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CONTROL
SYSTEMS

Linearization by input–output injections on homogeneous time scales

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Abstract. The problem of linearization by input–output (i/o) injections is addressed for nonlinear single-input single-output systems, defined on a homogeneous time scale. The paper provides conditions for the existence of a state transformation, bringing state equations into the observer form, which is linear up to some nonlinear input- and output-dependent functions, called i/o injections. These conditions are based on differential one-forms, associated with the i/o equation of the system.

Key words: nonlinear control system, time scale, observer form, differential one-forms.

1. INTRODUCTION

From a modelling point of view, dynamical systems on time scales incorporate both continuous- and discrete-time systems as special cases, allowing us to unify the study and consider the classical results as special cases from the new theory. On the other hand, the study of dynamical systems on time scales helps to reveal and explain discrepancies, occasionally appearing between the results obtained for continuous-time systems and their discrete-time counterparts (see, e.g., [1,12,13,18,20,21]). However, it is important to note that the discrete-time model in the time scale formalism is given in terms of the difference operator, and not in terms of the more conventional shift operator. The difference-based models, often referred to as delta-domain models, are not completely new for the description of discrete-time systems. They have been promoted during the last decades as the models closely linked to continuous-time systems, being less sensitive to round-off errors at higher sampling rates (e.g., [14,25]). More information on nonlinear control systems on time scales is available in [2–4,6,10,22].

The method of linearization of the nonlinear control systems by input–output (i/o) injections (alternatively, transformation of the state equations into the observer form) is the intermediate step in the observer construction. Design of the nonlinear observer for the system in the observer form (linear up to i/o injections) is relatively easy [11,15], allowing one to construct the nonlinear observer in such a way that the dynamics of the estimation error are linear, making it simple to guarantee that the error converges asymptotically to zero.

The purpose of this paper is to present necessary and sufficient conditions for the existence of the state transformation, allowing transformation of the single-input single-output (SISO) state equations, defined on a homogeneous time scale, into the observer form. The conditions are formulated within the algebraic

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framework, based on differential one-forms, and can be considered as the extension of the result from [11] to the case of a homogeneous time scale. The computation of one-forms is based on the i/o equation of the system and the conditions can be easily verified, whenever the i/o equation is found.

The paper is organized as follows. In Section 2 we give a brief exposition of the basic notions from the time scale calculus and an overview of the algebraic framework of differential forms on a homogeneous time scale. The problem of linearization by i/o injections is formulated in Section 3. Section 4 presents first a direct formula for computation of the differential one-forms, in terms of which the main result is formulated, and then provides necessary and sufficient conditions. The theory is illustrated by an example. Conclusions are drawn in Section 5.

2. PRELIMINARIES

2.1. Time scale calculus

In this subsection we recall only those facts that we need in this paper. For a general introduction to the time scale calculus see [9].

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. This paper is focused on the two most important for control theory instances of time scale, i.e. the continuous-time case, $\mathbb{T} = \mathbb{R}$ and the discrete-time case $\mathbb{T} = \tau\mathbb{Z} := \{\tau k : k \in \mathbb{Z}\}$ for $\tau > 0$. The examples of the other type of time scales, including the non-uniformly sampled time, can be found, for instance, in [6]. The most important notions of time scale calculus are the *forward jump operator* σ, the *backward jump operator* ρ, the *delta derivative* Δ, and the *graininess function* μ. Applications of σ, ρ, and Δ to the function $\xi : \mathbb{T} \to \mathbb{R}$, as well as the values of μ, are presented in Table 1 for two special cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \tau\mathbb{Z}$. A time scale $\mathbb{T}$ is called *homogeneous* if $\mu \equiv$ const and, as can be seen from Table 1, both time scales $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \tau\mathbb{Z}$ possess this property.

Hereinafter, we leave out the time argument t in order to simplify the exposition, so $\xi := \xi(t)$. We denote by $\xi^{\langle i \rangle}$ the delta derivative of an arbitrary order i. Moreover, for notational convenience, denote $\xi^{\langle i \dots n \rangle} := \left(\xi^{\langle i \rangle}, \dots, \xi^{\langle n \rangle}\right)$, for $0 \le i \le n$ and $\xi^{\langle 0 \rangle} := \xi$.

2.2. Algebraic framework

Consider the nonlinear SISO control system, defined on a homogeneous time scale $\mathbb{T}$, and described either by the state equations

$$\begin{aligned} x^{\Delta} &= f(x,u), \\ y &= h(x), \end{aligned} \tag{1}$$

or by the higher-order i/o delta-differential equation

$$y^{\langle n \rangle} = \phi\left(y, y^{\langle 1 \rangle}, \dots, y^{\langle n-1 \rangle}, u, u^{\langle 1 \rangle}, \dots, u^{\langle n-1 \rangle}\right), \tag{2}$$

where $x : \mathbb{T} \to \mathbb{X} \subseteq \mathbb{R}^n$ is an n-dimensional state vector, $u : \mathbb{T} \to \mathbb{U} \subseteq \mathbb{R}$ is an input, and $y : \mathbb{T} \to \mathbb{Y} \subseteq \mathbb{R}$ is an output. Moreover, $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$, $h : \mathbb{X} \to \mathbb{Y}$ and $\phi : \mathbb{Y}^n \times \mathbb{U}^n \to \mathbb{R}$ are assumed to be real analytic functions.

Table 1. Basic types of operators/functions

$\mathbb{T}$	$\xi^{\sigma}(t)$	$\xi^{\rho}(t)$	$\xi^{\Delta}(t)$	μ
$\mathbb{R}$	$\xi(t)$	$\xi(t)$	$\frac{\mathrm{d}\xi(t)}{\mathrm{d}t}$	0
$\tau\mathbb{Z}$	$\xi(t+\tau)$	$\xi(t-\tau)$	$\frac{\xi(t+\tau)-\xi(t)}{\tau}$	τ

The following assumption is specification of Theorem 3.1 from [23], where the multi-input multi-output case was considered.

Assumption 1. System (2) is submersive, i.e. the function ϕ in (2) satisfies the condition

$$\operatorname{rank}\left(1+\sum_{k=0}^{n-1}(-1)^{n-k-1}\mu^{n-k}\frac{\partial\phi}{\partial y^{\langle k\rangle}} \qquad \sum_{j=0}^{n-1}(-1)^{n-j}\mu^{n-j+1}\frac{\partial\phi}{\partial u^{\langle k\rangle}}\right)=1. \tag{3}$$

In this subsection we recall some facts from [5–7] focusing on equation (2). Let $\mathscr{K}^*$ denote the field of meromorphic functions in a finite number of independent system variables from the infinite set $\mathscr{C}^*=\left\{y,y^{\langle 1\rangle},\ldots,y^{\langle n-1\rangle};u^{\langle k\rangle},k\geq 0;\upsilon^{\rho^l},l\geq 1\right\}$, where ρ^l denotes the l-fold application of the backward jump operator ρ and the variable υ can be chosen either to be y or u. The choice can be briefly described as follows. If the first element of the matrix in (3) is not identically equal to zero, then one can choose $\upsilon=u$. In this case, using the i/o equation (2), the variables y^{ρ^l}, $l\geq 1$ can be expressed through the independent variables from $\mathscr{C}^*$. If the second element of the matrix in (3) is not identically equal to zero, then one can set $\upsilon=y$ and consider the variables u^{ρ^l}, $l\geq 1$ as dependent. If both elements of the matrix in (3) are not identically equal to zero, then one has freedom of choice. For $F\left(y^{\langle 0\ldots n-1\rangle},u^{\langle 0\ldots k\rangle}\right)\in\mathscr{K}^*$ the *forward-shift operator* $\sigma_\phi:\mathscr{K}^*\to\mathscr{K}^*$ is defined by

$$F^{\sigma_\phi}\left(y^{\langle 0\ldots n-1\rangle},u^{\langle 0\ldots k+1\rangle}\right):=F\left(\left(y^{\langle 0\ldots n-1\rangle}\right)^{\sigma_\phi},\left(u^{\langle 0\ldots k\rangle}\right)^{\sigma_\phi}\right),$$

where $\left(y^{\langle 0\ldots n-1\rangle}\right)^{\sigma_\phi}=\left(y+\mu y^{\langle 1\rangle},\ldots,y^{\langle n-2\rangle}+\mu y^{\langle n-1\rangle},y^{\langle n-1\rangle}+\mu\phi(\cdot)\right)$, $\left(u^{\langle 0\ldots k\rangle}\right)^{\sigma_\phi}=u^{\langle 0\ldots k\rangle}+\mu u^{\langle 1\ldots k+1\rangle}$ and $\phi(\cdot)$ is determined by (2). Hereinafter, we denote the n-fold application of the forward-shift operator by $F^{\sigma_\phi^n}:=\left(F^{\sigma_\phi^{n-1}}\right)^{\sigma_\phi}$. The *backward-shift operator* $\rho_\phi:\mathscr{K}^*\to\mathscr{K}^*$ is defined as the inverse of σ_ϕ, i.e. $\rho_\phi:=\sigma_\phi^{-1}$. Thus, denoting by ρ_ϕ^n the n-fold application of the backward-shift operator, we have $F=\left(F^{\rho_\phi}\right)^{\sigma_\phi}$ and $F^{\rho_\phi^n}=\left(F^{\rho_\phi^{n+1}}\right)^{\sigma_\phi}$. The *delta derivative operator* $\Delta_\phi:\mathscr{K}^*\to\mathscr{K}^*$ is defined by

$$F^{\Delta_\phi}\left(y^{\langle 0\ldots n-1\rangle},u^{\langle 0\ldots k+1\rangle}\right):=\begin{cases}\dfrac{1}{\mu}\left(F^{\sigma_\phi}(\cdot)-F(\cdot)\right) & \text{if } \mu\neq 0,\\ \displaystyle\sum_{l=0}^{n-1}\frac{\partial F(\cdot)}{\partial y^{\langle l\rangle}}y^{\langle l+1\rangle}+\sum_{k\geq 0}\frac{\partial F(\cdot)}{\partial u^{\langle k\rangle}}u^{\langle k+1\rangle} & \text{if } \mu=0,\end{cases}$$

where, according to (2), $y^{\langle n\rangle}$ should be replaced by ϕ, whenever it appears. Observe that for $\mu\neq 0$ the nth delta derivative can be computed by the formula

$$F^{\langle n\rangle}=\frac{1}{\mu^n}\sum_{k=0}^{n}(-1)^k C_n^k F^{\sigma_\phi^{n-k}}, \tag{4}$$

where C_n^k is a binomial coefficient, i.e. $C_n^k=n!/((n-k)!k!)$.

Proposition 2. [5] *For $F,G\in\mathscr{K}^*$ the delta derivative and forward-shift operators satisfy the following properties*:
(i) $F^{\sigma_\phi}=F+\mu F^{\Delta_\phi}$,
(ii) $(\alpha F+\beta G)^{\Delta_\phi}=\alpha F^{\Delta_\phi}+\beta G^{\Delta_\phi}$, for $\alpha,\beta\in\mathbb{R}$,
(iii) $(FG)^{\Delta_\phi}=F^{\sigma_\phi}G^{\Delta_\phi}+F^{\Delta_\phi}G$,
(iv) *on a homogeneous time scale operators Δ_ϕ and σ_ϕ commute, i.e.* $\left(F^{\sigma_\phi}\right)^{\Delta_\phi}=\left(F^{\Delta_\phi}\right)^{\sigma_\phi}$.

Generalization of (i) in Proposition 2 yields the n-fold application of operator σ_ϕ as

$$F^{\sigma_\phi^n} = \sum_{s=0}^{n} C_n^s \mu^s F^{\langle s \rangle}. \tag{5}$$

Consider the infinite set of symbols $\mathrm{d}\mathscr{C} = \left\{\mathrm{d}y, \mathrm{d}y^{\langle 1 \rangle}, \ldots, \mathrm{d}y^{\langle n-1 \rangle}, \mathrm{d}u^{\langle k \rangle}, k \geq 0\right\}$ and define $\mathscr{E} = \operatorname{span}_{\mathscr{K}^*} \mathrm{d}\mathscr{C}$. The elements of $\mathscr{E}$ are called the *differential one-forms*. Any element of $\mathscr{E}$ has the form

$$\omega = \sum_{l=0}^{n-1} A_l \mathrm{d}y^{\langle l \rangle} + \sum_{k \geq 0} B_k \mathrm{d}u^{\langle k \rangle},$$

where $A_l, B_k \in \mathscr{K}^*$ and only a finite number of coefficients B_k are nonzero.

For $F\left(y^{\langle 0 \ldots n-1 \rangle}, u^{\langle 0 \ldots k \rangle}\right) \in \mathscr{K}^*$ define the operator $\mathrm{d} : \mathscr{K}^* \to \mathscr{E}$ by

$$\mathrm{d}F := \sum_{l=0}^{n-1} \frac{\partial F}{\partial y^{\langle l \rangle}} \mathrm{d}y^{\langle l \rangle} + \sum_{k \geq 0} \frac{\partial F}{\partial u^{\langle k \rangle}} \mathrm{d}u^{\langle k \rangle}.$$

Starting from the space $\mathscr{E}$ it is possible to build up the structures used in exterior differential calculus. We refer to [7] for details, whereas here we just recall some basic notions. Define the set $\wedge \mathrm{d}\mathscr{C} = \{\mathrm{d}\zeta \wedge \mathrm{d}\eta : \mathrm{d}\zeta, \mathrm{d}\eta \in \mathrm{d}\mathscr{C}\}$, where $\wedge$ denotes the wedge product with the standard properties $\mathrm{d}\zeta \wedge \mathrm{d}\eta = -\mathrm{d}\eta \wedge \mathrm{d}\zeta$ and $\mathrm{d}\zeta \wedge \mathrm{d}\zeta = 0$ for $\mathrm{d}\zeta, \mathrm{d}\eta \in \mathrm{d}\mathscr{C}$. Introduce the space $\mathscr{E}^2 = \operatorname{span}_{\mathscr{K}^*} \wedge \mathrm{d}\mathscr{C}$ of two-forms. The operator $\mathrm{d} : \mathscr{E} \to \mathscr{E}^2$, called exterior derivative operator, is defined for $\omega = \sum_{\ell=1}^{k} a_\ell(\zeta_1, \ldots, \zeta_k) \mathrm{d}\zeta_\ell \in \mathscr{E}$, where $\zeta_1, \ldots, \zeta_k \in \mathscr{C}^*$, by the rule $\mathrm{d}\omega := \sum_{\ell,\ell'} \partial a_\ell / \partial \zeta_{\ell'} \, \mathrm{d}\zeta_\ell \wedge \mathrm{d}\zeta_{\ell'}$. The notion of two-form is generalized to the p-form and wedge product is defined for arbitrary p-forms.

One says that $\omega \in \mathscr{E}$ is an *exact* one-form if $\omega = \mathrm{d}F$ for some $F \in \mathscr{K}^*$. A one-form ω for which $\mathrm{d}\omega = 0$ is said to be *closed*. It is well known that an exact one-form is closed, whereas a closed one-form is only locally exact.

3. PROBLEM STATEMENT

Our purpose is to find the conditions under which there exists a (local) state transformation, i.e. diffeomorphism $\psi : \mathbb{X} \to \mathbb{X}$, defined by

$$z = \psi(x) \tag{6}$$

such that in the new state coordinates the state equations (1) are in the observer form

$$\begin{aligned} z_1^{\Delta} &= z_2 + \varphi_1(y, u), \\ &\vdots \\ z_{n-1}^{\Delta} &= z_n + \varphi_{n-1}(y, u), \\ z_n^{\Delta} &= \varphi_n(y, u), \\ y &= z_1, \end{aligned} \tag{7}$$

which is linear up to nonlinear functions $\varphi_1(y, u), \ldots, \varphi_n(y, u)$, called i/o injections.

State elimination

System (1) is called *generically (single-experiment) observable* if the rank of the observability matrix is generically equal to n [19], i.e. if

$$\operatorname{rank}_{\mathscr{K}^*} \left[\frac{\partial \left(h, h^{\langle 1 \rangle}, \ldots, h^{\langle n-1 \rangle}\right)}{\partial x} \right] = n. \tag{8}$$

Given a SISO observable nonlinear control system (1), defined on a homogeneous time scale, the problem is to find the higher-order i/o delta-differential equation (2), corresponding to (1). In general, the representation (2) is valid only locally. Compute,

$$\begin{aligned} y &= h(x), \\ y^{\Delta} &= h^{\Delta_\phi}(x,u), \\ &\vdots \\ y^{\langle n-1\rangle} &= h^{\langle n-1\rangle}\left(x,u,u^{\langle 1\rangle},\dots,u^{\langle n-2\rangle}\right). \end{aligned} \tag{9}$$

The set of equations (9) can be solved, under the observability assumption (8), with respect to the state variables

$$x = \zeta\left(y,y^{\langle 1\rangle},\dots,y^{\langle n-1\rangle},u,u^{\langle 1\rangle},\dots,u^{\langle n-2\rangle}\right). \tag{10}$$

Next, compute $y^{\langle n\rangle}$ and substitute x from (10) to get

$$y^{\langle n\rangle} = h^{\langle n\rangle}\left(\zeta\left(y,y^{\langle 1\rangle}\dots,y^{\langle n-1\rangle},u,u^{\langle 1\rangle},\dots,u^{\langle n-2\rangle}\right),u,u^{\langle 1\rangle},\dots,u^{\langle n-1\rangle}\right).$$

Note that the state equations (1) can be transformed into the observer form (7) with the state transformation (6), if the i/o equation (2), corresponding to (1), can be rewritten in the form

$$y^{\langle n\rangle} = (\varphi_1(y,u))^{\langle n-1\rangle} + \dots + (\varphi_{n-1}(y,u))^{\langle 1\rangle} + \varphi_n(y,u) \tag{11}$$

for some functions $\varphi_1(y,u),\dots,\varphi_n(y,u)$. The converse holds too, since (11) is always realizable into the extended observer form (7).

Indeed, if (2) has the form (11), one can define the new state variables as

$$\begin{aligned} z_1 &= y, \\ z_2 &= y^{\langle 1\rangle} - \varphi_1, \\ z_3 &= y^{\langle 2\rangle} - \varphi_1^{\langle 1\rangle} - \varphi_2, \\ &\vdots \\ z_n &= y^{\langle n-1\rangle} - \varphi_1^{\langle n-2\rangle} - \dots - \varphi_{n-2}^{\langle 1\rangle} - \varphi_{n-1}, \end{aligned} \tag{12}$$

yielding the state equations in the observer form (7). Note that, using the state equations (1), one can substitute the variables $y,y^{\langle 1\rangle},\dots,y^{\langle n-1\rangle}$ in such a way that the right-hand side of equation (12) depends only on x, meaning that (12) is the state transformation (6).

4. NECESSARY AND SUFFICIENT CONDITIONS

For $i=1,\dots,n$ define the differential one-forms

$$\omega_i = \sum_{j=0}^{i-1}(-1)^j C_{n-i+j}^{j}\left[\left(\left(\frac{\partial\phi}{\partial y^{\langle n-i+j\rangle}}\right)^{\rho_\phi^{n-i+j}}\right)^{\langle j\rangle}\mathrm{d}y + \left(\left(\frac{\partial\phi}{\partial u^{\langle n-i+j\rangle}}\right)^{\rho_\phi^{n-i+j}}\right)^{\langle j\rangle}\mathrm{d}u\right]. \tag{13}$$

The proof of Theorem 4 below is based on the following Proposition, which extends the results of [17] to the case of a homogeneous time scale, and the proof of which is given in Appendix.

Proposition 3. *Let $\Phi(\xi_1(t),\xi_2(t),\dots,\xi_r(t))$ be a composite function for which delta derivatives up to order $a+b$, where a and b are nonnegative integers, are defined. Then on a homogeneous time scale $\mathbb{T}$ for $l=1,2,\dots,r$ the following holds*:

$$\frac{\partial\left[(\Phi(\xi_1(t),\xi_2(t),\dots,\xi_r(t)))^{\langle a+b\rangle}\right]}{\partial\xi_l^{\langle a\rangle}(t)} = C_{a+b}^{b}\left(\left(\frac{\partial\Phi(\xi_1(t),\xi_2(t),\dots,\xi_r(t))}{\partial\xi_l(t)}\right)^{\langle b\rangle}\right)^{\sigma_\phi^a}. \tag{14}$$

Theorem 4. *The observable system of the form* (1) *can be transformed by the state transformation* (6) *into the observer form* (7) *if and only if for $i=1,\dots,n$*

$$\mathrm{d}\omega_i = 0, \tag{15}$$

where the one-forms ω_i are defined by (13).

Proof. Necessity: Assume that system (1) is transformable into the observer form (7). Consequently, the i/o equation (2), corresponding to the state equations (1), can be rewritten in the form (11), yielding

$$\phi = \sum_{k=1}^{n}\varphi_k^{\langle n-k\rangle}. \tag{16}$$

For the compactness of the proof denote

$$\omega_{i_y} := \sum_{j=0}^{i-1}(-1)^j C_{n-i+j}^{j}\left(\left(\frac{\partial\phi}{\partial y^{\langle n-i+j\rangle}}\right)^{\rho_\phi^{n-i+j}}\right)^{\langle j\rangle},$$
$$\omega_{i_u} := \sum_{j=0}^{i-1}(-1)^j C_{n-i+j}^{j}\left(\left(\frac{\partial\phi}{\partial u^{\langle n-i+j\rangle}}\right)^{\rho_\phi^{n-i+j}}\right)^{\langle j\rangle},$$

such that (13) may be rewritten as

$$\omega_i = \omega_{i_y}\mathrm{d}y + \omega_{i_u}\mathrm{d}u. \tag{17}$$

First, consider ω_{i_y}. Using (16), one obtains

$$\omega_{i_y} = \sum_{j=0}^{i-1}\sum_{k=1}^{n}(-1)^j C_{n-i+j}^{j}\left(\left(\frac{\partial\varphi_k^{\langle n-k\rangle}}{\partial y^{\langle n-i+j\rangle}}\right)^{\rho_\phi^{n-i+j}}\right)^{\langle j\rangle}.$$

Observe, that if $k > i-j$, then $n-k < n-i+j$ and so $\partial\varphi_k^{\langle n-k\rangle}/\partial y^{\langle n-i+j\rangle} = 0$. Therefore, instead of taking $k=1,\dots,n$, we can take $k=1,\dots,i-j$. Moreover, by Proposition 3 for $r=2$, $a=n-i+j$, and $b=i-j-k$, we have

$$\frac{\partial\varphi_k^{\langle n-k\rangle}}{\partial y^{\langle n-i+j\rangle}} = C_{n-k}^{i-j-k}\left(\left(\frac{\partial\varphi_k}{\partial y}\right)^{\langle i-j-k\rangle}\right)^{\sigma_\phi^{n-i+j}}.$$

Thus, one can write

$$\omega_{i_y} = \sum_{j=0}^{i-1}\sum_{k=1}^{i-j}(-1)^j C_{n-i+j}^{j}C_{n-k}^{i-j-k}\left(\frac{\partial\varphi_k}{\partial y}\right)^{\langle i-k\rangle}.$$

Changing the summation order $\sum_{j=0}^{i-1}\sum_{k=1}^{i-j} a_{j,k} = \sum_{k=1}^{i}\sum_{j=1}^{i-k+1} a_{j-1,k}$ and taking into account $C_{n-i+j-1}^{j-1} C_{n-k}^{i-j-k+1} = C_{n-k}^{i-k} C_{i-k}^{j-1}$, one obtains

$$\omega_{i_y} = \sum_{k=1}^{i} C_{n-k}^{i-k} \left(\frac{\partial \varphi_k}{\partial y}\right)^{\langle i-k\rangle} \sum_{j=1}^{i-k+1} (-1)^{j-1} C_{i-k}^{j-1}.$$

Note that for $i=1$ the above formula yields $\omega_{1_y} = \partial\varphi_1/\partial y$. In the case $i \geq 2$, one can separate the last addend of the sum ω_{i_y}, leading to

$$\omega_{i_y} = \frac{\partial \varphi_i}{\partial y} + \sum_{k=1}^{i-1} C_{n-k}^{i-k} \left(\frac{\partial \varphi_k}{\partial y}\right)^{\langle i-k\rangle} \sum_{j=1}^{i-k+1} (-1)^{j-1} C_{i-k}^{j-1}.$$

In [16] Lemma 1 says that for $k = 1,\dots,i-1$ and $i \geq 2$

$$\sum_{j=1}^{i-k+1} (-1)^{j-1} C_{i-k}^{j-1} = 0. \tag{18}$$

Then by (18), $\omega_{i_y} = \partial\varphi_i/\partial y$. In the same manner we get $\omega_{i_u} = \partial\varphi_i/\partial u$, for $i = 1,\dots,n$. Finally, from (17) we obtain

$$\omega_i = \mathrm{d}\varphi_i, \tag{19}$$

yielding (15).

Sufficiency: Assume that the conditions (15) are satisfied. Then locally there exist functions $\varphi_i(y,u)$, satisfying (19). Integrating the one-forms ω_i, the corresponding functions φ_i can be found, such that (16) holds and, as a consequence, the state equations in the observer form (7) can be constructed. □

Example 5. Consider the system

$$\begin{aligned}
x_1^\Delta &= \left(u + \frac{1}{u}\right)(x_2 + x_3) + x_1 + u - x_3,\\
x_2^\Delta &= u^2 - x_1 + (2 + x_3 + u)x_3 + (u + x_2 + 2x_3)x_2,\\
x_3^\Delta &= x_1 + ux_2 + (u-1)x_3,\\
y &= x_2 + x_3.
\end{aligned} \tag{20}$$

The i/o equation, corresponding to (20), is

$$\begin{aligned}
y^{\langle 3\rangle} = {}& \mu^2 y^{\langle 2\rangle}\left(2u^{\langle 2\rangle} + y^{\langle 2\rangle}\right) + 4\mu y^{\langle 2\rangle}\left(u^\Delta + y^\Delta\right) + 2y^{\langle 2\rangle}(u+y)\\
& + \mu y^\Delta\left(u^\Delta + 4u^{\langle 2\rangle}\right) + 2y^\Delta\left(y^\Delta + 2u^\Delta\right) + y\left(\frac{1}{u} + u^\Delta + 2u^{\langle 2\rangle}\right)\\
& + u^{\langle 2\rangle}\left(2u + 4\mu u^\Delta + \mu^2 u^{\langle 2\rangle}\right) + 2(u^\Delta)^2 + u(y^\Delta + 1).
\end{aligned}$$

Compute, according to (13),

$$\begin{aligned}
\omega_1 &= 2(u+y)\,\mathrm{d}y + 2(u+y)\,\mathrm{d}u,\\
\omega_2 &= u\,\mathrm{d}y + y\,\mathrm{d}u,\\
\omega_3 &= \frac{1}{u}\,\mathrm{d}y + \left(1 - \frac{y}{u^2}\right)\mathrm{d}u,
\end{aligned}$$

which leads to

$$\begin{aligned}
\mathrm{d}\omega_1 &= 2\mathrm{d}y\wedge\mathrm{d}y+2\mathrm{d}u\wedge\mathrm{d}y-2\mathrm{d}u\wedge\mathrm{d}y+2\mathrm{d}u\wedge\mathrm{d}u=0,\\
\mathrm{d}\omega_2 &= \mathrm{d}u\wedge\mathrm{d}y-\mathrm{d}u\wedge\mathrm{d}y=0,\\
\mathrm{d}\omega_3 &= -(1/u^2)\mathrm{d}u\wedge\mathrm{d}y+(1/u^2)\mathrm{d}u\wedge\mathrm{d}y+(2/u^3)\mathrm{d}u\wedge\mathrm{d}u=0,
\end{aligned}$$

meaning that the conditions of Theorem 4 are satisfied. Since in this case (19) holds whenever $u\neq 0$, integration of one-forms ω_i leads to

$$\begin{aligned}
\varphi_1(y,u) &= (y+u)^2,\\
\varphi_2(y,u) &= yu,\\
\varphi_3(y,u) &= \frac{y}{u}+u.
\end{aligned}$$

Using (12), one obtains the state transformation as

$$\begin{aligned}
z_1 &= x_2+x_3,\\
z_2 &= x_3,\\
z_3 &= x_1-x_3,
\end{aligned}$$

yielding the state equations in the observer form

$$\begin{aligned}
z_1^{\Delta} &= z_2+(y+u)^2,\\
z_2^{\Delta} &= z_3+yu,\\
z_3^{\Delta} &= \frac{y}{u}+u,\\
y &= z_1.
\end{aligned}$$

5. CONCLUSIONS

In the paper necessary and sufficient conditions for linearization of the nonlinear state equations, defined on a homogeneous time scale, by input-output (i/o) injections are given. For this aim the state transformation is used. The conditions are formulated in terms of differential one-forms, directly computable from the i/o equation of the given system. The main theorem states that the problem is solvable if and only if the exterior derivatives of these one-forms are equal to zero. Note that our conditions are simple and transparent but require that first the i/o equation of the control system has to be found. This can be done using the extension of the state elimination algorithm from [11] to the systems, defined on a homogeneous time scale, provided the control system is observable. Similarly to the continuous-time case, the i/o equations can always be found, at least locally. Nevertheless, it can sometimes be a difficult task, implying that the i/o equation cannot always be represented in terms of the elementary functions.

In order to provide the proof of the main theorem, the supporting proposition was stated and proved. The proposition is the extension of the theorem from [17] and it shows how the partial derivative of the total delta derivative of the composite function with a vector argument can be expressed through the total delta derivative of the partial derivative of the composite function. The proposition can also serve as a useful tool for research in the area of studying systems on homogeneous time scales.

It should be mentioned that though the results obtained in this paper basically recover those for the continuous-time case in [11], for the discrete-time systems, given in terms of the difference operator, the results are completely new. Moreover, unlike [11], where the step-by-step algorithm (requiring integration of one-forms) was employed, we suggested the direct formula for computation of the one-forms, necessary for conditions. To conclude, the theoretical results were obtained in a constructive way such that they can be implemented later in the software package NLControl (*Mathematica*-based package developed in the Institute of Cybernetics) (see [8]).

Regarding the future extension of the results of this paper, one may address the construction of the observer. Moreover, using both the state and the output transformations, like in [16], one may relax the conditions of Theorem 4. Note that, concerning this problem, in the discrete-time case simple necessary and sufficient conditions exist that are directly computable from the i/o equation and do not depend on an unknown single-variable output-dependent function [26], whereas in the continuous-time case derivation of similar conditions seems to be a difficult task. We expect that the unified formalism of time scale calculus will help to understand the reasons for this discrepancy and suggest a solution. Finally, one may employ the tools of differential geometry, like in [24] and [27], to derive the alternative conditions, which do not rely on the i/o equation of the system. These conditions would be preferable in the situations when the i/o equation is difficult to find.

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APPENDIX

PROOF OF PROPOSITION 3

Proof. Note that in [17] the formula (14) was proved for the case $\mathbb{T} = \mathbb{R}$. Therefore, we consider here only the case $\mathbb{T} = \tau\mathbb{Z}$, $\tau > 0$. First, consider the left-hand side of equation (14). Using (4) for $n = a+b$, the definition of the operator σ_ϕ and the chain rule for the partial derivative with respect to $\xi_l^{\langle a \rangle}$, one obtains

$$\frac{\partial\left[(\Phi(\xi_1,\xi_2,\ldots,\xi_r))^{\langle a+b\rangle}\right]}{\partial \xi_l^{\langle a\rangle}} = \frac{1}{\mu^{a+b}}\sum_{k=0}^{a+b}(-1)^k C_{a+b}^k \frac{\partial \Phi\left(\xi_1^{\sigma_\phi^{a+b-k}}, \xi_2^{\sigma_\phi^{a+b-k}}, \ldots, \xi_r^{\sigma_\phi^{a+b-k}}\right)}{\partial \xi_l^{\sigma_\phi^{a+b-k}}} \cdot \frac{\partial \xi_l^{\sigma_\phi^{a+b-k}}}{\partial \xi_l^{\langle a\rangle}},$$

which, according to (5) for $n = a+b-k$, yields

$$\frac{\partial\left[(\Phi(\xi_1,\xi_2,\ldots,\xi_r))^{\langle a+b\rangle}\right]}{\partial \xi_l^{\langle a\rangle}} = \frac{1}{\mu^{a+b}}\sum_{k=0}^{a+b}(-1)^k C_{a+b}^k \frac{\partial \Phi\left(\xi_1^{\sigma_\phi^{a+b-k}}, \xi_2^{\sigma_\phi^{a+b-k}}, \ldots, \xi_r^{\sigma_\phi^{a+b-k}}\right)}{\partial \xi_l^{\sigma_\phi^{a+b-k}}} \cdot \sum_{s=0}^{a+b-k} C_{a+b-k}^s \mu^s \frac{\partial \xi_l^{\langle s\rangle}}{\partial \xi_l^{\langle a\rangle}}.$$

Note that $\partial \xi_l^{\langle s\rangle}/\partial \xi_l^{\langle a\rangle} = 0$ for every s, except for $s = a$ when it equals 1. Furthermore, $s = a$ occurs only when $a+b-k \geq a$, implying $k \leq b$. Thus, one can write

$$\frac{\partial\left[(\Phi(\xi_1,\xi_2,\ldots,\xi_r))^{\langle a+b\rangle}\right]}{\partial \xi_l^{\langle a\rangle}} = \frac{1}{\mu^{b}}\sum_{k=0}^{b}(-1)^k C_{a+b}^k C_{a+b-k}^a \frac{\partial \Phi\left(\xi_1^{\sigma_\phi^{a+b-k}}, \xi_2^{\sigma_\phi^{a+b-k}}, \ldots, \xi_r^{\sigma_\phi^{a+b-k}}\right)}{\partial \xi_l^{\sigma_\phi^{a+b-k}}}.$$

Taking into account that by direct computations $C_{a+b}^{k}C_{a+b-k}^{a}=C_{a+b}^{b}C_{b}^{k}$ and using the properties

$$\frac{\partial F\left(\xi^{\sigma_\phi^i}\right)}{\partial \xi^{\sigma_\phi^i}}=\left(\frac{\partial F(\xi)}{\partial \xi}\right)^{\sigma_\phi^i} \text{ and } F^{\sigma_\phi^{i+j}}=\left(F^{\sigma_\phi^i}\right)^{\sigma_\phi^j},$$

one obtains

$$\frac{\partial\left[(\Phi(\xi_1,\xi_2,\ldots,\xi_r))^{\langle a+b\rangle}\right]}{\partial \xi_l^{\langle a\rangle}}=C_{a+b}^{b}\left(\frac{1}{\mu^b}\sum_{k=0}^{b}(-1)^k C_b^k\left(\frac{\partial\Phi(\xi_1,\xi_2,\ldots,\xi_r)}{\partial \xi_l}\right)^{\sigma_\phi^{b-k}}\right)^{\sigma_\phi^a},$$

which, according to (4) for $n=b$, confirms (14). This completes the proof. □

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Olekuvõrrandite lineariseerimine sisend-väljund-injektsioonide kaudu homogeensel ajaskaalal

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On uuritud homogeensel ajaskaalal defineeritud mittelineaarse juhtimissüsteemi olekuvõrrandite olekutaastaja kujule teisendamist. Alternatiivselt nimetatakse antud kujule teisendamist süsteemi lineariseerimiseks sisend-väljund-injektsioonide abil, sest sellisel kujul on süsteemi võrrandid lineaarsed kuni mõnede mittelineaarsete, ainult sisenditest ja väljunditest sõltuvate liidetavateni; viimaseid nimetatakse sisend-väljund-injektsioonideks. Olekutaastaja kujul esitatud süsteemi olekuid saab lihtsalt hinnata selliselt, et hindamisviga läheneb asümptootiliselt nullile. Olekute hindamine on oluline olukorras, kus need pole vahetult mõõdetavad või on mõõtmine liiga kallis. Analüüs ajaskaalal võimaldab ühildada pidevate ja diskreetsete dünaamiliste süsteemide uurimise. On leitud tarvilikud ja piisavad tingimused süsteemi võrrandite olekuteisendustega olekutaastaja kujule teisendamiseks. Tingimused on formuleeritud teatud (otseselt süsteemi sisend-väljund-võrrandist leitavate) diferentsiaalsete üks-vormide eksaktsuse kaudu. Kui tingimused on rahuldatud, siis üks-vormide integreerimisel leitakse funktsioonid (sisend-väljund-injektsioonid), mille abil saab olekuvõrrandite olekutaastaja kuju lihtsalt konstrueerida.