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MATHEMATICAL PHYSICS

A practical method for constructing a reflectionless potential with a given energy spectrum

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Abstract. A fully algebraic approach to constructing one-dimensional reflectionless potentials with any number (N) of bound states is described. A simple and easily applicable general formula is derived, using the methods of the theory of determinants. In particular, useful properties of special determinants – the alternants – have been exploited. The modified determinant that uniquely fixes the potential contains only 2^{N-1} terms, which is a huge win compared to the $N!$ terms of the original expansion. Moreover, the modified determinant can be very easily evaluated using the properties of alternants. To this end, two useful theorems have been proved. The main formula takes an especially simple form if one aims to reconstruct a symmetric reflectionless potential. Several examples are presented to illustrate the efficiency of the method.

Key words: inverse scattering, Gel′fand–Levitan–Marchenko equation, reflectionless potentials, alternants, elementary symmetric polynomials.

1. INTRODUCTION

The formulation and solution of inverse problems is an increasingly important field of scientific research. However, compared to a well-posed (in Hadamard's sense) direct or forward problem, the corresponding inverse problem is much more difficult and, as a rule, ill-posed. The inverse scattering problem can be considered an exception to this rule. Namely, in the simplest one-dimensional case, the inverse scattering theory provides strict mathematical criteria for the existence, uniqueness, and stability of the solution. It means that in this particular case the inverse problem is well-posed as well.

The related forward problem is the solution of the simplest time-independent Schrödinger equation

$$\Psi''(x) = \frac{V(x) - E}{C}\Psi(x),\ C \equiv \frac{\hbar^2}{2m} \tag{1}$$

for a given potential $V(x)$, and subjected to appropriate physical boundary conditions. Equation (1) can be easily solved numerically and thus, in principle, all spectral characteristics of the potential $V(x)$ can be accurately ascertained.

The inverse problem is to determine the **unknown** potential starting from the known spectral characteristics. This is a serious task even in this simple case due to the following problems:

1. It is not obvious what kind of input information is actually needed to solve the problem uniquely.
2. There must be a theoretical basis (a fundamental equation) which enables us to solve the problem.
3. Apart from the theoretical difficulties, another important question arises: how to obtain the necessary input data?
4. Even if the mentioned principal barriers could be overcome, the computational–technical solution of the problem is not at all trivial.

Problems 1 and 2 have been successfully solved in the early 1950s for the class of potentials on the half line: $x \in [0,\infty)$. The necessary and sufficient conditions for the unique solution of the inverse problem have been formulated in a series of outstanding theoretical works by Marchenko [1,2], Gel′fand and Levitan [3], Krein [4,5], and others (see, e.g., [6], section III.7 for an overview). In addition, three different methods to solve the problem have been worked out, based on the integral equations by Gel′fand–Levitan [3], Marchenko [2], and Krein [5].

The solution of the inverse scattering problem on the full line ($-\infty < x < \infty$) is a more challenging problem, which was first addressed by Kay [7], Kay and Moses [8], and a few years later in a series of papers by Faddeev [9–12]. These fundamental studies provide a full description of the solution procedure, while the correct necessary and sufficient criteria for the uniqueness of the solution were given by Marchenko (see [13], section III.5). These criteria apply to the following pair of integral equations [8,12,13]:

$$K_1(x,y) + A_1(x+y) + \int_x^\infty A_1(z+y)K_1(x,z)dz = 0, \tag{2}$$

$$K_2(x,y) + A_2(x+y) + \int_{-\infty}^x A_2(z+y)K_2(x,z)dz = 0, \tag{3}$$

which are formally very similar to the Marchenko equation on the half line (see [13], p. 218). On the other hand, their operator-theoretical content is closer to the Gel′fand–Levitan approach. Therefore, as a kind of compromise, Eqs (2)–(3) are often called Gel′fand–Levitan–Marchenko (GLM) equations. The kernel A_1 in Eq. (2) (where $y > x$) is completely specified by the so-called right scattering data, while the kernel A_2 in Eq. (3) ($y < x$) is determined by the left scattering data. These two sets of input data are equivalent: the left data are uniquely determined by the right ones and *vice versa*. In the following analysis we will rely on Eq. (2). If one is able to solve this integral equation, then the potential is given by

$$V(x) = -2C\frac{dK_1(x,x)}{dx}. \tag{4}$$

Now, let us briefly discuss Problem 3 in the above list. Unfortunately, the criteria for the uniqueness of the solution are so strict that it is nearly impossible to get all the necessary input data experimentally. However, if one sets an additional constraint that the resulting potential must be reflectionless, the inverse scattering problem can be solved much more easily. Moreover, a **symmetric** reflectionless potential is uniquely determined if its full spectrum of bound states is known. In the following analysis it is assumed that the potential $V(x)$ that corresponds to Eq. (4) is reflectionless by definition.

As the general principles of building confining reflectionless potentials are long known [12,14], a natural question arises: is there any need to revisit the topic? A motivation comes from Problem 4 stated above: if the total number of bound states is large, the technical side of the procedure becomes important. This in turn motivates the development of more efficient algorithms. In this paper, an easily applicable analytic algorithm is derived, which enables us to calculate a reflectionless potential with an arbitrary number (N) of bound states.

The paper is organized as follows. In Section 2, the general principles are briefly described, which form the overall basis for the approach. Sections 3 and 4 make an excursion to the theory of determinants, the benefits of which are described and illustrated in Section 5. Finally, Section 6 concludes the work.

2. RECONSTRUCTION OF REFLECTIONLESSS POTENTIALS: UNIVERSAL RECIPE

Suppose we are given $2N$ parameters for an unknown reflectionless potential $V(x)$: the positions of N discrete energy levels $E_n = -C\kappa_n^2$ and N norming constants C_n $(n = 1,2,...,N)$ for the Jost solution $\Psi_1(i\kappa_n,x)$ of Eq. (1), so that $\Psi_1(i\kappa_n,x) \to \exp(-\kappa_n x)$ as $x \to +\infty$, and

$$\int_{-\infty}^{\infty} \Psi_n^2(x)dx = 1,\ \Psi_n \equiv C_n\Psi_1(i\kappa_n,x). \tag{5}$$

Then it can be shown [14] that

$$\Psi_n(x) = -\frac{1}{\Lambda_n(x)}\frac{\det\left[A^{(n)}\right]}{\det(A)}. \tag{6}$$

Here A is a symmetric matrix with the following elements:

$$A_{mn} = \delta_{mn} + \frac{\Lambda_m(x)\Lambda_n(x)}{\kappa_m + \kappa_n}, \tag{7}$$

$A^{(n)}$ is obtained from A by replacing the nth column with its derivative, and

$$\Lambda_n(x) \equiv C_n \exp(-\kappa_n x). \tag{8}$$

A simple formula can also be obtained for the potential [12,14]:

$$V(x) = -2C\frac{d^2}{dx^2}\left\{\ln\left[\det(A)\right]\right\}, \tag{9}$$

which is uniquely determined by parameters κ_n and C_n.

In principle, using Eqs (7)–(9), one can reconstruct any reflectionlesss potential with given discrete energy levels and norming constants. In practice, however, the direct use of Eq. (9) is only justified if $\det(A)$ can be easily calculated, which means that N must not be very large ($N \leq 3$). With increasing N the problem becomes more and more troublesome, because the expansion of $\det(A)$ contains $N!$ terms. For example, if $N = 10$, there would be $10! = 3\ 628\ 800$ terms, which means that ascertaining the potential is not at all trivial. Fortunately, as will be demonstrated in the forthcoming sections, there is no need to explicitly use Eq. (9). This general formula can be essentially simplified, so that the corresponding modified expansion will contain only 2^{N-1} terms.

Note that the potential remains unchanged if $\det(A)$ is multiplied by a function $\exp(\alpha x+\beta)$, where α and β are arbitrary constants. Consequently, we can multiply, for example, any row of the initial matrix by a function $\exp[\kappa_n(x-x_n)]$ $(n = 1,2,...,N)$, where x_n are new parameters, equivalent to norming constants:

$$\exp(2\kappa_n x_n) \equiv \frac{C_n^2}{2\kappa_n}, \tag{10}$$

and thus, according to definition (8),

$$\frac{\Lambda_n^2(x)}{2\kappa_n} = \exp[-2\kappa_n(x-x_n)]. \tag{11}$$

As a result, we get another matrix which contains full information for reconstructing the potential according to Eq. (9):

$$\tilde{A}_N = \tilde{B}_N + \tilde{C}_N,\ V(x) = -2C\frac{d^2}{dx^2}\left\{\ln\left[\det(\tilde{A}_N)\right]\right\}. \tag{12}$$

Here

$$\tilde{B}_N \equiv \begin{pmatrix} e^{\kappa_1(x-x_1)} & 0 & ... & 0 \\ 0 & e^{\kappa_2(x-x_2)} & ... & 0 \\ 0 & ... & ... & ... \\ 0 & ... & ... & e^{\kappa_N(x-x_N)} \end{pmatrix} \tag{13}$$

and

$$\tilde{C}_N \equiv \begin{pmatrix} e^{-\kappa_1(x-x_1)} & \frac{2\sqrt{\kappa_1\kappa_2}}{\kappa_1+\kappa_2}e^{-\kappa_2(x-x_2)} & ... & \frac{2\sqrt{\kappa_1\kappa_N}}{\kappa_1+\kappa_N}e^{-\kappa_n(x-x_N)} \\ \frac{2\sqrt{\kappa_1\kappa_2}}{\kappa_1+\kappa_2}e^{-\kappa_1(x-x_1)} & e^{-\kappa_2(x-x_2)} & ... & \frac{2\sqrt{\kappa_2\kappa_N}}{\kappa_2+\kappa_N}e^{-\kappa_n(x-x_N)} \\ ... & ... & ... & ... \\ \frac{2\sqrt{\kappa_1\kappa_N}}{\kappa_1+\kappa_N}e^{-\kappa_1(x-x_1)} & \frac{2\sqrt{\kappa_2\kappa_N}}{\kappa_2+\kappa_N}e^{-\kappa_2(x-x_2)} & ... & e^{-\kappa_N(x-x_N)} \end{pmatrix}. \tag{14}$$

Remark. A subscript was added to denote the number of the bound states (and the rank of the matrix).

From now on, the determinants having the structure $\det(A)\cdot\exp(\alpha x+\beta)$ with the elements A_{mn} defined in Eq. (7) will be sometimes called τ-functions (as is common in soliton theory). In the next two sections it will be shown that such determinants can be easily calculated even for an arbitrarily large N.

3. GENERAL FORMULA FOR THE τ-FUNCTIONS

To evaluate a non-trivial determinant, one can use the Laplace expansion (see [15], p. 487) in terms of the fixed row (or column) indices. For example, choosing a set of indices m_1, m_2, ..., m_k for an arbitrary $N\times N$-matrix A_N, so that $1\le m_1<m_2<...\le N$, we get

$$\det(A_N)=\sum_{1\le n_1...\le N}\det A(m_1m_2...m_k|n_1n_2...n_k)\cdot\det\widehat{A}(m_1m_2...m_k|n_1n_2...n_k). \tag{15}$$

Here $A(m_1m_2...m_k|n_1n_2...n_k)$ is a $k\times k$-submatrix of A_N that lies on the intersection of rows m_1, m_2, ..., m_k and columns n_1, n_2, ..., n_k, while

$$\det\widehat{A}(m_1m_2...m_k|n_1n_2...n_k)=(-1)^{m_1+...+m_k+n_1...+n_k}M(m_1m_2...m_k|n_1n_2...n_k),$$

and $M(m_1m_2...m_k|n_1n_2...n_k)$ is a minor obtained from $\det(A_N)$ by deleting rows m_1, m_2, ..., m_k and columns n_1, n_2, ..., n_k.

Consequently, applying Eq. (15) to the τ-function defined by Eqs (12)–(14) yields

$$\begin{aligned}\tau_N \equiv \det\left(\tilde{A}_N\right) &= a_0\exp(\alpha_0) \\ &+\sum_{i=1}^{N}a_i\exp(\alpha_i)+\sum_{1\le i<j\le N}a_{ij}\exp(\alpha_{ij})+\sum_{1\le i<j<k\le N}a_{ijk}\exp(\alpha_{ijk})+...+a_{123...N}\exp(\alpha_{123...N}),\end{aligned} \tag{16}$$

where the coefficients a_0, a_i, a_{ij}, a_{ijk},... as well as the corresponding arguments of the exponents can be easily fixed with the help of Eqs (12)–(14). Indeed,

$$\begin{aligned}&\alpha_0=\sum_{l=1}^{N}\kappa_l(x-x_l),\ \alpha_i=\sum_{l=1}^{N}(-1)^{\delta_{il}}\kappa_l(x-x_l),\ \alpha_{ij}=\sum_{m=1}^{N}(-1)^{\delta_{im}+\delta_{jm}}\kappa_m(x-x_m), \\ &\alpha_{ijk}=\sum_{n=1}^{N}(-1)^{\delta_{in}+\delta_{jn}+\delta_{kn}}\kappa_n(x-x_n),...,\alpha_{123...N}=\sum_{n=1}^{N}(-1)^{\delta_{1n}+\delta_{2n}+...\delta_{Nn}}\kappa_n(x-x_n)=-\alpha_0,\end{aligned} \tag{17}$$

δ_{il} being the Kronecker symbol.

To further simplify Eq. (16), let us group the terms into pairs, so that the arguments of the corresponding exponents differ only by sign. For example, the first pair is formed of the terms with coefficients $a_0 = 1$ and

$$a_{123\ldots N} \equiv A_0^2 = \begin{vmatrix} 1 & \frac{2\sqrt{\kappa_1\kappa_2}}{\kappa_1+\kappa_2} & \ldots & \frac{2\sqrt{\kappa_1\kappa_N}}{\kappa_1+\kappa_N} \\ \frac{2\sqrt{\kappa_1\kappa_2}}{\kappa_1+\kappa_2} & 1 & \ldots & \frac{2\sqrt{\kappa_2\kappa_N}}{\kappa_2+\kappa_N} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{2\sqrt{\kappa_1\kappa_N}}{\kappa_1+\kappa_N} & \frac{2\sqrt{\kappa_2\kappa_N}}{\kappa_2+\kappa_N} & \ldots & 1 \end{vmatrix}.$$

Note that

$$\det\left(\tilde{B}_N\right) = a_0\exp(\alpha_0),\ \det\left(\tilde{C}_N\right) = a_{123\ldots N}\exp(-\alpha_0).$$

Here we defined a new coefficient A_0, whose subscript "0" emphasizes that the expression $-\alpha_0 = -\kappa_1(x-x_1) - \kappa_2(x-x_2) - \ldots - \kappa_N(x-x_N)$ contains no terms (0 terms) with plus sign. The same logic can be applied to all terms of Eq. (16). For example, the appropriate partner for the term $a_1\exp(\alpha_1)$ is $a_{234\ldots N}\exp(-\alpha_1)$, where

$$a_1 = 1,\ a_{234\ldots N} \equiv A_1^2 = \begin{vmatrix} 1 & \frac{2\sqrt{\kappa_2\kappa_3}}{\kappa_2+\kappa_3} & \ldots & \frac{2\sqrt{\kappa_2\kappa_N}}{\kappa_2+\kappa_N} \\ \frac{2\sqrt{\kappa_2\kappa_3}}{\kappa_2+\kappa_3} & 1 & \ldots & \frac{2\sqrt{\kappa_3\kappa_N}}{\kappa_3+\kappa_N} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{2\sqrt{\kappa_2\kappa_N}}{\kappa_2+\kappa_N} & \frac{2\sqrt{\kappa_3\kappa_N}}{\kappa_3+\kappa_N} & \ldots & 1 \end{vmatrix}.$$

Analogously, we can form a pair from $a_{12}\exp(\alpha_{12})$ and $a_{345\ldots N}\exp(-\alpha_{12})$, where

$$a_{12} = \begin{vmatrix} 1 & \frac{2\sqrt{\kappa_1\kappa_2}}{\kappa_1+\kappa_2} \\ \frac{2\sqrt{\kappa_1\kappa_2}}{\kappa_1+\kappa_2} & 1 \end{vmatrix},\ a_{345\ldots N} = \begin{vmatrix} 1 & \frac{2\sqrt{\kappa_3\kappa_4}}{\kappa_3+\kappa_4} & \ldots & \frac{2\sqrt{\kappa_3\kappa_N}}{\kappa_3+\kappa_N} \\ \frac{2\sqrt{\kappa_3\kappa_4}}{\kappa_3+\kappa_4} & 1 & \ldots & \frac{2\sqrt{\kappa_4\kappa_N}}{\kappa_4+\kappa_N} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{2\sqrt{\kappa_3\kappa_N}}{\kappa_3+\kappa_N} & \frac{2\sqrt{\kappa_4\kappa_N}}{\kappa_4+\kappa_N} & \ldots & 1 \end{vmatrix}.$$

The principle is simple: the terms $a_{n_1n_2\ldots n_i}\exp(\alpha_{n_1n_2\ldots n_i})$ and $a_{n_{i+1}n_{i+2}\ldots n_N}\exp(-\alpha_{n_1n_2\ldots n_i})$ are the partners in the sense explained above. In addition, as we will see below, it is convenient to define a relevant coefficient

$$A^2_{n_1n_2\ldots n_i} \equiv a_{n_1n_2\ldots n_i} \times a_{n_{i+1}n_{i+2}\ldots n_N}, \tag{18}$$

where the indices point at the terms with the plus sign on the right side of the expression

$$-\alpha_{n_1\ldots n_i} = \kappa_{n_1}(x-x_{n_1}) + \ldots + \kappa_{n_i}(x-x_{n_i}) - \kappa_{n_{i+1}}(x-x_{n_{i+1}}) - \ldots. \tag{19}$$

Looking at the structure of the matrix $\tilde{A}_N$ (see Eqs (12)–(14)), it is obvious that all these plus sign terms can only originate from the expansion of $\det\left(\tilde{B}_N\right)$ and they correspond to the product $(\tilde{B}_N)_{n_1n_1} \times (\tilde{B}_N)_{n_2n_2} \times ... \times (\tilde{B}_N)_{n_in_i}$. The terms with minus sign on the right side of Eq. (19) are related to the expansion of $\det\left(\tilde{C}_N\right)$, without any contribution from $\det\left(\tilde{B}_N\right)$.

On the basis of the above arguments, the following conclusions can be made:

- All terms on the right side of Eq. (16) can be grouped into pairs. There is only one term, $a_{123...N}\exp = (-\alpha_0)\det\left(\tilde{C}_N\right)$ (with partner $\det\left(\tilde{B}_N\right) = a_0\exp(\alpha_0)$) which is entirely formed of the elements of the matrix $\tilde{C}_N$. Any other term (both partners) contains some diagonal elements of the matrix $\tilde{B}_N$ as well.
- Any term $a_{n_1n_2...n_i}\exp(\alpha_{n_1n_2...n_i})$ $(i = 1,2,...)$ can be obtained by replacing all elements of the rows and columns $n_{i+1}, n_{i+2}, ..., n_N$ of the matrix $\tilde{C}_N$ with the corresponding elements of the matrix $\tilde{B}_N$ (mostly with zeros). As a result, one gets a modified matrix $\widehat{C}_N$, while $a_{n_1n_2...n_i}\exp(\alpha_{n_1n_2...n_i}) = \det\left(\widehat{C}_N\right)$. The Laplace expansion (15) of this determinant for the fixed rows $n_{i+1}, n_{i+2}, ..., n_N$ contains only one term!
- For any term $a_{n_1n_2...n_i}\exp(\alpha_{n_1n_2...n_i})$ of the expansion (16) there is a partner

$$a_{n_{i+1}n_{i+2}...n_N}\exp(-\alpha_{n_1n_2...n_i}) = \det\left(\overline{C}_N\right),$$

where $\overline{C}_N$ can be obtained by replacing all elements of the rows and columns $n_1, n_2, ..., n_i$ of the matrix $\tilde{C}_N$ with the corresponding elements of the matrix $\tilde{B}_N$. The Laplace expansion of $\det\left(\overline{C}_N\right)$ for the rows $n_1, n_2, ..., n_i$ also contains only one term:

$$\det\left(\overline{C}_N\right) = \exp(-\alpha_{12...i}) \begin{vmatrix} 1 & \frac{2\sqrt{\kappa_{n_{i+1}}\kappa_{n_{i+2}}}}{\kappa_{n_{i+1}}+\kappa_{n_{i+2}}} & \dots & \frac{2\sqrt{\kappa_{n_{i+1}}\kappa_{n_N}}}{\kappa_{n_{i+1}}+\kappa_{n_N}} \\ \frac{2\sqrt{\kappa_{n_{i+1}}\kappa_{n_{i+2}}}}{\kappa_{n_{i+1}}+\kappa_{n_{i+2}}} & 1 & \dots & \frac{2\sqrt{\kappa_{n_{i+2}}\kappa_{n_N}}}{\kappa_{n_{i+2}}+\kappa_{n_N}} \\ \dots & \dots & \dots & \dots \\ \frac{2\sqrt{\kappa_{n_{i+1}}\kappa_{n_N}}}{\kappa_{n_{i+1}}+\kappa_{n_N}} & \frac{2\sqrt{\kappa_{n_{i+2}}\kappa_{n_N}}}{\kappa_{n_{i+2}}+\kappa_{n_N}} & \dots & 1 \end{vmatrix}.$$

It means, for example, that

$$\begin{aligned} A^2_{12...i} &= a_{12...i} \times a_{i+1,i+2,...N} \\ &= \begin{vmatrix} 1 & \frac{2\sqrt{\kappa_1\kappa_2}}{\kappa_1+\kappa_2} & \dots & \frac{2\sqrt{\kappa_1\kappa_i}}{\kappa_1+\kappa_i} \\ \frac{2\sqrt{\kappa_1\kappa_2}}{\kappa_1+\kappa_2} & 1 & \dots & \frac{2\sqrt{\kappa_2\kappa_i}}{\kappa_2+\kappa_i} \\ \dots & \dots & \dots & \dots \\ \frac{2\sqrt{\kappa_1\kappa_i}}{\kappa_1+\kappa_i} & \frac{2\sqrt{\kappa_2\kappa_i}}{\kappa_2+\kappa_i} & \dots & 1 \end{vmatrix} \times \begin{vmatrix} 1 & \frac{2\sqrt{\kappa_{i+1}\kappa_{i+2}}}{\kappa_{i+1}+\kappa_{i+2}} & \dots & \frac{2\sqrt{\kappa_{i+1}\kappa_N}}{\kappa_{i+1}+\kappa_N} \\ \frac{2\sqrt{\kappa_{i+1}\kappa_{i+2}}}{\kappa_{i+1}+\kappa_{i+2}} & 1 & \dots & \frac{2\sqrt{\kappa_{n_{i+2}}\kappa_{n_N}}}{\kappa_{n_{i+2}}+\kappa_{n_N}} \\ \dots & \dots & \dots & \dots \\ \frac{2\sqrt{\kappa_{i+1}\kappa_N}}{\kappa_{i+1}+\kappa_N} & \frac{2\sqrt{\kappa_{n_{i+2}}\kappa_{n_N}}}{\kappa_{n_{i+2}}+\kappa_{n_N}} & \dots & 1 \end{vmatrix}. \end{aligned} \quad (20)$$

- One should avoid re-use of the terms: an already existing pair must not be included again! It means that the members of the modified expansion (16) are identified by no more than $[N/2]$ indices (square brackets denote the integer part of $N/2$). It is convenient to group the members on the basis of the number of indices, so that there will be $0, 1, 2, ..., [N/2]$ different indices.

As the final result of the above analysis, we get the following general formula:

$$\begin{aligned}\tau_N = \det\left(\tilde{A}_N\right) &= \sum_{i=0}^{N} A_i\left[\exp(\alpha_i+\beta_i)+\exp(-\alpha_i-\beta_i)\right]\\ &+ ... + \sum_{1\le i_1<...<i_{[N/2]}\le N} A_{i_1...i_{[N/2]}}\left[\exp(\alpha_{i_1...i_{[N/2]}}+\beta_{i_1...i_{[N/2]}})+\exp(-\alpha_{i_1..i_{[N/2]}}-\beta_{i_1...i_{[N/2]}})\right]\\ &= \sum_{i=0}^{N} 2A_i\cosh(\alpha_i+\beta_i)+\sum_{1\le i_1<i_2\le N} 2A_{i_1i_2}\cosh(\alpha_{i_1i_2}+\beta_{i_1i_2})\\ &+ ... + \sum_{1\le i_1<...<i_{[N/2]}\le N} 2A_{i_1i_2...i_{[N/2]}}\cosh(\alpha_{i_1i_2...i_{[N/2]}}+\beta_{i_1i_2...i_{[N/2]}}),\end{aligned} \tag{21}$$

where

$$\exp(\beta_{n_1n_2...n_i}) \equiv \frac{a_{n_1n_2...n_i}}{a_{n_{i+1}n_{i+2}...n_N}}. \tag{22}$$

It is easy to be convinced that the expansion (21) contains exactly 2^{N-1} terms in total (apart from inessential factor 2). Indeed, there are $N+1 = C_0^N + C_1^N$ terms with just one index and C_k^N different terms with $k>1$ indices $(i_1i_2...i_k)$, while

$$C_0^N + C_1^N + C_2^N + C_3^N + ... + C_{[N/2]}^N = \frac{(1+1)^N}{2} = 2^{N-1},\ C_k^N \equiv \frac{N!}{(N-k)!k!},$$

according to Newton's binomial theorem. Here we took into consideration that only half of this formal series is actually needed.

4. ALTERNANTS OF τ-FUNCTIONS

We have shown that not only $\det\left(\tilde{A}_N\right)$ itself but also the coefficients A_i, $A_{i_1i_2}$,..., $A_{i_1i_2...i_{[N/2]}}$ in the expansion (21) are τ-functions. Consequently, the solution of the inverse scattering problem has been reduced to evaluating a number of determinants

$$D(\kappa_1,\kappa_2,..,\kappa_n) \equiv \begin{vmatrix} 1 & \frac{2\sqrt{\kappa_1\kappa_2}}{\kappa_1+\kappa_2} & ... & \frac{2\sqrt{\kappa_1\kappa_n}}{\kappa_1+\kappa_n} \\ \frac{2\sqrt{\kappa_1\kappa_2}}{\kappa_1+\kappa_2} & 1 & ... & \frac{2\sqrt{\kappa_2\kappa_n}}{\kappa_2+\kappa_n} \\ ... & ... & ... & ... \\ \frac{2\sqrt{\kappa_1\kappa_n}}{\kappa_1+\kappa_n} & \frac{2\sqrt{\kappa_2\kappa_n}}{\kappa_2+\kappa_n} & ... & 1 \end{vmatrix}, \tag{23}$$

fixed by the parameters $\kappa_1<\kappa_2<...<\kappa_n$, with n being an appropriate natural number. We are now going to derive a simple formula for calculating such τ-functions. First, let us set a one-to-one correspondence between each row of the determinant and a fixed parameter

$$q_i = \sqrt{\kappa_i}\ (i=1,2,...,n). \tag{24}$$

For example, the modified elements of the first row of Eq. (23) will be

$$1 = \frac{2q_1\sqrt{\kappa_1}}{q_1^2+\kappa_1},\ \frac{2\sqrt{\kappa_1\kappa_2}}{\kappa_1+\kappa_2} = \frac{2q_1\sqrt{\kappa_2}}{q_1^2+\kappa_2},...,\frac{2\sqrt{\kappa_1\kappa_n}}{\kappa_1+\kappa_n} = \frac{2q_1\sqrt{\kappa_n}}{q_1^2+\kappa_n}.$$

The usefulness of this trick soon becomes evident, although there seems to be only a formal change:

$$D(\kappa_1,\kappa_2,..,\kappa_n) \equiv \begin{vmatrix} \frac{2q_1\sqrt{\kappa_1}}{q_1^2+\kappa_1} & \frac{2q_1\sqrt{\kappa_2}}{q_1^2+\kappa_2} & \cdots & \frac{2q_1\sqrt{\kappa_n}}{q_1^2+\kappa_n} \\ \frac{2q_2\sqrt{\kappa_1}}{q_2^2+\kappa_1} & \frac{2q_2\sqrt{\kappa_2}}{q_2^2+\kappa_2} & \cdots & \frac{2q_2\sqrt{\kappa_n}}{q_2^2+\kappa_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{2q_n\sqrt{\kappa_1}}{q_n^2+\kappa_1} & \frac{2q_n\sqrt{\kappa_2}}{q_n^2+\kappa_2} & \cdots & \frac{2q_n\sqrt{\kappa_n}}{q_n^2+\kappa_n} \end{vmatrix}$$

$$= 2^n\cdot\kappa_1\kappa_2...\kappa_n \begin{vmatrix} \frac{1}{q_1^2+\kappa_1} & \frac{1}{q_1^2+\kappa_2} & \cdots & \frac{1}{q_1^2+\kappa_n} \\ \frac{1}{q_2^2+\kappa_1} & \frac{1}{q_2^2+\kappa_2} & \cdots & \frac{1}{q_2^2+\kappa_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{q_n^2+\kappa_1} & \frac{1}{q_n^2+\kappa_2} & \cdots & \frac{1}{q_n^2+\kappa_n} \end{vmatrix}. \tag{25}$$

Here we separated a factor $2q_i$ from each row and $\sqrt{\kappa_i}$ from each column, i.e., $2q_i\sqrt{\kappa_i}=2\kappa_i$ $(i=1,2,...,n)$ from each such pair.

Next, let us transform Eq. (25) into a polynomial, multiplying each row by $\prod_j\left(q_i^2+\kappa_j\right)$ $(i=1,2,...,n)$. The result is

$$D(\kappa_1,\kappa_2,..,\kappa_n)=\frac{1}{\prod\limits_{1\le i<j\le n}(\kappa_i+\kappa_j)^2}\times D_n, \tag{26}$$

where a new determinant

$$D_n\equiv\begin{vmatrix} \prod\limits_{j\neq1}\left(q_1^2+\kappa_j\right) & \prod\limits_{j\neq2}\left(q_1^2+\kappa_j\right) & ... & \prod\limits_{j\neq n}\left(q_1^2+\kappa_j\right) \\ \prod\limits_{j\neq1}\left(q_2^2+\kappa_j\right) & \prod\limits_{j\neq2}\left(q_2^2+\kappa_j\right) & ... & \prod\limits_{j\neq n}\left(q_2^2+\kappa_j\right) \\ ... & ... & ... & ... \\ \prod\limits_{j\neq1}\left(q_n^2+\kappa_j\right) & \prod\limits_{j\neq2}\left(q_n^2+\kappa_j\right) & ... & \prod\limits_{j\neq n}\left(q_n^2+\kappa_j\right) \end{vmatrix} \tag{27}$$

was introduced. As can be seen, the factor $2^n\cdot\kappa_1\kappa_2...\kappa_n$ was cancelled out from Eq. (26).

We can see that the elements of the columns of D_n correspond to different values of the same function, while any row is characterized by a single fixed parameter. Indeed, Eq. (27) can be expressed as

$$D_n=\begin{vmatrix} F_1(q_1) & F_2(q_1) & ... & F_n(q_1) \\ F_1(q_2) & F_2(q_2) & ... & F_n(q_2) \\ ... & ... & ... & ... \\ F_1(q_n) & F_2(q_n) & ... & F_n(q_n) \end{vmatrix}, \tag{28}$$

where

$$\begin{aligned} F_1(x)&\equiv\left(x^2+\kappa_2\right)\left(x^2+\kappa_3\right)...\left(x^2+\kappa_n\right), \\ F_2(x)&\equiv\left(x^2+\kappa_1\right)\left(x^2+\kappa_3\right)...\left(x^2+\kappa_n\right), \\ F_3(x)&\equiv\left(x^2+\kappa_1\right)\left(x^2+\kappa_2\right)\left(x^2+\kappa_4\right)...\left(x^2+\kappa_n\right), \\ &... \\ F_n(x)&\equiv\left(x^2+\kappa_1\right)\left(x^2+\kappa_2\right)\left(x^2+\kappa_3\right)...\left(x^2+\kappa_{n-1}\right). \end{aligned} \tag{29}$$

A determinant that has a structure of Eq. (28) is called *alternant* (see [16], p. 161). The best-known alternant is Vandermonde's determinant (for the same set of variables)

$$V_n \equiv \begin{vmatrix} 1 & q_1 & q_1^2 & ... & q_1^{n-1} \\ 1 & q_2 & q_2^2 & ... & q_2^{n-1} \\ ... & ... & ... & ... & ... \\ 1 & q_{n-1} & q_{n-1}^2 & ... & q_{n-1}^{n-1} \\ 1 & q_n & q_n^2 & ... & q_n^{n-1} \end{vmatrix}, \tag{30}$$

which can be easily evaluated (see [17], p. 16):

$$V_n(q_1, q_2, ..., q_n) = \prod_{1 \le i < j \le n} (q_j - q_i). \tag{31}$$

An important point is that the factor $V_n(q_1, q_2, ..., q_n)$ can be separated from any nth-order alternant. Indeed, the argument q_n may only appear in the nth row of Eq. (28): if we put it into any other row, the determinant would be identically zero. It means that D_n has a factor $\prod_{i=1}^{n-1}(q_n - q_i)$. Analogous reasoning applied to q_{n-1} shows that D_n also has a factor $\prod_{i=1}^{n-2}(q_{n-1} - q_i)$, etc. Putting it all together, we conclude that an nth-order alternant always has a factor $V_n(q_1, q_2, ..., q_n)$.

To continue the analysis, let us recall some useful properties of the elementary symmetric functions:

$$\begin{aligned} &\sigma_0 \equiv 1, \\ &\sigma_1 \equiv q_1 + q_2 + ... + q_n, \\ &\qquad ... \\ &\sigma_k \equiv \sum_{1 \le i_1 < i_2 \cdots < i_k \le n} q_{i_1} q_{i_2} \cdots q_{i_k}, \\ &\qquad ... \\ &\sigma_n \equiv q_1 q_2 \cdots q_n. \end{aligned} \tag{32}$$

Here σ_k ($k \ge 2$) is the sum of all possible products of exactly k variables arranged in the ascending order of their indices. According to the Fundamental Theorem for symmetric polynomials (see [18], p. 312), any such polynomial can be uniquely expressed as a polynomial in $\sigma_1, \sigma_2, ..., \sigma_n$. This in turn is a basis for the following important theorem:

Theorem 1. *Let $|A_n|$ be an nth-order alternant generated by the functions*

$$F_j(x) = a_{0j} + a_{1j} \cdot x + a_{2j} \cdot x^2 + ... + a_{nj} \cdot x^n, \ j = 1, 2, ..., n, \tag{33}$$

where the parameters a_{ij} do not depend on x, and define

$$S_k \equiv (-1)^k \sigma_k(q_1, q_2, ..., q_n). \tag{34}$$

Then

$$\frac{|A_n|}{V_n(q_1, q_2, ..., q_n)} = \begin{vmatrix} a_{01} & a_{11} & a_{21} & ... & a_{n1} \\ a_{02} & a_{12} & a_{22} & ... & a_{n2} \\ ... & ... & ... & ... & ... \\ a_{0n} & a_{1n} & a_{2n} & ... & a_{nn} \\ S_n & S_{n-1} & S_{n-2} & ... & S_0 \end{vmatrix} \equiv \Delta_{n+1}. \tag{35}$$

Proof. Let us introduce an auxiliary polynomial

$$F(x) \equiv (x - q_1)(x - q_2)...(x - q_n) \tag{36}$$

and form an $(n+1)$th-order Vandermonde's determinant, adding a new (arbitrary) variable q_{n+1}, so that

$$V_{n+1}(q_1,q_2,...,q_n,q_{n+1}) = \begin{vmatrix} 1 & 1 & ... & 1 & 1 \\ q_1 & q_2 & ... & q_n & q_{n+1} \\ ... & ... & ... & ... & ... \\ q_1^{n-1} & q_2^{n-1} & ... & q_n^{n-1} & q_{n+1}^{n-1} \\ q_1^n & q_2^n & ... & q_n^n & q_{n+1}^n \end{vmatrix}$$

(compared with Eq. (30), the row and column indices are interchanged). Multiplying Δ_{n+1} by V_{n+1} and using Eqs (33), (34), (36), we get

$$\begin{aligned} \Delta_{n+1} \times V_{n+1} &= \begin{vmatrix} F_1(q_1) & F_1(q_2) & ... & F_1(q_n) & F_1(q_{n+1}) \\ F_2(q_1) & F_2(q_2) & ... & F_2(q_n) & F_2(q_{n+1}) \\ ... & ... & ... & ... & ... \\ F_n(q_1) & F_n(q_2) & ... & F_n(q_n) & F_n(q_{n+1}) \\ F(q_1) & F(q_2) & ... & F(q_n) & F(q_{n+1}) \end{vmatrix} \\ &= \begin{vmatrix} F_1(q_1) & F_1(q_2) & ... & F_1(q_n) & F_1(q_{n+1}) \\ F_2(q_1) & F_2(q_2) & ... & F_2(q_n) & F_2(q_{n+1}) \\ ... & ... & ... & ... & ... \\ F_n(q_1) & F_n(q_2) & ... & F_n(q_n) & F_n(q_{n+1}) \\ 0 & 0 & ... & 0 & F(q_{n+1}) \end{vmatrix} \end{aligned}$$

or, in a more compact form,

$$\Delta_{n+1} \times V_{n+1} = (q_{n+1}-q_1)...(q_{n+1}-q_n) \times |A_n|, \tag{37}$$

where we took into consideration that $F(q_i) = 0$ if $i = 1,2,...,n$.

From Eq. (31) one concludes that

$$V_{n+1} = (q_{n+1}-q_1)(q_{n+1}-q_2)...(q_{n+1}-q_n) \cdot V_n,$$

so that both sides of Eq. (37) contain a common divisor $(q_{n+1}-q_1)(q_{n+1}-q_2)...(q_{n+1}-q_n) \neq 0$. Consequently, $\Delta_{n+1}V_n = |A_n|$, q.e.d.

4.1. Relationship to the inverse scattering problem

Let us apply Theorem 1 to the alternant (28) generated by the functions (29), which are polynomials in a variable x^2. Using Eqs (33) and (34), one gets

$$\begin{aligned} a_{01} &= \sigma_{n-1}(\kappa_2,\kappa_3,...,\kappa_n), \\ a_{11} &= \sigma_{n-2}(\kappa_2,\kappa_3,...,\kappa_n),...,a_{n-1,1}=1,\ a_{n1}=0, \\ a_{02} &= \sigma_{n-1}(\kappa_1,\kappa_3,...,\kappa_n), \\ a_{12} &= \sigma_{n-2}(\kappa_1,\kappa_3,...,\kappa_n),...,a_{n-1,2}=1,\ a_{n2}=0, \\ &... \\ a_{0n} &= \sigma_{n-1}(\kappa_1,\kappa_2,...,\kappa_{n-1}), \\ a_{1n} &= \sigma_{n-2}(\kappa_1,\kappa_2,...,\kappa_{n-1}),...,a_{n-1,n}=1,\ a_{nn}=0. \end{aligned}$$

Consequently,

$$D_n = V_n(q_1,q_2,...,q_n)\times|R_n|, \tag{38}$$

$$|R_n| \equiv \begin{vmatrix} \sigma_{n-1}(\kappa_2,\kappa_3,...,\kappa_n) & \sigma_{n-2}(\kappa_2,\kappa_3,...,\kappa_n) & ... & 1 & 0 \\ \sigma_{n-1}(\kappa_1,\kappa_3,...,\kappa_n) & \sigma_{n-2}(\kappa_1,\kappa_3,...,\kappa_n) & ... & 1 & 0 \\ ... & ... & ... & ... & ... \\ \sigma_{n-1}(\kappa_1,\kappa_2,...,\kappa_{n-1}) & \sigma_{n-2}(\kappa_1,\kappa_2,...,\kappa_{n-1}) & ... & 1 & 0 \\ S_n & S_{n-1} & ... & S_1 & S_0 \end{vmatrix}$$

$$= \begin{vmatrix} \sigma_{n-1}(\kappa_2,\kappa_3,...,\kappa_n) & \sigma_{n-2}(\kappa_2,\kappa_3,...,\kappa_n) & ... & 1 \\ \sigma_{n-1}(\kappa_1,\kappa_3,...,\kappa_n) & \sigma_{n-2}(\kappa_1,\kappa_3,...,\kappa_n) & ... & 1 \\ ... & ... & ... & ... \\ \sigma_{n-1}(\kappa_1,\kappa_2,...,\kappa_{n-1}) & \sigma_{n-2}(\kappa_1,\kappa_2,...,\kappa_{n-1}) & ... & 1 \end{vmatrix}.$$

Here we defined a new determinant $|R_n|$ which seems to be another alternant, so we can apply Theorem 1 to evaluate it. To be convinced that $|R_n|$ is indeed an alternant, we have to specify the generating functions. Obviously,

$$F_n(x) = 1 = \sigma_0(\kappa_1,\kappa_2,...,\kappa_n),\ F_{n-1}(x) = \sigma_1(\kappa_1,\kappa_2,...,\kappa_n) - x. \tag{39}$$

Also, it is easy to prove that

$$F_{n-2}(x) = \sigma_2 - xF_{n-1}(x) = \sigma_2 - \sigma_1 x + x^2, \tag{40}$$

where we dropped the arguments to get a more compact formula. Let us agree that if no arguments are explicitly given, the corresponding function depends on n arguments: $\kappa_1,\kappa_2,...,\kappa_n$. Taking, e.g., $x=\kappa_1$, we can check the validity of Eq. (40). Indeed,

$$\begin{aligned} F_{n-2}(\kappa_1) &= \sigma_2 - \sigma_1\kappa_1 + \kappa_1^2 \\ &= \sum_{1\le i_1<i_2\le n}(\kappa_{i_1}\kappa_{i_2}) - \kappa_1(\kappa_1+\kappa_2+...+\kappa_n) + \kappa_1^2 = \sigma_{n-1}(\kappa_2,\kappa_3,...,\kappa_n), \end{aligned}$$

as needed according to Eq. (38). Continuing in the same manner, we get the following result:

$$\begin{gathered} F_{n-3}(x) = \sigma_3 - xF_{n-2}(x) = \sigma_3 - \sigma_2 x + \sigma_1 x^2 - x^3, \\ ... \\ F_{n-i}(x) = \sigma_{n-i} - xF_{n-i-1}(x) = \sum_{j=0}^{i}(-1)^j\sigma_{i-j}x^j \to \\ F_k(x) = \sum_{j=0}^{n-k}(-1)^j\sigma_{n-k-j}x^j = \sum_{j=0}^{n-k}a_{jk}x^j, \end{gathered} \tag{41}$$

$$a_{jk} = \begin{cases} (-1)^j\sigma_{n-k-j},\ j\le n-k, \\ 0,\ j>n-k. \end{cases} \tag{42}$$

Thus $|R_n|$ is indeed an alternant with generating functions (33). Consequently, according to Eqs (34), (35), and (42),

$$\frac{|R_n|}{V_n} = \begin{vmatrix} a_{01} & a_{11} & a_{21} & ... & a_{n1} \\ a_{02} & a_{12} & a_{22} & ... & a_{n2} \\ ... & ... & ... & ... & ... \\ a_{0n} & a_{1n} & a_{2n} & ... & a_{nn} \\ S_n & S_{n-1} & S_{n-2} & ... & S_0 \end{vmatrix}$$

$$= \begin{vmatrix} \sigma_{n-1} & -\sigma_{n-2} & \sigma_{n-3} & ... & (-1)^{n-2}\sigma_1 & (-1)^{n-1} & 0 \\ \sigma_{n-2} & -\sigma_{n-3} & \sigma_{n-4} & ... & (-1)^{n-2} & 0 & 0 \\ \sigma_{n-3} & -\sigma_{n-4} & \sigma_{n-5} & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & 0 & 0 & 0 \\ \sigma_1 & -1 & 0 & ... & 0 & 0 & 0 \\ 1 & 0 & 0 & ... & 0 & 0 & 0 \\ (-1)^n\sigma_n & (-1)^{n-1}\sigma_{n-1} & (-1)^{n-1}\sigma_{n-2} & ... & \sigma_2 & -\sigma_1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \sigma_{n-1} & -\sigma_{n-2} & \sigma_{n-3} & ... & (-1)^{n-2}\sigma_1 & (-1)^{n-1} \\ \sigma_{n-2} & -\sigma_{n-3} & \sigma_{n-4} & ... & (-1)^{n-2} & 0 \\ \sigma_{n-3} & -\sigma_{n-4} & \sigma_{n-5} & ... & 0 & 0 \\ ... & ... & ... & ... & 0 & 0 \\ \sigma_1 & -1 & 0 & ... & 0 & 0 \\ 1 & 0 & 0 & ... & 0 & 0 \end{vmatrix}, \tag{43}$$

which means that

$$\frac{|R_n|}{V_n} = 1. \tag{44}$$

Indeed, we can transform Eq. (43), repeatedly using cofactor expansion in terms of the last column and applying the general definition

$$\det(A) = \sum_p \sigma(p) a_{1p_1} a_{2p_2} ... a_{np_n}. \tag{45}$$

Here the sum involves all possible permutations $p = (p_1, p_2, ..., p_n)$ of indices $(1,2,...,n)$, $\sigma(p) = (-1)^{N_p}$ and N_p is the number of pairwise interchanges needed to restore the natural order $(1,2,...,n)$. For example, $\sigma(3,2,1) = -1$, but $\sigma(4,3,2,1) = 1$. As a result of the described operation, we obtain

$$\frac{|R_n|}{V_n} = \sigma(n, n-1, ..., 1) \cdot (-1)^{[n/2]}.$$

On the other hand, $\sigma(n, n-1, ..., 1) = (-1)^{[n/2]}$, since $N_p = [n/2]$. Consequently,

$$|R_n| = V_n \cdot (-1)^{[n/2]} \cdot (-1)^{[n/2]} = V_n.$$

In summary, we have obtained a very simple and universal recipe for calculating determinants defined by Eq. (27):

$$D_n = (V_n)^2 = \prod_{1 \le i < j \le n} (\kappa_j - \kappa_i)^2. \tag{46}$$

Combining Eqs (26) and (46), we can formulate a general and important result:

Theorem 2. *Let $\kappa_1, \kappa_2, .., \kappa_n$ be arbitrary positive real numbers arranged in the ascending order, so that $\kappa_1 < \kappa_2 < ... < \kappa_n$, and let $D(\kappa_1, \kappa_2, .., \kappa_n)$ be a determinant, defined by Eq. (23). Then*

$$D(\kappa_1, \kappa_2, .., \kappa_n) = \prod_{1 \le i < j \le n} \left(\frac{\kappa_j - \kappa_i}{\kappa_j + \kappa_i} \right)^2, \tag{47}$$

where the product contains all possible combinations of the pairs (κ_j, κ_i) *satisfying the condition* $1 \le i < j \le n$.

5. SYMMETRIC REFLECTIONLESS POTENTIALS

The excursion to theory of determinants concluded with a surprisingly simple final result. Indeed, as is seen from Eqs (18), (20), and (22), all coefficients in Eq. (21) can be evaluated with the help of Eq. (47), which means that the general formula for τ-functions can be essentially simplified. For example,

$$A_0 = \sqrt{D(\kappa_1, \kappa_2, .., \kappa_n)} = \prod_{1 \le i < j \le n} \left(\frac{\kappa_j - \kappa_i}{\kappa_j + \kappa_i} \right), \tag{48}$$

$$\frac{A_i}{A_0} = \sqrt{\frac{D(\kappa_1, \kappa_2, .., \kappa_{i-1}\kappa_{i+1}\kappa_{i+2}...\kappa_n)}{D(\kappa_1, \kappa_2, .., \kappa_n)}} = \prod_{j \ne i} \left| \frac{\kappa_j + \kappa_i}{\kappa_j - \kappa_i} \right|, \tag{49}$$

etc. Thus Eq. (21) transforms to

$$\begin{aligned} \tau_N = & \frac{\det\left(\tilde{A}_N\right)}{2A_0} = \cosh(\alpha_0 + \beta_0) + \sum_i \prod_{j \ne i} \left| \frac{\kappa_j + \kappa_i}{\kappa_j - \kappa_i} \right| \cosh(\alpha_i + \beta_i) \\ & + \sum_{1 \le i_1 < i_2 \le N} \prod_{j \ne i_1, i_2} \left| \frac{\kappa_j + \kappa_{i_1}}{\kappa_j - \kappa_{i_1}} \right| \cdot \left| \frac{\kappa_j + \kappa_{i_2}}{\kappa_j - \kappa_{i_2}} \right| \cosh(\alpha_{i_1 i_2} + \beta_{i_1 i_2}) \\ & + ... + \sum_{1 \le i_1 ... < i_{[N/2]} \le N} \prod_{j \ne i_1, i_2, ..., i_{[N/2]}} \left| \frac{\kappa_j + \kappa_{i_1}}{\kappa_j - \kappa_{i_1}} \right| \cdot \left| \frac{\kappa_j + \kappa_{i_2}}{\kappa_j - \kappa_{i_2}} \right| ... \left| \frac{\kappa_j + \kappa_{i_{[N/2]}}}{\kappa_j - \kappa_{i_{[N/2]}}} \right| \cosh(\alpha_{i_1 ... i_{[N/2]}} + \beta_{i_1 ... i_{[N/2]}}), \end{aligned} \tag{50}$$

and Eq. (12) can be rewritten as

$$V(x) = -2C \frac{d^2}{dx^2} \left\{ \ln \tau_N(x) \right\}. \tag{51}$$

Equations (50)–(51) express the main result of this paper.

In general, as mentioned, the reflectionless potential is uniquely determined if $2N$ parameters κ_n and C_n $(n = 1, 2, ..., N)$ are known. However, if one sets an additional constraint

$$V(-x) = V(x),$$

the number of necessary input parameters is twofold reduced. In other words, a symmetric reflectionless potential is uniquely determined by its N binding energies [19] $E_n = -C\kappa_n^2$. Let us analyse this in more detail.

Obviously, Eq. (50) can only be symmetric if the arguments of all cosh functions are of the linear form $ax + b$ with $b \equiv 0$. It means, for example, that

$$\sum_{i=1}^{N} \kappa_i x_i = \beta_0 = \sum_{1 \le i < j \le N} c_{ij}, \tag{52}$$

$$\kappa_1 x_1 - \kappa_2 x_2 - ... - \kappa_N x_N = 2\kappa_1 x_1 - \beta_0 = -\beta_1 = \sum_{j=2}^{N} c_{1j} - \beta_0, \tag{53}$$

where

$$c_{ij} \equiv \ln \left| \frac{\kappa_j + \kappa_i}{\kappa_j - \kappa_i} \right|, \tag{54}$$

and consequently,

$$2\kappa_1 x_1 = \sum_{j=2}^{N} c_{1j}.$$

Using Eqs (17) and Eq. (22), we get a similar expression for any other combination $\kappa_i x_i$. From Eqs (10) and (54) we therefore obtain the following symmetricity conditions:

$$\exp(2\kappa_i x_i) = \frac{C_i^2}{2\kappa_n} = \prod_{j\neq i}^{N} \left| \frac{\kappa_j + \kappa_i}{\kappa_j - \kappa_i} \right|, \; i = 1,2,...,N \geq 2. \tag{55}$$

If $N = 1$, then

$$\exp(2\kappa_1 x_1) = \frac{C_1^2}{2\kappa_1} = 1. \tag{56}$$

It can be easily shown that Eqs (55) are indeed the symmetricity conditions for τ_N and $V(x)$. To this end, in full analogy with Eq. (53), we can write

$$\begin{aligned}
\kappa_1 x_1 - \kappa_2 x_2 - ... - \kappa_N x_N &= \sum_{j\neq 1}^{N} c_{1j} - \beta_0, \\
-\kappa_1 x_1 + \kappa_2 x_2 - ... - \kappa_N x_N &= \sum_{j\neq 2}^{N} c_{2j} - \beta_0, \\
&... \\
-\kappa_1 x_1 + \kappa_2 x_2 - ... + \kappa_N x_N &= \sum_{j\neq N}^{N} c_{Nj} - \beta_0.
\end{aligned} \tag{57}$$

Summing these equations, we get

$$-(N-2)\sum_{i=1}^{N} \kappa_i x_i = -(N-2)\beta_0,$$

which coincides with Eq. (52).

Remark. If $N = 2$, then the first two equations of the system (57) are not linearly independent, since

$$\beta_1 = \beta_0 - \sum_{j\neq 1}^{N} c_{1j} = \beta_0 - \sum_{j\neq 2}^{N} c_{2j} = \beta_2 = 0,$$

so that $\kappa_1 x_1 = \kappa_2 x_2$. Consequently, in this (and only in this) special case Eqs (52)–(53) must be treated as the actual symmetricity conditions, while Eq. (55) still remains valid.

The next step is to complement Eq. (57), for example, with another condition

$$2(\kappa_1 x_1 + \kappa_2 x_2) = \sum_{j\neq 1}^{N} c_{1j} + \sum_{j\neq 2}^{N} c_{2j}, \tag{58}$$

which is a direct conclusion from Eq. (55). Thus

$$\begin{aligned}
2(\kappa_1 x_1 + \kappa_2 x_2) - \sum_{i=1}^{N} \kappa_i x_i &= \kappa_1 x_1 + \kappa_2 x_2 - \kappa_3 x_3 - ... - \kappa_N x_N \\
&= \sum_{j\neq 1}^{N} c_{1j} + \sum_{j\neq 2}^{N} c_{2j} - \beta_0 = \beta_{12},
\end{aligned}$$

which means that

$$\cosh(\alpha_{12} + \beta_{12}) = \cosh\left[(\kappa_1 + \kappa_2 - \kappa_3 - ... - \kappa_N)\, x\right]$$

is a symmetric function. Analogously, one can prove that any other term $\cosh(\alpha_{i_1 i_2 ...} + \beta_{i_1 i_2 ...})$ in Eq. (50) is a symmetric function as well. This in turn proves that the norming constants C_n of a symmetric reflectionless potential are uniquely determined by the given binding energies E_n.

5.1. Some practical examples

To illustrate the results, let us take, for example, $N = 4$. Then Eq. (50) reads

$$\begin{aligned}
T_4 = &\cosh(\alpha_0+\beta_0) + \left(\frac{\kappa_2+\kappa_1}{\kappa_2-\kappa_1}\right)\left(\frac{\kappa_3+\kappa_1}{\kappa_3-\kappa_1}\right)\left(\frac{\kappa_4+\kappa_1}{\kappa_4-\kappa_1}\right)\cosh(\alpha_1+\beta_1)\\
&+\left(\frac{\kappa_2+\kappa_1}{\kappa_2-\kappa_1}\right)\left(\frac{\kappa_3+\kappa_2}{\kappa_3-\kappa_2}\right)\left(\frac{\kappa_4+\kappa_2}{\kappa_4-\kappa_2}\right)\cosh(\alpha_2+\beta_2) + \left(\frac{\kappa_3+\kappa_1}{\kappa_3-\kappa_1}\right)\left(\frac{\kappa_3+\kappa_2}{\kappa_3-\kappa_2}\right)\\
&\times\left(\frac{\kappa_4+\kappa_3}{\kappa_4-\kappa_3}\right)\cosh(\alpha_3+\beta_3) + \left(\frac{\kappa_4+\kappa_1}{\kappa_4-\kappa_1}\right)\left(\frac{\kappa_4+\kappa_2}{\kappa_4-\kappa_2}\right)\left(\frac{\kappa_4+\kappa_3}{\kappa_4-\kappa_3}\right)\cosh(\alpha_4+\beta_4)\\
&+\left(\frac{\kappa_3+\kappa_1}{\kappa_3-\kappa_1}\right)\left(\frac{\kappa_4+\kappa_1}{\kappa_4-\kappa_1}\right)\left(\frac{\kappa_3+\kappa_2}{\kappa_3-\kappa_2}\right)\left(\frac{\kappa_4+\kappa_2}{\kappa_4-\kappa_2}\right)\cosh(\alpha_{12}+\beta_{12})\\
&+\left(\frac{\kappa_2+\kappa_1}{\kappa_2-\kappa_1}\right)\left(\frac{\kappa_4+\kappa_1}{\kappa_4-\kappa_1}\right)\left(\frac{\kappa_3+\kappa_2}{\kappa_3-\kappa_2}\right)\left(\frac{\kappa_4+\kappa_3}{\kappa_4-\kappa_3}\right)\cosh(\alpha_{13}+\beta_{13})\\
&+\left(\frac{\kappa_2+\kappa_1}{\kappa_2-\kappa_1}\right)\left(\frac{\kappa_3+\kappa_1}{\kappa_3-\kappa_1}\right)\left(\frac{\kappa_4+\kappa_2}{\kappa_4-\kappa_2}\right)\left(\frac{\kappa_4+\kappa_3}{\kappa_4-\kappa_3}\right)\cosh(\alpha_{14}+\beta_{14}),
\end{aligned} \tag{59}$$

while the symmetricity conditions, according to Eqs (53)–(55), can be given as

$$\begin{aligned}
\kappa_1x_1-\kappa_2x_2-\kappa_3x_3-\kappa_4x_4 &= -\beta_1 = \ln\left(\frac{\kappa_3-\kappa_2}{\kappa_3+\kappa_2}\times\frac{\kappa_4-\kappa_2}{\kappa_4+\kappa_2}\times\frac{\kappa_4-\kappa_3}{\kappa_4+\kappa_3}\right),\\
-\kappa_1x_1+\kappa_2x_2-\kappa_3x_3-\kappa_4x_4 &= -\beta_2 = \ln\left(\frac{\kappa_3-\kappa_1}{\kappa_3+\kappa_1}\times\frac{\kappa_4-\kappa_1}{\kappa_4+\kappa_1}\times\frac{\kappa_4-\kappa_3}{\kappa_4+\kappa_3}\right),\\
-\kappa_1x_1-\kappa_2x_2+\kappa_3x_3-\kappa_4x_4 &= -\beta_3 = \ln\left(\frac{\kappa_2-\kappa_1}{\kappa_2+\kappa_1}\times\frac{\kappa_4-\kappa_1}{\kappa_4+\kappa_1}\times\frac{\kappa_4-\kappa_2}{\kappa_4+\kappa_2}\right),\\
-\kappa_1x_1-\kappa_2x_2-\kappa_3x_3+\kappa_4x_4 &= -\beta_4 = \ln\left(\frac{\kappa_2-\kappa_1}{\kappa_2+\kappa_1}\times\frac{\kappa_3-\kappa_1}{\kappa_3+\kappa_1}\times\frac{\kappa_3-\kappa_2}{\kappa_3+\kappa_2}\right).
\end{aligned} \tag{60}$$

Summing the corresponding sides of Eqs (60), we get

$$\kappa_1x_1+\kappa_2x_2+\kappa_3x_3+\kappa_4x_4+\beta_0 = 0,$$

which means that

$$\cosh(\alpha_0+\beta_0) = \cosh\left[(\kappa_1+\kappa_2+\kappa_3+\kappa_4)x\right]$$

is a symmetric function.

Analogously, subtracting the sides of the last two equations from the corresponding sides of the first two equations of (60), we get

$$\kappa_1x_1+\kappa_2x_2-\kappa_3x_3-\kappa_4x_4+\beta_{12} = 0,$$

which means that $\cosh(\alpha_{12}+\beta_{12})$ is a symmetric function. Continuing in a similar manner, it is easy to be convinced that $\cosh(\alpha_{13}+\beta_{13})$ and $\cosh(\alpha_{14}+\beta_{14})$ are symmetric functions as well.

Example 1. To be more specific, let $x_0 = 1/\kappa_1$ be the length unit and $E_0 = C\kappa_1^2$, the energy unit (i.e., $x_0 = 1$ and $E_0 = 1$). The simplest and the best-known symmetric potential then corresponds to

$$\kappa_n = n,\ n = 1,2,3,4.$$

Therefore, according to Eq. (59),

$$\begin{aligned}T_4 =& \cosh\left[(\kappa_1+\kappa_2+\kappa_3+\kappa_4)x\right]+\frac{3\cdot4\cdot5}{1\cdot2\cdot3}\cosh\left[(\kappa_1-\kappa_2-\kappa_3-\kappa_4)x\right]+\frac{3\cdot5\cdot6}{1\cdot1\cdot2}\\ &\times\cosh\left[(\kappa_2-\kappa_1-\kappa_3-\kappa_4)x\right]+\frac{4\cdot5\cdot7}{2\cdot1\cdot1}\cosh\left[(\kappa_3-\kappa_1-\kappa_2-\kappa_4)x\right]+\frac{5\cdot6\cdot7}{3\cdot2\cdot1}\\ &\times\cosh\left[(\kappa_4-\kappa_1-\kappa_2-\kappa_3)x\right]+\frac{4\cdot5\cdot5\cdot6}{2\cdot3\cdot1\cdot2}\cosh\left[(\kappa_1+\kappa_2-\kappa_3-\kappa_4)x\right]+\frac{3\cdot5\cdot5\cdot7}{1\cdot3\cdot1\cdot1}\\ &\times\cosh\left[(\kappa_1+\kappa_3-\kappa_2-\kappa_4)x\right]+\frac{3\cdot4\cdot6\cdot7}{1\cdot2\cdot2\cdot1}\cosh\left[(\kappa_1+\kappa_4-\kappa_2-\kappa_3)x\right]\\ =&\cosh(10x)+10\cosh(8x)+45\cosh(6x)+120\cosh(4x)+210\cosh(2x)+126. \end{aligned}\tag{61}$$

At first sight Eq. (61) may seem impractical. However, using the standard transformation formulas (obtained from the corresponding trigonometric formulas by replacing $x\to ix$)

$$\begin{aligned}\cosh(10x) &= 512\cosh^{10}(x)-1280\cosh^8(x)+1120\cosh^6(x)-400\cosh^4(x)+50\cosh^2(x)-1,\\ \cosh(8x) &= 128\cosh^8(x)-256\cosh^6(x)+160\cosh^4(x)-32\cosh^2(x)+1,\\ \cosh(6x) &= 32\cosh^6(x)-48\cosh^4(x)+18\cosh^2(x)-1,\\ \cosh(4x) &= 8\cosh^4(x)-8\cosh^2(x)+1,\\ \cosh(2x) &= 2\cosh^2(x)-1,\end{aligned}$$

the result is as follows:

$$T_4 = 512\cosh^{10}(x).$$

Thus

$$[\ln T_4(x)]' = \frac{T_4'}{T_4} = 10\tanh(x)$$

and

$$V(x) = -\frac{20}{\cosh^2(x)} = -\frac{N(N+1)}{\cosh^2(x)},$$

exactly as needed.

Example 2. Next, let us construct a reflectionless approximant to a symmetric rectangular potential with four energy levels (see Fig. 1). These levels can be determined from (see, e.g., [20], Sec. II.9)

$$\kappa_i x_i\tan(\kappa_i x_i) = \sqrt{\frac{U_0a^2}{C}-(\kappa_i x_i)^2},\tag{62}$$

$$-\frac{\kappa_i x_i}{\tan(\kappa_i x_i)} = \sqrt{\frac{U_0a^2}{C}-(\kappa_i x_i)^2},\tag{63}$$

where a and U_0 denote the half-width and the depth of the potential well, respectively. Equation (62) fixes the symmetric and (63) the antisymmetric solutions to the Schrödinger equation. Again, it is convenient to

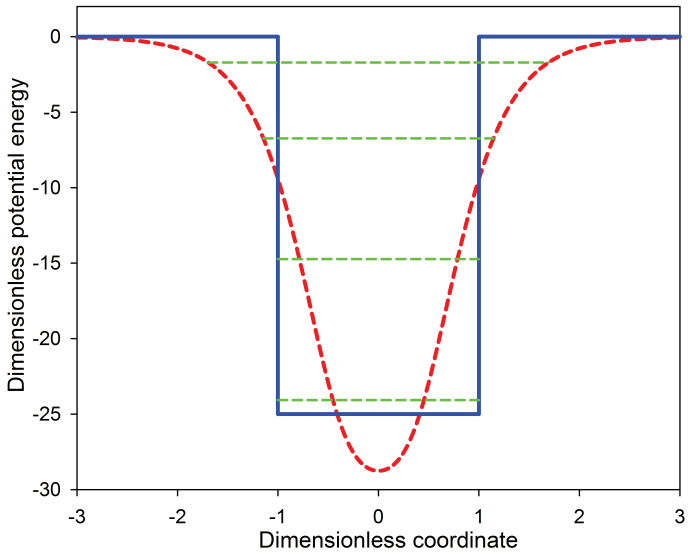

Fig. 1. The isospectral reflectionless approximant (dashed curve) to a symmetric rectangular potential (solid line). The common energy levels are shown by horizontal dashed lines.

use dimensionless units for the length and energy, taking $x_0 = a = 1$ and $E_0 = C/a^2 = 1$. In addition, let us fix

$$\sqrt{\frac{U_0 a^2}{C}} = 5.$$

Then the system has four discrete levels (as assumed) corresponding to

$$\begin{aligned} \kappa_1 &= 1.3064400089,\ \kappa_2 = 2.5957390789, \\ \kappa_3 &= 3.8374671080,\ \kappa_4 = 4.9062951521. \end{aligned} \tag{64}$$

In this case Eq. (59) cannot be further simplified, but this is not a serious problem. Indeed, let us define the coefficients A_i and B_i such that

$$T_4 = \sum_i A_i \cosh(B_i x).$$

Then the corresponding potential becomes

$$V(x) = -2C \left\{ \frac{\sum_i A_i B_i^2 \cosh(B_i x)}{\sum_i A_i \cosh(B_i x)} - \left[\frac{\sum_i A_i B_i \sinh(B_i x)}{\sum_i A_i \cosh(B_i x)} \right]^2 \right\}. \tag{65}$$

The result for the input data (64) can be seen in Fig. 1.

Example 3. Figure 2 demonstrates two isospectral potentials corresponding to the following set of input parameters:

$$\kappa_1 = 1/8,\ \kappa_2 = 3/8,\ \kappa_3 = 5/8,\ \kappa_4 = 7/8, \tag{66}$$

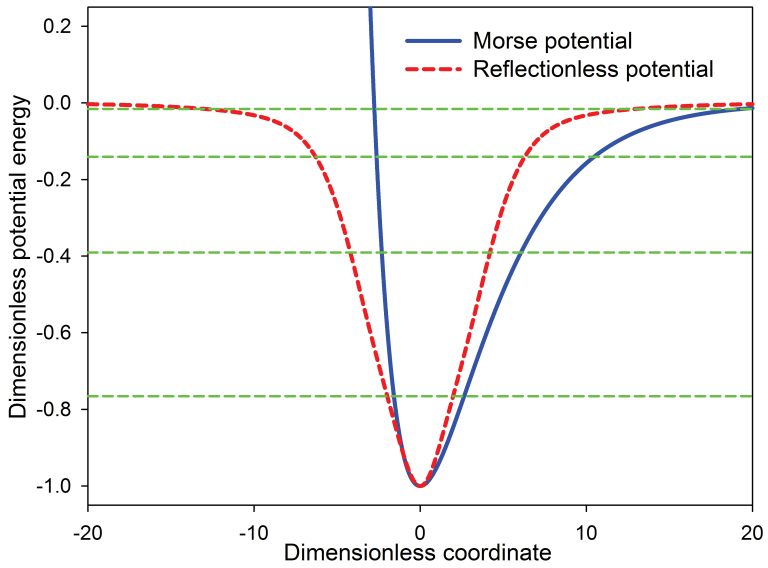

Fig. 2. The isospectral reflectionless substitute to a Morse potential (see the explanations to Example 3). The common energy levels are shown by horizontal dashed lines.

the four energy levels being $E_n = -C\kappa_n^2$ $(n = 1,2,3,4)$ as previously. The solid curve in this figure corresponds to a Morse potential [21]

$$\frac{V(x)}{D} = \exp\left(-\frac{2\alpha x}{x_0}\right) - 2\exp\left(-\frac{\alpha x}{x_0}\right), \tag{67}$$

taking $x_0 \equiv \sqrt{\hbar^2/(2mD)} = 1$, $E_0 = D = 1$, and consequently, $C = 1$. The energy eigenvalues read

$$E_n = -D\left(1 - \frac{n+1/2}{a}\right)^2, \tag{68}$$

where $a \equiv \sqrt{D/C}/\alpha$. In Fig. 2, the value $a = 4$ has been chosen so that $\alpha = 1/4$ in our dimensionless units. As in the previous example, the dashed curve shows the symmetric reflectionless potential derived by Eq. (65) from the input parameters (66).

6. CONCLUSION

The main result of this work is a general formula for calculating τ-functions. This important formula, Eq. (50), is a direct conclusion from Theorem 2 that has been proved with the help of well-known methods of the theory of determinants. We demonstrated that τ-functions can be expanded in terms of special determinants called alternants [16]. Any alternant has a divisor – the Vandermonde's determinant of the same order, while the quotient can be uniquely expressed as a polynomial in elementary symmetric functions (32) (see Theorem 1). Moreover, in the case of alternants related to the inverse scattering problem this quotient equals unity, i.e., the alternant itself equals the Vandermonde determinant. These useful properties of alternants are the key to a very simple final result expressed by Eq. (50).

Using Eqs (50)–(51), one can reconstruct any reflectionless one-dimensional potential on the full line $(-\infty < x < \infty)$, provided that the $2N$ input parameters κ_n and C_n $(n = 1,2,...,N)$ are known. Moreover, if the result is expected to be a symmetric function of the coordinate x, then the problem can be uniquely solved on the basis of the N binding energies $E_n = -C\kappa_n^2$. Compared to the direct use of Eq. (9), the described

approach significantly reduces computational efforts. Indeed, the expansions (21) and (50) contain only 2^{N-1} members, while Eq. (9) requires the evaluation of a determinant with $N!$ members.

The efficiency of the method has been explicitly demonstrated for the case $N = 4$, and there is no doubt that the algorithm can be successfully applied to a much higher number (in principle, to an arbitrary number) of given binding energies. The described approach can also be applied to building N-soliton solutions to the Korteweg–de Vries equation, but this would be a subject for another paper.

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Praktiline meetod etteantud energiaspektriga mittepeegeldava potentsiaali konstrueerimiseks

Matti Selg

On kirjeldatud puhtalgebralist meetodit, mis võimaldab konstrueerida kuitahes suure hulga (N) etteantud diskreetsete energianivoodega mittepeegeldava potentsiaali. Konstruktsiooni aluseks on üldine valem (valem (9) selles artiklis), mis taandab potentsiaali konstrueerimise uuritava süsteemi jaoks iseloomuliku $N \times N$-determinandi arvutamisele. Kuna taolise determinandi arendus sisaldab $N!$ liiget, siis osutub ülesanne üsna keerukaks juba suhteliselt väikese arvu nivoodega süsteemi ($N > 3$) jaoks. Seesuguste determinantide arvutamist saab aga väga oluliselt lihtsustada, kuna need on avaldatavad spetsiifiliste omadustega determinantide – alternantide – kaudu.

Artiklis on tõestatud kaks üldist teoreemi ja tuletatud lihtne ning kergesti kasutatatav valem alternantide arvutamiseks. Kirjeldatud protseduuri tulemusena saadakse modifitseeritud determinant, mille arendus sisaldab ainult 2^{N-1} liiget (vt valemeid (50)–(51)). Arvutustehnilises mõttes on see tohutu võit. Tõepoolest, kui näiteks $N = 10$, siis on arenduses $10! = 3\,628\,800$ asemel ainult $2^9 = 512$ liiget, ja ka märksa suurema nivoode arvu korral on arvutused tänapäevaste vahendite abil kergesti teostatavad.

Kui seada lisatingimus, et otsitav potentsiaal on reaalteljel $(-\infty, \infty)$ sümmeetriline, lihtsustub probleem veelgi. Sel juhul on potentsiaali kuju N energianivoo kaudu üheselt määratud (üldjuhul tuleb fikseerida $2N$ parameetrit). Meetodi tõhusust sümmeetriliste potentsiaalide konstrueerimisel on mitme näite abil ka vahetult demonstreeritud. Väärib märkimist, et töö tulemusi saab kasutada mitte ainult hajumisteooria pöördülesande kontekstis, vaid ka Kortewegi-de Vriesi võrrandi N-solitoni lahendite kindlakstegemiseks.