbb{C}$, hom$A$ the subset of continuous homomorphisms in Hom$A$,

$$\mathrm{sp}_A(a) = \left\{ \lambda \in \mathbb{C} \setminus \{0\} : \frac{a}{\lambda} \notin \mathrm{Qinv}A \right\} \cup \{0\}$$

(in unital case

$$\mathrm{sp}_A(a) = \left\{ \lambda \in \mathbb{C} : a - \lambda e_A \notin \mathrm{Inv}A \right\})$$

the *spectrum* of $a \in A$ and

$$\mathrm{r}_A(a) = \sup\left\{ |\lambda| : \lambda \in \mathrm{sp}_A(a) \right\}$$

the *spectral radius* of $a \in A$. Then

$$\left\{ \varphi(a) : \varphi \in \mathrm{hom}A \right\} \subseteq \left\{ \phi(a) : \phi \in \mathrm{Hom}A \right\} \subseteq \mathrm{sp}_A(a)$$

for each $a \in A$. In the particular case when

$$\mathrm{sp}_A(a) = \left\{ \varphi(a) : \varphi \in \mathrm{hom}A \right\}$$

for each $a \in A$, we will say that A has the *functional spectrum*.

An element $a \in A$ is called *bounded* if the *radius of boundedness*

$$\beta_A(a) = \inf\left\{ \lambda > 0 : \left(\frac{a}{\lambda}\right)^n \text{ is bounded in } A \right\}$$

$$= \inf\left\{ \lambda > 0 : \left(\frac{a}{\lambda}\right)^n \text{ vanishes in } A \right\}$$

is finite. There are topological algebras for which $\mathrm{r}_A(A) = \beta_A(a)$ for every $a \in A$ and those for which $\mathrm{r}_A(A) \neq \beta_A(a)$ for every $a \in A$.

The main properties of the topology defined by a bornology on a topological linear space and the main properties of Mackey Q-algebras are presented in the present paper. The relationships of Mackey Q-algebras with other classes of topological algebras are described. In addition to other results, it is shown that every Mackey Q-algebra is an advertibly Mackey complete algebra, every strongly sequential Mackey Q-algebra is a Q-algebra, every infrasequential Mackey Q-algebra is an advertibly complete algebra, and every infrasequential advertive Hausdorff algebra is a Mackey Q-algebra.

2. BORNOLOGY

We recall first the main notions connected with bornology. A *bornology* (see, for example [12]) on a nonempty set X is a collection $\mathscr{B}$ of subsets of X that satisfies the following conditions:

(a) $X = \bigcup\limits_{B \in \mathscr{B}} B$;

(b) if $B \in \mathscr{B}$ and $C \subseteq B$, then $C \in \mathscr{B}$;

(c) if $B_1, B_2 \in \mathscr{B}$, then $B_1 \cup B_2 \in \mathscr{B}$.

A set X with a bornology $\mathscr{B}$ on X is called a *bornological space* and denoted by $(X, \mathscr{B})$. In the case where X is a vector space over $\mathbb{K}$, a bornology $\mathscr{B}$ on X is called a *linear* or *vector bornology* if the following

conditions are satisfied:
(d) if $B_1, B_2 \in \mathscr{B}$, then $B_1 + B_2 \in \mathscr{B}$;

(e) if $B \in \mathscr{B}$ and $\lambda \in \mathbb{K}$, then $\lambda B \in \mathscr{B}$;

(f) $b(B) = \bigcup\limits_{|\lambda|\leqslant 1} \lambda B \in \mathscr{B}$ for every $B \in \mathscr{B}$.

Moreover, when X is an algebra over $\mathbb{K}$, a bornology $\mathscr{B}$ on X is called an *algebra bornology* if $\mathscr{B}$ satisfies the conditions $(a)-(f)$ and
(g) if $a \in X$ and $B \in \mathscr{B}$, then $aB, Ba \in \mathscr{B}$.
The condition (g) means that the multiplication in X is separately bounded.

Let now $\mathscr{B}$ be a linear bornology on X. It is said that a net $(x_\lambda)_{\lambda\in\Lambda}$ in X converges to $x_0 \in X$ *bornologically* if there is a balanced set $B \in \mathscr{B}$ and for every $\varepsilon > 0$ an index $\lambda_0 \in \Lambda$ such that $x_\lambda - x_0 \in \varepsilon B$ whenever $\lambda \succ \lambda_0$. In the particular case when (X, τ) is a topological linear space and $\mathscr{B}_\tau$ the *von Neumann bornology* on X (that is, the set of all bounded subsets of (X, τ)), the bornological convergence in X is called the *Mackey convergence*. It is easy to see that every Mackey convergent net in (X, τ) is topologically convergent (that is, convergent in the topology τ), but not conversely. In particular, when the space (X, τ) is metrizable, then every topologically convergent net in (X, τ) is Mackey convergent as well (see, for example [14, pp. 27–28]).

Let now (A, τ) be a topological algebra. It is known (see [14, p. 21, Example 4]) that $\mathscr{B}_\tau$ on A is a linear bornology. To show that also the condition (g) holds, let $a \in A$ and $B \in \mathscr{B}_\tau$. Then for every neighbourhood O of zero in A there exists another neigbourhood O' of zero such that $aO', O'a \subseteq O$ and $\lambda' > 0$ such that $B \subseteq \lambda' O'$. Thus, $aB \subseteq \lambda' aO' \subseteq \lambda' O$. Hence $aB \in \mathscr{B}_\tau$. Similarly, $Ba \in \mathscr{B}_\tau$. Consequently, $\mathscr{B}_\tau$ is an algebra bornology on every topological algebra (A, τ).

A linear bornology $\mathscr{B}$ on A is called a *convex bornology* if $\Gamma_1(B)$ for every element $B \in \mathscr{B}$. Moreover, a topological linear space (X, τ) is called *Mackey complete* if every Mackey convergent Cauchy net in X is topologically convergent. So, every complete topological linear space is Mackey complete, but there exist Mackey complete topological linear spaces that are not complete. We will speak about *sequentially Mackey complete* spaces if instead of nets only sequences are considered.

3. TOPOLOGY DEFINED BY BORNOLOGY

Let $(X, \mathscr{B})$ be a bornological linear space (that is, $\mathscr{B}$ is a linear bornology on X) and let $\tau_\mathscr{B}$ denote the set

$$\{U \subset X : \forall\, u \in U \text{ and } \forall \text{ balanced } B \in \mathscr{B}\ \exists\ \lambda > 0 \text{ such that } u + \lambda B \subset U\} \cup \{\emptyset\}.$$

It is easy to see that $\tau_\mathscr{B}$ is a topology on X. This topology, introduced in 1950 by Arnold in [8], is called the *topology defined by the bornology* $\mathscr{B}$. In the case where X is a topological linear space and $\mathscr{B}$ is the von Neumann bornology on X, the topology $\tau_\mathscr{B}$ is called the *Mackey closure topology* and is denoted by $\tau_{\mathscr{B}_\tau}$. When (X, τ) is a T_1-space, then $(X, \tau_{\mathscr{B}_\tau})$ is also a T_1-space, but $\tau_{\mathscr{B}_\tau}$ is not necessarily a Hausdorff topology.

Let $(X, \mathscr{B})$ be a bornological linear space. Next we present the main properties of the topology $\tau_\mathscr{B}$, from which several are known for sequences or in the case where X has additional restrictions.

Property 1. *Let $(X, \mathscr{B})$ be a bornological linear space. If $O \in \tau_\mathscr{B}$, $x \in X$ and $\mu \in \mathbb{K} \setminus \{0\}$, then $\mu O, O + x \in \tau_\mathscr{B}$.*

To show that this property is true, let $a \in \mu O$ and $B \in \mathscr{B}$ be balanced. Then $\frac{1}{\mu} a \in O$ and $\frac{1}{\mu} B \in \mathscr{B}$ are balanced. Hence, there is a $\lambda > 0$ such that $\frac{1}{\mu} a + \lambda \frac{1}{\mu} B \subset O$. This means that $a + \lambda B \subset \mu O$. Consequently, $\mu O \in \tau_\mathscr{B}$. Let now $a \in O + x$ and $B \in \mathscr{B}$ be balanced. Then there is a $\nu > 0$ such that $(a - x) + \nu B \subset O$. This means that $a + \nu B \subset O + x$. So, $O + x \in \tau_\mathscr{B}$. Hence, the homothety and the shift are homeomorphisms in bornological linear spaces in the topology $\tau_\mathscr{B}$ like in the case of topological linear spaces.

By Property 1, it is easy to show that the addition and the multiplication by scalars in every topological linear space (X,τ) are separately continuous maps in the Mackey closure topology. Indeed, for any $x \in X$ and $\mu \in \mathbb{K}\setminus\{0\}$, let f_x and g_μ denote the maps, defined by $f_x(x') = x+x'$ and $g_\mu(x) = \mu x$ for each $x,x' \in X$. Then

$$f_x(O-x) = O \text{ and } g_\mu\Big(\frac{1}{\mu}O\Big) = O$$

for any neighbourhood O of zero in X in the Mackey closure topology and any nonzero number μ. Since $O-x, \frac{1}{\mu}O \in \tau_{\mathscr{B}_\tau}$ by Property 1, the addition and the multiplication by scalars in topological linear spaces in the Mackey closure topology are separately continuous maps but, in general, not jointly continuous. Therefore every topological linear space (X,τ) in the Mackey closure topology is only a pseudotopological linear space (that is, a vector space with topology in which the addition and the multiplication by scalars are only separately continuous maps).

Let now (A,τ) be a topological algebra, $a \in A$, $O \in \tau_{\mathscr{B}_\tau}$ be balanced, h_a be the map defined by $h_a(\lambda) = \lambda a$ for each $\lambda \in \mathbb{K}$, and $\mu > 0$ be such number that $\mu b(\{a\}) = \theta_A + \mu b(\{a\} \subset O$ because $\theta_A \in O$. Since

$$h_a(O_\mu) = O_\mu a = \mu O_1 a \subset \mu b(\{a\}) \subset O$$

and

$$O_\mu = \{\lambda \in \mathbb{K} : |\lambda| < 1\}$$

is an open subset in $\mathbb{K}$, then g_a is a continuous map in the Mackey closure topology. Consequently, the multiplication in any topological algebra is separately continuous in the Mackey closure topology.

Property 2. *A base of neighbourhoods of zero in the topology $\tau_{\mathscr{B}}$ on X consists of balanced neighbourhoods of zero.*

The proof of this statements is given in [4, Lemma 3.1].

Property 3. *Let $(X,\mathscr{B})$ be a bornological linear space and U a subset of X. Then $U \in \tau_{\mathscr{B}}$ if and only if for every $x \in U$ and every net $(x_\lambda)_{\lambda\in\Lambda}$ in X that bornologically converges to x, there exists an index $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$ whenever $\lambda \succ \lambda_0$ and U is $\tau_{\mathscr{B}}$-closed if and only if from the bornological convergence of a net $(x_\lambda)_{\lambda\in\Lambda}$ in U to $x \in X$ follows that $x \in U$.*

Let $U \in \tau_{\mathscr{B}}$, $x \in U$ and $(x_\lambda)_{\lambda\in\Lambda}$ be a net in X that bornologically converges to x. Then for every balanced $B \in \mathscr{B}$ there is a number $\mu_B > 0$ such that $x+\mu_B B \subset U$, and there exists a balanced set $B' \in \mathscr{B}$ and $\lambda_0 \in \Lambda$ such that $x_\lambda - x \in \mu_{B'}B'$ whenever $\lambda \succ \lambda_0$. Terefore, $x_\lambda \in x+\mu_{B'}B' \subset U$ whenever $\lambda \prec \lambda_0$.

Suppose that $U \notin \tau_{\mathscr{B}}$. Then there are an element $x_0 \in U$ and a balanced subset $B_0 \in \mathscr{B}$ such that $x_0 + \mu B_0 \not\subset U$ for every $\mu > 0$. Therefore, for every $\mu > 0$ there is an element $b_\mu \in B_0$ such that $x_0 + \mu b_\mu \notin U$. Thus, we have a net $(x_0 + \mu b_\mu)_{\mu>0}$ in X that bornologically converges to x_0 because $x_0 + \mu b_\mu - x_0 \in \varepsilon\frac{\mu}{\varepsilon}B_0 \subset \varepsilon B_0$ whenever $\mu > \varepsilon$.

Let $U \subset X$ be a $\tau_{\mathscr{B}}$-closed set in X and let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in U that bornologically converges to $x \in X$. Suppose that $x \notin U$. Since $X\setminus U \in \tau_{\mathscr{B}}$, there exists an index $\lambda_0 \in \Lambda$ such that $x_\lambda \in X\setminus U$ whenever $\lambda \succ \lambda_0$. As this is not possible, then $x \in U$.

Suppose now that the set U is not $\tau_{\mathscr{B}}$-closed in X. Then $X\setminus U \notin \tau_{\mathscr{B}}$. Therefore, there are an element $x \in X\setminus U$ and a balanced set $B \in \mathscr{B}$ such that $x+\mu B \not\subset X\setminus U$ for every $\mu > 0$. Hence, for every $\mu > 0$ there exists an element $b_\mu \in B$ such that $x+\mu b_\mu \notin X\setminus U$. Hence $x+\mu b_\mu \in U$ for every $\mu > 0$. So we have a net $(x+\mu b_\mu)_{\mu>0}$ in U that bornologically converges to $x \notin U$.

Property 4. *Every neighbourhood of zero in a bornological linear space $(X,\mathscr{B})$ is an absorbing set in the topology $\tau_{\mathscr{B}}$.*

Let O be an open subset of X in the topology $\tau_{\mathscr{B}}$, $x \in X$ be fixed, g_x the map defined by $g_x(\lambda) = \lambda x$ for each $\lambda \in \mathbb{K}$, and B a balanced bounded subset in $\mathbb{K}$. Then there is a number $M > 0$ such that $B \subset O_M = \{\lambda \in \mathbb{K} : |\lambda| \leqslant M\}$. Since $O_M = MO_1$ and $Mb(\{x\}) \in \mathscr{B}$ (see the definition of linear bornology) is balanced, then for every $\lambda_0 \in g_x^{-1}(O)$ there exists a number $\mu > 0$ such that $g_x(\lambda_0) + \mu Mb(\{x\}) \subset O$. Now, from

$$g_x(\lambda_0 + \mu B) = g_x(\lambda_0) + \mu g_x(B) \subset g_x(\lambda_0) + \mu M O_1 x \subset g_x(\lambda_0) + \mu Mb(\{x\}) \subset O$$

follows that $\lambda_0 + \mu B \subset g_x^{-1}(O)$. That is, $g_x^{-1}(O)$ is an open set in $\mathbb{K}$ in the topology defined by the bornology of $\mathbb{K}$ (hence, $g_x^{-1}(O)$ is an open set in the topology of $\mathbb{K}$ as well by Proposition 7 below). Consequently, there is a number $\varepsilon_x > 0$ such that $O_{\varepsilon_x} \subset g_x^{-1}(O)$. Hereby, $\mu x \in O$ whenever $|\mu| \leqslant \varepsilon_x$. This means that O is an absorbing set.

Property 5. *Let $(X, \mathscr{B})$ be a bornological linear space. Then every net $(x_\lambda)_{\lambda\in\Lambda}$ that bornologically converges to $x \in X$, converges to x also in the topology $\tau_{\mathscr{B}}$.*

Let O be a neighbourhood of zero in the topology $\tau_{\mathscr{B}}$. Then there is a set $G \in \tau_{\mathscr{B}}$ such that $G \subset O$. Therefore, for every balanced set $B \in \mathscr{B}$ there is a number $\mu_B > 0$ such that $\mu_B B \subset G$. Since the net $(x_\lambda)_{\lambda\in\Lambda}$ bornologically converges to x, there exist a balanced set $B_0 \in \mathscr{B}$ and an index $\lambda_0 \in \Lambda$ such that $x_\lambda - x \in \mu_{B_0} B_0 \subset G \subset O$ whenever $\lambda \succ \lambda_0$. This means that the net $(x_\lambda)_{\lambda\in\Lambda}$ converges to x in the topology $\tau_{\mathscr{B}}$.

Property 6. *Let (X, τ) be a topological linear space and $\mathscr{B}_\tau$ the von Neumann bornology on X. Then $\tau \subseteq \tau_{\mathscr{B}_\tau}$.*

Let $O \in \tau$, $x \in O$ and $B \in \mathscr{B}_\tau$ be balanced. Then there are a neighbourhood U of zero in X such that $x + U \subset O$ and a number $\mu > 0$ such $B \subset \mu U$. Since $x + \frac{1}{\mu} B \subset x + U \subset O$, then $O \in \tau_{\mathscr{B}_\tau}$.

Property 7. *Let (X, τ) be a topological linear space in which every topologically convergent net is Mackey convergent. Then $\tau = \tau_{\mathscr{B}_\tau}$.*

It is enough to show only that $\tau_{\mathscr{B}_\tau} \subset \tau$. For this, let $O \in \tau_{\mathscr{B}_\tau}$. Then $X \setminus O$ is $\tau_{\mathscr{B}_\tau}$-closed. Let x_0 be an arbitrary element in the closure of $X \setminus O$ in the topology τ. Then there is a net $(x_\lambda)_{\lambda\in\Lambda}$ in X that converges to x_0 in the topology τ. Hence this net Mackey converges to x_0 as well. Then $x_0 \in X \setminus O$ by Property 3. Thus $X \setminus O$ is τ-closed. Consequently, $O \in \tau$.

4. MACKEY Q-ALGEBRA

In 1985 Akkar [6] named a unital algebra A with convex bornology $\mathscr{B}$ a *bornological Q-algebra* when the set Inv$A \in \tau_{\mathscr{B}}$. The first example of such bornological algebra was given by Hogbe-Nlend in [13, Proposition II.1.1]. In 1992 Oudadess in [17] called such commutative unital complete locally m-convex algebras (A, τ) for which Inv$A \in \tau_{\mathscr{B}_\tau}$ *Mackey Q-algebras*.

We will say that a topological algebra (A, τ) is a *Mackey Q-algebra* when the set Qinv$A$ (when A is a unital algebra, then Inv$A$) is open in the Mackey closure topology $\tau_{\mathscr{B}_\tau}$. Since $\tau \subset \tau_{\mathscr{B}_\tau}$ by Property 6, and $\tau = \tau_{\mathscr{B}_\tau}$ by Property 7, if A is metrizable, then every Q-algebra is a Mackey Q-algebra and every metrizable Mackey Q-algebra is a Q-algebra. There are Mackey Q-algebras that are not Q-algebras. For example (see [7, Proposition 1]), the algebra $C(X)$ of all $\mathbb{K}$-valued continuous functions on a pseudocompact non-compact Hausdorff space X endowed with the compact-open topology is a Mackey Q-algebra because the spectrum $\mathrm{sp}_{C(X)}(f) = f^\beta(\beta X)$ for every $f \in C(X)$ (here βX denotes the Stone-Čech compactification of X and f^β the continuous extension of f on βX) is bounded. By Theorem 1, the boundedness of the spectrum is equivalent to the statement that $C(X)$ is a Mackey Q-algebra. However, it is not a Q-algebra. Some other examples could be found in [6,10,13].

Next we describe the main properties of Mackey Q-algebras.

Theorem 1. *Let (A,τ) be a topological algebra over $\mathbb{C}$. Then the following statements are equivalent* (statements (1)–(5) are equivalent also for topological algebras over $\mathbb{R}$):
(1) *(A,τ) is a Mackey Q-algebra;*
(2) $\mathrm{Qinv}A$ *is a neighbourhood of zero in the Mackey closure topology;*
(3) *the interior of* $\mathrm{Qinv}A$ *is not empty in the Mackey closure topology;*
(4) *θ_A is an interior point of* $\mathrm{Qinv}A$ *in the Mackey closure topology;*
(5) $\mathrm{Qinv}A$ *is a bornivore in* (A,τ) (that is, $\mathrm{Qinv}A$ absorbs every bounded subset of (A,τ))*;*
(6) *the spectral radius r_A, as a map, transforms every τ-bounded subset of A to bounded subset in $\mathbb{R}^+$;*
(7) $A = \{a \in A : \mathrm{r}_A(a)$ is finite$\}$*;*
(8) $S = \{a \in A;\ \mathrm{r}_A(a) \leqslant 1\}$ *is a neighbourhood of zero in the Mackey closure topology;*
(9) *there is a balanced neighbourhood V of zero in A in the Mackey closure topology such that $r_A(a) \leqslant g_V(a)$ for each $a \in A$* (here g_V, denotes the gauge function of V, which exists because V is an absorbing set in A by Property 4).

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial.

(4) $\Rightarrow$ (1). See the proof of Lemma 3.1 in [4, p. 202].

(1) $\Rightarrow$ (5). Let $B \in \mathscr{B}_\tau$. Since $\mathscr{B}_\tau$ is a linear bornology, then $b(B) \in \mathscr{B}_\tau$ is balanced and $B \subset b(B)$. By statement (1), the set $\mathrm{Qinv}A \in \tau_{\mathscr{B}_\tau}$. Therefore, there is a positive number μ such that $\mu b(B) \subset \mathrm{Qinv}A$. Hence, $\mathrm{Qinv}A$ is a bornivore in (A,τ) because $\mu B \subset \mu b(B) \subset \mathrm{Qinv}A$.

(5) $\Rightarrow$ (1). Let $a \in \mathrm{Qinv}A$ and B be a balanced set in $\mathscr{B}_\tau$. Since $\mathscr{B}_\tau$ is an algebra bornology, then $B - Ba_q^{-1}$ is τ-bounded. Thus, by statement (5), there exists a $\lambda > 0$ such that $\lambda(B - Ba_q^{-1}) \subset \mathrm{Qinv}A$. Taking this into account, we have

$$a + \lambda B = \left[(a+\lambda B)\circ a_q^{-1}\right]\circ a = \lambda(B - Ba_q^{-1})\circ a \subset \mathrm{Qinv}A \circ a \subset \mathrm{Qinv}A.$$

Hence, $\mathrm{Qinv}A$ is open in the Mackey closure topology.

(5) $\Rightarrow$ (6). Let B be a τ-bounded subset in A. Then, by statement (5), there exists a number $\lambda > 0$ such that $B \subset \lambda \mathrm{Qinv}A$. Therefore, $\mathrm{r}_A(b) < \lambda$ for each $b \in B$. Hence, r_A transforms B to a bounded set in $\mathbb{R}^+$.

(6) $\Rightarrow$ (7). Since every one-point set is τ-bounded, then from statement (6) follows (7).

(7) $\Rightarrow$ (5). Let $\mathrm{Qinv}A$ not be a bornivore in (A,τ). Then there is a τ-bounded set B in A and for every $\lambda_0 > 0$ a number λ with $|\lambda| \geqslant \lambda_0$ such that $\frac{B}{\lambda} \not\subset \mathrm{Qinv}A$. Hence, there is an element $b \in B$ such that $\frac{b}{\lambda} \notin \mathrm{Qinv}A$. That is, $\lambda \in \mathrm{sp}_A(b)$ when $|\lambda| \geqslant \lambda_0$. Consequently, $\mathrm{r}_A(b)$ is not finite. Thus, from statement (7) follows (5).

(2) $\Rightarrow$ (8). Let $\mathrm{Qinv}A$ be a neighbourhood of zero in A in the Mackey closure topology. Then, by Property 2, there is a balanced neighbourhood O of zero such that $O \subset \mathrm{Qinv}A$. Suppose that there is an element $x \in O \setminus S$. Then $\mathrm{r}_A(x) > 1$. Hence, there is an element $\lambda \in \mathrm{sp}_A(x)$ such that $|\lambda| > 1$. However, on the other hand, $\frac{x}{\lambda} \in \frac{1}{\lambda}O \subset O \subset \mathrm{Qinv}A$ because $\frac{1}{|\lambda|} < 1$ and O is balanced. That is, $\lambda \notin \mathrm{sp}_A(a)$. Hence, $O \subset S$. Consequently, S is a neighbourhood of zero in the Mackey closure topology.

(8) $\Rightarrow$ (2). Suppose that there is an element $x \in S$ such that $\frac{x}{2} \notin \mathrm{Qinv}A$. Then from $2 \in \mathrm{sp}_A(x)$ follows that $\mathrm{r}_A(x) \geqslant 2$. Hence, $\frac{S}{2} \subset \mathrm{Qinv}A$. This means that $\mathrm{Qinv}A$ is open in the Mackey closure topology.

(2) $\Rightarrow$ (9). Let $a \in A$. By statement (2) the set $\mathrm{Qinv}A$ is a neighbourhood of zero in the Mackey closure topology. Then, by Property 2, there is a balanced neighbourhood V of zero in the Mackey closure topology such that $V \subset \mathrm{Qinv}A$. If $\mu > 0$ is an arbitrary number such that $a \in \mu V$, then $\mu \notin \mathrm{sp}_A(a)$. Therefore, $\mathrm{r}_A(a) < \mu$ for each $\mu > 0$ such that $a \in \mu V$. Hence,

$$\mathrm{r}_A(a) \leqslant \inf\{\mu > 0 : a \in \mu V\} = g_V(a).$$

(9) $\Rightarrow$ (2). Suppose that there is a balanced neighbourhood V of zero such that $\mathrm{r}_A(a) \leqslant g_V(a)$ for each $a \in A$. If $a \in \frac{1}{2}V$ is an arbitrary element, then

$$\mathrm{r}_A(a) \leqslant g_V(a) \leqslant \frac{1}{2} < 1.$$

Hence, $1 \notin \mathrm{sp}_A(a)$. Therefore, $\frac{1}{2}V \subset \mathrm{Qinv}A$. This means that Qinv$A$ is a neighbourhood of zero in A in the Mackey closure topology. □

In particular, when the spectrum of every element of a topological algebra is functional, we have the following equivalent conditions. To describe these, we recall that a map f from one topological linear space X to another topological linear space Y is *Mackey continuous* at $x \in X$ if for every net $(x_\lambda)_{\lambda\in\Lambda}$, which Mackey converges to x_0, the net $(f(x_\lambda))_{\lambda\in\Lambda}$ Mackey converges to $f(x_0)$ and f is a *Mackey continuous map* if f is Mackey continuous at every point of X. Moreover, a collection $\mathscr{F}$ of maps f from a topological linear space X into another topological linear space Y is *equibounded* if the union $\bigcup\{f(B) : f \in \mathscr{F}\}$ is bounded in Y for each bounded subset B in X.

Theorem 2. *Let (A,τ) be a topological algebra with functional spectrum. Then the following statements are equivalent:*
(1) *(A,τ) is a Mackey Q-algebra*;
(2) *the spectrum* $\mathrm{sp}_A(a)$ *of a is a bounded set for every $a \in A$*;
(3) *the specrum* $\mathrm{sp}_A(a)$ *of a is a compact set for every $a \in A$*;
(4) *the spectral radius* r_A *is a bounded map*;
(5) *the spectral radius* r_A *is Mackey continuous* at zero;
(6) *the spectral radius* r_A *is a Mackey continuous map*;
(7) *the set* $\{a \in A : \mathrm{r}_A(a) < 1\} \in \tau_{\mathscr{B}_\tau}$;
(8) *the set* Hom$A$ *is equibounded.*

Proof. (1) ⇒ (2). The proof[1] of this conclusion is given in [5, Proposition 3], but not in detail. Therefore, we present here a new, more detailed proof.

First we remark that the set Qinv$A$ is a neighbourhood of zero in the Mackey closure topology by statement (1). Therefore (by Property 2), there is a balanced neighbourhood O of zero in the Mackey closure topology such that $O \subset \mathrm{Qinv}A$. As O is an absorbing set in A by Property 4, then there is a number $\varepsilon_a > 0$ such that $\mu a \in O$ whenever $0 < |\mu| \leqslant \varepsilon_a < 1$.

Let now $\rho \in (0, \varepsilon_a)$. If there is an element $a \in A$ such that $\mathrm{r}_A(a) > \frac{1}{\rho}$, then there is a number $\mu_o \in \mathrm{sp}_A(a)$ such that $|\mu_0| > \frac{1}{\rho}$. But now

$$\frac{a}{\mu_0} = \rho\left(\frac{1}{\rho\mu_0}\right)a \in \left(\frac{1}{\rho\mu_0}\right)O \subset O \subset \mathrm{Qinv}A$$

because $\rho < 1$, $\left|\frac{1}{\rho\mu_o}\right| < 1$, and O is balanced. Hence $\mu_0 \notin \mathrm{sp}_A(a)$, which is impossible. So, $\mathrm{r}_A(a) \leqslant \frac{1}{\rho} < \infty$ for every $a \in A$. Consequently, the spectrum $\mathrm{sp}_A(a)$ of every $a \in A$ is bounded.

(2) ⇒ (1). By statement (2) the spectral radius $\mathrm{r}_A(a)$ of every $a \in A$ is finite. Therefore, (A,τ) is a Mackey Q-algebra by Theorem 1.

(2) ⇒ (3). Suppose that $\mathrm{sp}_A(a)$ is not closed in $\mathbb{C}$ for some $a \in A$. Then there exists a $\mu_a \in \mathbb{C}$ such that $\mu_a \in \mathrm{cl}_{\mathbb{C}}(\mathrm{sp}_A(a)) \setminus \mathrm{sp}_A(a)$ because which $\frac{1}{\mu_a}a \in \mathrm{Qinv}A$ (here $\mu_a \neq 0$ because $0 \in \mathrm{sp}_A(a)$). By assumption, (A,τ) is an algebra with functional spectrum. Therefore, hom$A$ is not empty and

$$\mathrm{sp}_A(a) = \{\varphi(a) : \varphi \in \mathrm{hom}A\} \cap S,$$

where $S = \{0\}$ if $a \notin \cap\{\ker\varphi : \varphi \in \mathrm{hom}A\}$ and otherwise S is an empty set. Now there exists a sequence (φ_n) in hom$A$ such that the sequence $(\varphi_n(a))$ converges to μ_a. It is well known (see, for example [18, Theorem 1.6.11]) that

$$\mathrm{sp}_A(a_q^{-1}) = \left\{\frac{\lambda}{\lambda - 1} : \lambda \in \mathrm{sp}_A(a)\right\}.$$

[1] The proof does not use the restriction that the spectrum of every element is functional.

Therefore,

$$\mathrm{sp}_A\left[\left(\frac{a}{\mu_a}\right)_q^{-1}\right]=\left\{\frac{\varphi(a)}{\varphi(a)-\mu_a}:\varphi\in\mathrm{hom}A\right\}.$$

Thus

$$\mathrm{sp}_A\left[\left(\frac{a}{\mu_a}\right)_q^{-1}\right]$$

is not bounded. This is not possible by statement (2). Hence, $\mathrm{sp}_A(a)$ is closed in $\mathbb{C}$ for each $a\in A$. So, the spectrum of each $a\in A$ is a compact set in $\mathbb{C}$.

(3) $\Rightarrow$ (2) is trivial.

(1) $\Rightarrow$ (4) is true by Theorem 1.

(4) $\Rightarrow$ (5). Let $(a_\lambda)_{\lambda\in\Lambda}$ be a net in A that Mackey converges to zero. Then there exists a balanced $B\in\mathscr{B}_\tau$ and for every $\varepsilon>0$ there exists an index $\lambda_\varepsilon\in\Lambda$ such that $a_\lambda\in\varepsilon B$ whenever $\lambda\succ\lambda_\varepsilon$. Since $\mathrm{r}_A(a_\lambda)\in\varepsilon\mathrm{r}_A(B)\subset\varepsilon b(\mathrm{r}_A(B))$ whenever $\lambda\succ\lambda_\varepsilon$ and $\mathrm{r}_A(B)$ is bounded in $[0,\infty)$ by statement (4) (hence, $b(\mathrm{r}_A(B))$ is balanced and bounded), then the net $(\mathrm{r}_A(a_\lambda))_{\lambda\in\Lambda}$ Mackey converges to $0=\mathrm{r}_A(\theta_A)$. Therefore, r_A is Mackey continuous at θ_A (without the restriction that the spectrum of every element is functional).

(5) $\Rightarrow$ (6). Let $(a_\lambda)_{\lambda\in\Lambda}$ be a net in A that Mackey converges to $a\in A$. Then the net $(a_\lambda-a)_{\lambda\in\Lambda}$ Mackey converges to θ_A. Hence, by statement (5), the net $(\mathrm{r}_A(a_\lambda-a))_{\lambda\in\Lambda}$ Mackey converges (hence, converges also in topology of $\mathbb{R}^+$) to $\mathrm{r}_A(\theta_A)=0$. Since A has the functional spectrum, then r_A is a subadditive map. Hereby,

$$|\mathrm{r}_A(a)-\mathrm{r}_A(b)|\leqslant\mathrm{r}_A(a-b)$$

for all $a,b\in A$. Taking this into account, the net $(\mathrm{r}(a_\lambda))_{\lambda\in\Lambda}$ converges to $\mathrm{r}_A(a)$. Because $[0,\infty)$ is a metric space, the net $(\mathrm{r}(a_\lambda))_{\lambda\in\Lambda}$ Mackey converges to $\mathrm{r}_A(a)$ too. This means that r_A is a Mackey continuous map.

(6) $\Rightarrow$ (7). Let $U=A\setminus\{a\in A:\mathrm{r}_A(a)<1\}$ and $(a_\lambda)_{\lambda\in\Lambda}$ be a net in U that Mackey converges to a_0. Then $\mathrm{r}_A(a_\lambda)\geqslant 1$ for each $\lambda\in\Lambda$. By statement (6), $(\mathrm{r}_A(a_\lambda))_{\lambda\in\Lambda}$ Mackey converges (hence, converges in the topology τ as well) to $\mathrm{r}_A(a_0)\geqslant 1$ or $a_0\in U$. Hence, U is Mackey closed by Property 3. Consequently, the set $\{a\in A:\mathrm{r}_A(a)<1\}$ is open in the Mackey closure topology.

(7) $\Rightarrow$ (1). By statement (7), the set $O=\{a\in A:\mathrm{r}_A(a)<1\}$ is a neighbourhood of zero in A in the Mackey closure topology. Since $O\subset\mathrm{Qinv}A$, then θ_A is an interior point of $\mathrm{Qinv}A$. Hence, (A,τ) is a Mackey Q-algebra by Theorem 1.

(4) $\Rightarrow$ (8). Since A has a functional spectrum for every $a\in A$ and

$$\{\varphi(a):\varphi\in\mathrm{hom}A\}\subseteq\{\psi(a):\psi\in\mathrm{Hom}A\}\subset\mathrm{sp}_A(a)$$

for each $a\in A$, then

$$\mathrm{r}_A(a)=\sup\{|\psi(a)|:\psi\in\mathrm{Hom}A\}$$

for each $a\in A$. Hence,

$$\bigcup_{\psi\in\mathrm{Hom}A}\psi(B)$$

is bounded in $(0,\infty)$ for each bounded subset B in A by statement (4). Consequently, $\mathrm{Hom}A$ is an equibounded set.

(8) $\Rightarrow$ (4). By statement (8), for every bounded set B in A there is a number $M_B>0$ such that $|\psi(a)|<M_B$ for all $a\in B$ and $\psi\in\mathrm{Hom}A$. Therefore, the set $\mathrm{r}_A(B)$ is bounded. Hence, r_A is a bounded map. □

To prove other properties of Mackey Q-algebras, we need

Lemma 1. *If I is a non-dense left* (*right or two-sided*) *ideal in A, then the closure of I in the Mackey closure topology is a left* (*respectively, right or two-sided*) *ideal of A.*

Proof. Let I be a left ideal in A and J the closure of I in the Mackey closure topology. Then J is closed with respect to the addition and the multiplication by scalars (see [12, Corollary 1, p. 15]. To show that J is closed with respect to the multiplication with elements of A, let $a \in A$ and g_a denote the map defined by $g_a(b) = ab$ for each $b \in A$. Since g_a is continuous in the Mackey closure topology, then $g_a(J) \subset \mathrm{cl}(g_a(I)) \subset J$ (here $\mathrm{cl}(S)$ denotes the closure of $S \subset A$ in the Mackey closure topology). Hence, $AJ \subset J$. Because $J \neq A$, J is a left ideal in A. □

The proofs for right and two-sided ideals are similar.

Theorem 3. *Let A be a Mackey Q-algebra. The following statements hold:*
(1) *every maximal regular left* (*right or two-sided*) *ideal of A is closed in the Mackey closure topology;*
(2) *the Jacobson radical* $\mathrm{Rad}A$ *is closed in the Mackey closure topology;*
(3) *if* $\mathrm{Hom}A$ *is not empty, then* $\mathrm{Hom}A = \mathrm{hom}A$*;*
(4) *if $\beta_A(a) = \mathrm{r}_A(a)$ for each $a \in A$, then every element in A is bounded.*

Proof. (1) Let M be a maximal regular left ideal in A, u the right unit in A modulo M, and N the closure of M in the Mackey closure topology on A. Suppose that $N = A$. Since A is a Mackey Q-algebra, then $\mathrm{Qinv}A$ is a neighbourhood of zero in the Mackey closure topology by statement (2) of Theorem 1. Therefore $(-u + \mathrm{Qinv}A) \cap M$ is not empty. Hence, there is an element $m \in M$ such that $m + u \in \mathrm{Qinv}A$. Let b denote the quasi-inverse of $m + u$. Since $b \circ (m+u) = \theta_A$, then $u = -(b - bu) - m + bm \in M$. By this, $A = M$, which is not possible. Hence, N is a left regular ideal in A by Lemma 1. As M is maximal and $M \subset N$, then $M = N$. Hence, M is closed.

The proofs for right and two-sided cases are similar.

(2) It is well known (see, for example [18, Theorem 2.3.2]) that the Jacobson radical $\mathrm{Rad}A$ is the intersection of all maximal regular left (or right) ideals of A. Therefore, $\mathrm{Rad}A$ is closed in A by statement (1) of Theorem 3.

(3) Since $\ker\psi$ is a maximal regular two-sided ideal in A for each $\psi \in \mathrm{Hom}A$, then $\ker\psi$ is closed. Therefore, ψ is continuous [19, Chapter III, statement 1.3].

(4) It is known that in every Mackey Q-algebra A over $\mathbb{C}$ the inequality $\rho_A(a) \leqslant \beta_A(a)$ holds for each $a \in A$ (see [2, Theorem 4.2]). If now A is a Mackey Q-algebra in which $\mathrm{sp}_A(a) = \beta_A(a)$ for each $a \in A$, then, by statement (7) of Theorem 1, there is an $M_a > 0$ such that

$$\beta_A\left(\frac{a}{M_a}\right) = \mathrm{r}\left(\frac{a}{M_a}\right) < 1.$$

□

Hence, a is bounded in A.

5. ADVERTIBLY MACKEY COMPLETE ALGEBRAS

The class of advertibly complete locally m-convex algebras was introduced by Warner in [20]. It is known (see [16, Theorem 6.4, p. 45]) that every Q-algebra is advertibly complete. An example of an advertibly complete locally m-convex algebra, which is neither a complete algebra nor a Q-algebra, is given in [20, p. 8].

Let now A be a topological algebra. A net $(a_\lambda)_{\lambda\in\Lambda}$ is *advertibly convergent* (*advertibly Mackey convergent*) in A if there is an element $a \in A$ such that the nets $(a \circ a_\lambda)_{\lambda\in\Lambda}$ and $(a_\lambda \circ a)_{\lambda\in\Lambda}$ converge (respectively, Mackey converge) to θ_A. In the particular case when A is a unital algebra, a net $(a_\lambda)_{\lambda\in\Lambda}$ is *advertibly convergent* (*advertibly Mackey convergent*) in A if there is an element $a \in A$ such that the nets $(aa_\lambda)_{\lambda\in\Lambda}$ and $(a_\lambda a)_{\lambda\in\Lambda}$ converge (respectively, Mackey converge) to e_A. A topological algebra A is *advertibly complete* (*advertibly Mackey complete*) when every advertiby convergent (respectively, advertibly Mackey convergent) Cauchy net converges in A. Hence, every Mackey complete algebra is an advertibly Mackey complete algebra. Moreover, every advertibly complete algebra is an advertibly Mackey complete algebra. Indeed, let $(a_\lambda)_{\lambda\in\Lambda}$ be an advertibly Mackey convergent Cauchy net in an advertibly complete algebra A. Then $(a_\lambda)_{\lambda\in\Lambda}$ is also an advertibly convergent Cauchy net that converges in A.

Proposition 1. *Let A be a Mackey Q-algebra. Then every advertibly Mackey convergent net in A is Mackey convergent.*

Proof. Let $(a_\lambda)_{\lambda\in\Lambda}$ be an advertibly Mackey convergent net in A. Then there exists an element $a\in A$ such that the nets $(a_\lambda\circ a)_{\lambda\in\Lambda}$ and $(a\circ a_\lambda)_{\lambda\in\Lambda}$ Mackey converge to θ_A. Hence, there exists a balanced and bounded subset B of A and for any given $\varepsilon>0$ an index $\lambda_\varepsilon\in\Lambda$ such that $a\circ a_\lambda\in\varepsilon B$ and $a_\lambda\circ a\in\varepsilon B$ whenever $\lambda\succeq\lambda_\varepsilon$. Since A is a Mackey Q-algebra, then Qinv$A$ is a bornivore in A by Theorem 1. Therefore, there is a number $\mu>0$ such that $B\subset\mu\mathrm{Qinv}A$. As B is balanced, then

$$a\circ a_\lambda\in\mu^{-1}(\varepsilon\mu)B\subset\mu^{-1}B\subset\mathrm{Qinv}A$$

and

$$a_\lambda\circ a\in\mu^{-1}(\varepsilon\mu)B\subset\mu^{-1}B\subset\mathrm{Qinv}A$$

for $\varepsilon\leqslant\frac{1}{\mu}$ whenever $\lambda\succeq\lambda_\varepsilon$.

Let now $\lambda_0\in\Lambda$ be such that $\lambda_0\succeq\lambda_\varepsilon$, $x\in A$ be the quasi-inverse of $a\circ a_{\lambda_0}$, and $y\in A$ the quasi-inverse of $a_{\lambda_0}\circ a$. Then from

$$a\circ a_{\lambda_0}\circ x=\theta_A=y\circ a_{\lambda_0}\circ a$$

follows that $a_{\lambda_0}\circ x$ is the right quasi-inverse of a and $y\circ a_{\lambda_0}$ the left quasi-inverse of a. As these are equal, then $a\in\mathrm{Qinv}A$. Let a_q^{-1} denote the quasi-inverse of a. Then

$$(a_\lambda-a_q^{-1})\circ a=a_\lambda\circ a-a_q^{-1}\circ a+a=a_\lambda\circ a+a\subset\varepsilon B+a$$

whenever $\lambda\succeq\lambda_\varepsilon$. Taking this into account,

$$a_\lambda-a_q^{-1}=(a_\lambda-a_q^{-1})\circ\theta_A=[(a_\lambda-a_q^{-1})\circ a]\circ a_q^{-1}\subset$$

$$(\varepsilon B+a)\circ a_q^{-1}=\varepsilon(B-Ba_q^{-1})$$

whenever $\lambda\succeq\lambda_\varepsilon$, where $B-Ba_q^{-1}$ is a balanced and bounded set in A. Consequently, every advertibly Mackey convergent net $(a_\lambda)_{\lambda\in\Lambda}$ Mackey converges in A. □

Corollary 1. *Every Mackey Q-algebra is advertibly Mackey complete.*

Corollary 2. *Let A be a Mackey Q-algebra. If every topologically convergent net is Mackey convergent in A* (as in the metrizable case)*, then A is an advertibly complete algebra.*

Proof. Let $(a_\lambda)_{\lambda\in\Lambda}$ be an advertibly convergent Cauchy net. Then there is an element $a\in A$ such that $(a\circ a_\lambda)_{\lambda\in\Lambda}$ and $(a_\lambda\circ a)_{\lambda\in\Lambda}$ topologically converge to zero in A. By assumption, these nets Mackey converge to zero as well in A. Since A is a Mackey Q-algebra, the net $(a_\lambda)_{\lambda\in\Lambda}$ is Mackey convergent in A by Proposition 1. Therefore, $(a_\lambda)_{\lambda\in\Lambda}$ topologically converges in A. Hence, A is an advertibly complete algebra. □

6. CONNECTIONS OF MACKEY Q-ALGEBRAS WITH OTHER CLASSES OF TOPOLOGICAL ALGEBRAS

6.1. A topological algebra $(A;\tau)$ is called a *strongly sequential algebra* (*strongly Mackey sequential algebra*) if there is a neighbourhood O of zero in the topology τ (in the Mackey closure topology) such that the sequence (a^n) converges (Mackey converges) to zero for all $a \in O$. For example, the algebra $\mathscr{K}(\mathbb{R})$ (of all continuous $\mathbb{C}$-valued functions with compact support, endowed with the strict inductive limit topology defined by Banach algebras

$$K_n = \big\{f \in C\big([-n,n]\big) : f(-n) = f(n) = 0\big\},$$

where $n \in \mathbb{N}$) is a strongly sequential algebra. Here

$$O = \Big\{f \in \mathscr{K}(\mathbb{R}) : \sup_{v\in[-n,n]} |f(x)| < 1 \text{ for each } n \in \mathbb{N}\Big\}$$

is a neighbourhood of zero in $\mathscr{K}(\mathbb{R})$ such that $f^k \in O$ for each $f \in O$ if $k \in \mathbb{N}$. Hence (f^k) converges to zero.

It is easy to see that every strongly Mackey sequential algebra is a strongly sequential algebra and both topological algebras coincide in the metrizable case.

To show that next results hold without any kind of convexity of a topological algebra, we need the following results.

Lemma 2 ([17, Proposition 6.1]). *A topological algebra A is strongly sequential if and only if the radius map β_A of boundedness is continuous at zero.*

Proposition 2. *If A is an advertive Hausdorff algebra over $\mathbb{C}$, then $\rho_A(a) \leqslant \beta_A(a)$ for each $a \in A$.*

Proof. Let A be an advertive Hausdorff algebra, that is $\mathrm{Tqinv}A = \mathrm{Inv}A$, and let $a \in A$. If $\beta_A(a) = \infty$, then $\rho_A(a) \leqslant \beta_A(a)$ holds. Let now $\beta_A(a) \in \mathbb{R}^+$ and λ be an arbitrary number in $\mathbb{C}$ such that $\beta_A(a) < |\lambda|$. Then

$$\lim_{n\to\infty} \left(\frac{a}{\lambda}\right)^n = \theta_A.$$

We put

$$S_n = -\sum_{k=1}^{n} \left(\frac{a}{\lambda}\right)^k$$

for each $n \in \mathbb{N}$. Since

$$S_n \circ \frac{a}{\lambda} = \frac{a}{\lambda} \circ S_n = \left(\frac{a}{\lambda}\right)^{n+1}$$

for each $n \in \mathbb{N}$, the sequence (S_n) converges advertibly. Hence (see [1, Proposition 1]), the sequence (S_n) converges in A, because A is an advertive Hausdorff algebra. Consequently, $\frac{a}{\lambda} \in \mathrm{Qinv}A$. This means that $\lambda \notin \mathrm{sp}_A(a)$ or $\rho_A(a) \leqslant |\lambda|$ for each $\lambda \in \mathbb{C}$ with $\beta_A(a) < |\lambda|$. Consequently, $\rho_A(a) \leqslant \beta_A(a)$ for every $a \in A$. □

Proposition 3. *Let A be a strongly sequential algebra over $\mathbb{C}$. Then $\rho_A(a) \leqslant \beta_A(a)$ for each $a \in A$ if and only if A is a Q-algebra.*

Proof. If A is a Q-algebra, then $\rho_A(a) \leqslant \beta_A(a)$ for each $a \in A$ by Proposition 2, because every Q-algebra is advertive (see [1, Proposition 2]). Conversely, let $\rho_A(a) \leqslant \beta_A(a)$ for each $a \in A$. Since A is strongly sequential, then β_A is continuous at zero by Lemma 1. Therefore, ρ_A is also continuous at zero. Hence $\{a \in A : \rho_A(a) \leqslant 1\}$ is a neighbourhood of zero in A. Consequently, A is a Q-algebra by [16, Lemma 4.2, p. 59]. □

It is known that $\rho_A(a) \leqslant \beta_A(a)$ for each a in every Mackey Q-algebra A over $\mathbb{C}$ (see [2, Corollary 4.3]). Therefore, we have

Corollary 3. *Every strongly sequential Mackey Q-algebra over $\mathbb{C}$ and every advertive Mackey Q-algebra over $\mathbb{C}$ is a Q-algebra.*

Let now A be a strongly Mackey sequential algebra with $\rho_A(a) \leqslant \beta_A(a)$ for each $a \in A$. Then there is a neighbourhood O of zero in Mackey closure topology such that (a^n) Mackey converges (hence topologically as well) to zero for each $a \in O$. Therefore, $\rho_A(a) \leqslant \beta_A(a) < 1$ for each $a \in O$, because of which $O \subset \mathrm{Qinv}A$. Consequently, Qinv$A$ is open in the Mackey closure topology. Hence, we have

Proposition 4. *Every strongly Mackey sequential algebra over $\mathbb{C}$ with $\rho_A(a) \leqslant \beta_A(a)$ for each $a \in A$ is a Mackey Q-algebra.*

6.2. A topological algebra A is called an *infrasequential algebra* if for every bounded subset B of A there is a number $\lambda > 0$ such that the sequence $((\lambda a)^n)$ converges to zero for all $a \in B$. For example, every normed algebra A is an infrasequential algebra, because for every bounded set $\{a \in A : ||a|| < M\}$ with $M > 0$, $0 < \varepsilon < 1$, and $\varepsilon_0 \in (0, \varepsilon)$ we have

$$\left|\left|\left(\frac{\varepsilon_0}{M}a\right)^n\right|\right| \leqslant \frac{\varepsilon_0^n}{M^n}||a||^n < \varepsilon_0^n < \varepsilon$$

for each $n \in \mathbb{N}$. Hence, $\left(\left(\frac{\varepsilon_0 a}{M}\right)^n\right)$ converges to zero.

The next result is well known in the case of locally convex and locally k-convex algebras. We show that this result holds without any kind of convexity of a topological algebra.

Proposition 5. *Every infrasequential algebra over $\mathbb{C}$ with $\rho_A(a) \leqslant \beta_A(a)$ for each $a \in A$ is a Mackey Q-algebra.*

Proof. Let A be an infrasequential algebra and B a bounded subset in A. Then there is a $\lambda > 0$ such that $(\lambda a)^n$ converges to zero for each $a \in B$. Hence, $\rho_A(a) \leqslant \beta_A(a) < \frac{1}{\lambda}$ for each $a \in B$. Consequently, $\lambda B \subset \mathrm{Qinv}A$. It means that Qinv$A$ is a bornivore. Hence, A is a Mackey Q-algebra by Theorem 1. □

By Propositions 2 and 5 we have the following result.

Corollary 4. *Every infrasequential advertive Hausdorff algebra over $\mathbb{C}$ is a Mackey Q-algebra.*

6.3. We say that a topological algebra A is a *netial algebra* if for each net $(a_\lambda)_{\lambda\in\Lambda}$, converging to zero, there exists an index $\lambda_0 \in \Lambda$ such that the sequence $(a_{\lambda_0}^n)$ converges to zero. In a particular case when $\Lambda = \mathbb{N}$, we speak about *sequential algebras*. An example of sequential algebras is $C([0,1])$ of continuous $\mathbb{C}$-valued functions with the uniform convergence topology on compact subsets of $[0,1]$ (see [15, Example 3.22]).

Proposition 6. *Every netial (sequential) algebra over $\mathbb{C}$ with $\rho_A(a) \leqslant \beta_A(a)$ for each $a \in A$ is an advertibly complete (respectively, sequentially advertibly complete) algebra.*

Proof. Let A be a netial algebra and $(a_\lambda)_{\lambda\in\Lambda}$ an arbitrary advertibly convergent Cauchy net in A. Then there is an element $a \in A$ such that $(a \circ a_\lambda)_{\lambda\in\Lambda}$ and $(a_\lambda \circ a)_{\lambda\in\Lambda}$ converge to zero in A. Since A is a netial algebra, then there is $\lambda_0 \in \Lambda$ such that the sequences $((a_{\lambda_0} \circ a)^n)$ and $((a \circ a_{\lambda_0})^n)$

converge to zero. Therefore $\rho_A(a \circ a_{\lambda_0}) \leqslant \beta_A(a \circ a_{\lambda_0}) < 1$ and $\rho_A(a_{\lambda_0} \circ a) \leqslant \beta_A(a_{\lambda_0} \circ a) < 1$. Hence, $a \circ a_{\lambda_0}, a_{\lambda_0} \circ a \in \mathrm{Qinv}A$. Similarly as in the proof of Proposition 1, $a \in \mathrm{Qinv}A$ and $(a_\lambda)_{\lambda\in\Lambda}$ converges to the quasi-inverse a_q^{-1} of a. Consequently, A is an advertibly complete algebra.

The proof for the case when $\Lambda = \mathbb{N}$ is similar. □

Corollary 5. *All netial advertive Hausdorff algebras over $\mathbb{C}$ and all netial Mackey Q-algebras over $\mathbb{C}$ are advertibly complete algebras.*

7. REMARK

Several results of the present paper are known, but mainly in the unital case, unital locally convex or unital locally k-convex case with $k \in [0,1]$, or in the case of unital algebras with convex bornology. Namely, Theorem 1 is known partly in the case of commutative unital complete locally m-convex algebras (see [6, Corollary 6], where the equivalence of statements (1), (2), and (6) is shown) and in the case of unital algebras with convex bornology (see [7, Proposition 2], where the equivalence of statements (1) and (3) is shown, and [6, Corollary 4], where it is shown that from statement (1) follows (7)). Theorem 2 is known partly in the case of the unital algebra with convex bornology (see [6, Theorem 3], where the equivalence of statements (1) and (5) is proved) and in the case of commutative unital complete locally m-convex algebras (see [6, Theorem, p. 61] and [7, Corollary 3], where the equivalence of statements (1)–(6) and (8) is shown, and [17, Proposition 3.3], where the equivalence of statements (1) and (3) is proved) and in the case of unital commutative locally convex algebras of special types (see [9, Proposition 5], where the equivalence of statements (1) and (3) is proved). Theorem 3 is known partly in the case of unital complete bornological multiplicative convex algebras (see [13, Proposition II.1.3 and Corollary, p. 32], where statements (1) and (2) are proved) and in the case of unital algebras with convex bornology (see [6, Proposition 6], where statement (1) is proved). Proposition 1 is known in the case of unital algebras (see [11, Proposition 4.1]). Proposition 3 is known (see [3, Corollary 4.3] and in the case of unital locally pseudoconvex algebras in [11, Proposition 2.4]). Corollary 3 is proved in the case of unital locally k-convex algebras with $k \in (0,1]$ in [11, Proposition 5.3]; Proposition 5 is proved in the case of unital locally k-convex algebras with $k \in (0,1]$ in [11, Proposition 5.6] and Corollary 5 partly in the case of unital sequential locally k-convex algebras in [11, Corollary 4.5].

8. CONCLUSION

The present paper shows that all these results hold in a general case without the assumption of unity and any kind of convexity for topological algebras.

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REFERENCES

1. Abel, M. Advertive topological algebras. In *General Topological Algebras*. Tartu, 1999, 14–24; *Math. Stud.* (Tartu), 1. Est. Math. Soc., Tartu, 2001.
2. Abel, M. Topological algebras with pseudoconvexly bounded elements. In *Topological Algebras, Their Applications, and Related Topics*. *Banach Center Publications*, **67**, 21–33. Polish Academy of Sciences, Warsaw, 2005.
3. Abel, M. Topological algebras with idempotently pseudoconvex von Neumann bornology. *Contemp. Math.*, 2007, **427**, 15–29.

4. Abel, M. Structure of locally idempotent algebras. *Banach J. Math. Anal.*, 2007, **1**(2), 195–207.
5. Abel, M. Topological algebras with all elements bounded. In *Proceedings of the 8th International Conference of Topological Algebras and Their Applications*, 2015. De Gruyter Proc. Math. (accepted).
6. Akkar, M. Sur le groupe des éléments inversibles d'une algèbre bornologique convexe, Q-algèbres bornologiques convexes. *C. R. Acad. Sc. Paris*, Ser. I, 1985, **300**(2), 35–38.
7. Akkar, M. Caractérisation des algèbres localement m-convexes dont l'ensemble des caractères est équiborné. *Colloq. Math.*, 1995, **68**(1), 59–65.
8. Arnold, B. H. Topologies defined by bounded sets. *Duke Math. J.*, 1951, **18**, 631–642.
9. Bedda, A. and Cheikh, O. H. Boundedness of characters in locally A-convex algebras. *Rend. Circ. Mat. Palermo* Ser. II, 1998, **47**, 91–98.
10. El Adlouni, H. and Oubbi, L. Groupe des inversibles et rayon spectral dans les algèbres topologiques. *Rend. Circ. Mat. Palermo*, 2000, **49**, 527–539.
11. El Kinani, A. Advertible complétute et structure de Q-algèbre. *Rend. Circ. Mat. Palermo*, 2001, **50**, 427–442.
12. Hogbe-Nlend, H. *Théories des bornologies et applications.* Lecture Notes Math., **213**. Springer-Verlag, Berlin, 1971.
13. Hogbe-Nlend, H. Les fondements de la théorie spectrale des algèbres bornologiques. *Bol. Soc. Brasil. Mat.*, 1972, **3**(1), 19–56.
14. Hogbe-Nlend, H. *Bornologies and Functional Analysis. Introductory Course on the Theory of Duality Topology-Bornology and Its Use in Functional Analysis*. North-Holland Math. Studies, **62**. North-Holland Publ. Co., Amsterdam, 1977.
15. Husain, T. *Multiplicative Functionals on Topological Algebras*. Res. Notes in Math., **85**. Bitman, Boston, 1983.
16. Mallios, A. *Topological Algebras. Selected Topics.* North-Holland Math. Studies, **124**. North-Holland Publ. Co., Amsterdam, 1986.
17. Oudadess, M. A note on m-convex and pseudo-Banach structures. *Rend. Circ. Mat. Palermo*, 1992, **41**(1), 105–110.
18. Rickart, C. E. *General Theory of Banach Algebras.* D. van Nostrand Co., Inc., Princeton, NJ, 1960.
19. Schaefer, H. H. *Topological Vector Spaces*. Second edition. Graduate Texts in Math., **3**. Springer-Verlag, New York, 1999.
20. Warner, S. Polynomial completeness in locally multiplicatively-complete algebras. *Duke Math. J.*, 1956, **23**, 1–11.

Mackey Q-algebra

Mati Abel

On kirjeldatud bornoloogia abil defineeritud topoloogia ja Mackey Q-algebra põhiomadusi ning teiste topoloogiliste algebrate ja Mackey Q-algebrate vahelisi vahekordi.