Proceedings of the Estonian Academy of Sciences,
2014, **63**, 4, 355–371
doi: 10.3176/proc.2014.4.01
Available online at www.eap.ee/proceedings

MATHEMATICS

On endomorphisms of groups of order 32 with maximal subgroups $C_8 \times C_2$

Piret Puusemp and Peeter Puusemp*

Department of Mathematics, Tallinn University of Technology, Ehitajate tee 5, 19086 Tallinn, Estonia

Received 9 June 2014, revised 18 September 2014, accepted 26 September 2014, available online 20 November 2014

Abstract. It is proved that each group of order 32 which has a maximal subgroup isomorphic to $C_8 \times C_2$ is determined by its endomorphism semigroup in the class of all groups.

Key words: group, semigroup, endomorphism semigroup.

1. INTRODUCTION

It is well known that all endomorphisms of an Abelian group form a ring and many of its properties can be characterized by this ring. An excellent overview of the present situation in the theory of endomorphism rings of groups is given by Krylov et al. [6]. All endomorphisms of an arbitrary group form only a semigroup. The theory of endomorphism semigroups of groups is quite modestly developed. In a number of our papers we have made efforts to describe some properties of groups by the properties of their endomorphism semigroups. For example, we have proved that many well-known classes of groups are determined by their endomorphism semigroups in the class of all groups. Note that if G is a fixed group and an isomorphism of semigroups $\mathrm{End}(G)$ and $\mathrm{End}(H)$, where H is an arbitrary group, always implies an isomorphism of G and H, then we say that the group G is determined by its endomorphism semigroup in the class of all groups. Some of such groups are finite Abelian groups ([7], Theorem 4.2), generalized quaternion groups ([8], Corollary 1), torsion-free divisible Abelian groups ([11], Theorem 1), etc. On the other hand, there exist many examples of groups that are not determined by their endomorphism semigroups in the class of all groups. For example, the following result of Corner is well known [2]: any countable, reduced, torsion-free, associative ring with unity is an endomorphism ring for a continual number of countable, reduced, torsion-free Abelian groups. There exist finite groups that are semidirect products of cyclic groups and are not determined by their endomorphism semigroups in the class of all groups [10].

We know a complete answer to this problem for finite groups of order less than 32. It was proved in [12] that among the finite groups of order less than 32 only the alternating group A_4 (also called the tetrahedral group) and the binary tetrahedral group $\langle a, b \mid b^3 = 1,\, aba = bab \rangle$ are not determined by their endomorphism semigroups in the class of all groups. These two groups are non-isomorphic, but their endomorphism semigroups are isomorphic. In the light of this result it is natural to consider next the groups of order 32.

All groups of order 32 were described by Hall and Senior [5]. There exist exactly 51 non-isomorphic groups of order 32. In [5], these groups are numbered by $1, 2, \ldots, 51$. We shall mark these groups

* Corresponding author, peeter.puusemp@ttu.ee

by $\mathscr{G}_1, \mathscr{G}_2, \ldots, \mathscr{G}_{51}$, respectively. The groups $\mathscr{G}_1$–$\mathscr{G}_7$ are Abelian, and, therefore, are determined by their endomorphism semigroups in the class of all groups ([7], Theorem 4.2). In [3], it was proved that the groups of order 32 presentable in the form $(C_4 \times C_4) \leftthreetimes C_2$ (C_k – the cyclic group of order k) are determined by their endomorphism semigroups in the class of all groups. The groups of this type are $\mathscr{G}_3, \mathscr{G}_{14}, \mathscr{G}_{16}, \mathscr{G}_{31}, \mathscr{G}_{34}, \mathscr{G}_{39}, \mathscr{G}_{41}$. In [4], it was proved that the groups of order 32 presentable in the form $(C_8 \times C_2) \leftthreetimes C_2$ are determined by their endomorphism semigroups in the class of all groups. The groups of this type are $\mathscr{G}_4, \mathscr{G}_{17}, \mathscr{G}_{20}, \mathscr{G}_{26}, \mathscr{G}_{27}$. In [14], Theorem 1.1, it was proved that the groups of order 32 which have a maximal subgroup isomorphic to $C_4 \times C_2 \times C_2$ are determined by their endomorphism semigroup in the class of all groups. The groups of this type are $\mathscr{G}_2$–$\mathscr{G}_4$, $\mathscr{G}_8$–$\mathscr{G}_{14}$, $\mathscr{G}_{16}$, $\mathscr{G}_{18}$, $\mathscr{G}_{20}$, $\mathscr{G}_{36}$–$\mathscr{G}_{38}$.

In this paper, we consider the groups of order 32 that have a maximal subgroup isomorphic to $C_8 \times C_2$ and prove the following theorem:

Theorem 1.1. *Each group of order 32 which has a maximal subgroup isomorphic to $C_8 \times C_2$ is determined by its endomorphism semigroup in the class of all groups.*

The groups of order 32 which have a maximal subgroup isomorphic to $C_8 \times C_2$ are:

$$\mathscr{G}_4, \mathscr{G}_5, \mathscr{G}_6, \mathscr{G}_{17}, \mathscr{G}_{19}, \mathscr{G}_{20}, \mathscr{G}_{21}, \mathscr{G}_{22}, \mathscr{G}_{26}, \mathscr{G}_{27}, \mathscr{G}_{28}, \mathscr{G}_{29}, \mathscr{G}_{30}, \mathscr{G}_{32}.$$

To prove the theorem, the characterization of these groups by their endomorphism semigroups will be given. These characterization properties that are preserved by isomorphisms of endomorphism semigroups will then be used in the proofs.

We shall use the following notations:
G – a group;
$\mathrm{End}(G)$ – the endomorphism semigroup of G;
C_k – the cyclic group of order k;
$Q_n = \langle a, b \mid a^{2^n} = 1,\ a^{2^{n-1}} = b^2,\ b^{-1}ab = a^{-1} \rangle$ – the generalized quaternion group ($n \geq 2$);
$Q = Q_2$ – the quaternion group;
$\mathbb{Z}_k$ – the ring of residual classes modulo k;
$\langle K, \ldots, g, \ldots \rangle$ – the subgroup generated by subsets $K, \ldots$ and elements $g, \ldots$;
$[a, b] = a^{-1}b^{-1}ab$ ($a, b \in G$);
G' – the commutator-group of G;
$\widehat{g}$ – the inner automorphism of G, generated by an element $g \in G$;
$I(G)$ – the set of all idempotents of $\mathrm{End}(G)$;
$K(x) = \{z \in \mathrm{End}(G) \mid zx = xz = z\}$;
$J(x) = \{z \in \mathrm{End}(G) \mid zx = xz = 0\}$;
$H(x) = \{z \in \mathrm{End}(G) \mid xz = z,\ zx = 0\}$;
$o(g)$ – the order of an element $g \in G$.

The sets $K(x)$ and $J(x)$ are subsemigroups of $\mathrm{End}(G)$. We shall write the mapping right from the element on which it acts.

2. GROUPS THAT HAVE A MAXIMAL SUBGROUP $C_8 \times C_2$

According to Hall and Senior [5], the groups of order 32 that have a maximal subgroup isomorphic to $C_8 \times C_2$ are:
- $\mathscr{G}_4 = C_2 \times C_2 \times C_8$, $\mathscr{G}_5 = C_4 \times C_8$, $\mathscr{G}_6 = C_2 \times C_{16}$,
- $\mathscr{G}_{17} = \langle a, b, c \mid a^8 = b^2 = c^2 = 1,\ ab = ba,\ ac = ca,\ c^{-1}bc = ba^4 \rangle = (\langle a \rangle \times \langle b \rangle) \leftthreetimes \langle c \rangle = (C_8 \times C_2) \leftthreetimes C_2$,
- $\mathscr{G}_{19} = \langle a, b \mid a^4 = b^8 = 1,\ ab^2 = b^2a,\ b^{-1}ab = ab^4 \rangle = \langle a, b \mid a^4 = b^8 = 1,\ a^{-1}ba = b^5 \rangle = \langle b \rangle \leftthreetimes \langle a \rangle$
  $= C_8 \leftthreetimes C_4$,
- $\mathscr{G}_{20} = \langle a, b, c \mid a^8 = b^2 = c^2 = 1,\ ab = ba,\ bc = cb,\ c^{-1}ac = ab \rangle = (\langle a \rangle \times \langle b \rangle) \leftthreetimes \langle c \rangle = (C_8 \times C_2) \leftthreetimes C_2$,

• $\mathscr{G}_{21} = \langle a, b \mid a^4 = b^8 = 1,\ b^{-1}ab = a^{-1}\rangle = \langle a\rangle \leftthreetimes \langle b\rangle = C_4 \leftthreetimes C_8$,
• $\mathscr{G}_{22} = \langle a, b \mid a^{16} = b^2 = 1,\ b^{-1}ab = a^9\rangle = \langle a\rangle \leftthreetimes \langle b\rangle = C_{16} \leftthreetimes C_2$,
• $\mathscr{G}_{26} = \langle a, b, c \mid a^8 = b^2 = c^2 = 1,\ ab = ba,\ c^{-1}ac = a^{-1},\ c^{-1}bc = a^4b\rangle = (\langle a\rangle \times \langle b\rangle) \leftthreetimes \langle c\rangle$
$= (C_8 \times C_2) \leftthreetimes C_2 = \langle a, d, c \mid a^8 = 1,\ a^4 = d^2, d^{-1}ad = a^{-1},\ c^2 = 1,\ cd = dc,\ c^{-1}ac = a^{-1}\rangle$
$= \langle a, d\rangle \leftthreetimes \langle c\rangle = Q_3 \leftthreetimes C_2 \ \ (d = cba^6)$,
• $\mathscr{G}_{27} = \langle a, b, c \mid a^8 = b^2 = c^2 = 1,\ ab = ba,\ bc = cb,\ c^{-1}ac = a^{-1}b\rangle = (\langle a\rangle \times \langle b\rangle) \leftthreetimes \langle c\rangle = (C_8 \times C_2) \leftthreetimes C_2$,
• $\mathscr{G}_{28} = \langle a, b, c \mid a^8 = b^2 = 1,\ ab = ba,\ c^2 = a^4,\ bc = cb,\ c^{-1}ac = a^{-1}b\rangle$,
• $\mathscr{G}_{29} = \langle a, b \mid a^8 = b^4 = 1,\ b^{-1}ab = a^{-1}\rangle = \langle a\rangle \leftthreetimes \langle b\rangle = C_8 \leftthreetimes C_4$,
• $\mathscr{G}_{30} = \langle a, b, c \mid a^8 = b^2 = 1,\ ab = ba,\ c^2 = b,\ c^{-1}ac = a^3\rangle = \langle a, c \mid a^8 = c^4 = 1,\ c^{-1}ac = a^3\rangle$
$= \langle a\rangle \leftthreetimes \langle c\rangle = C_8 \leftthreetimes C_4$,
• $\mathscr{G}_{32} = \langle a, b, c \mid a^8 = b^2 = 1,\ ab = ba,\ c^2 = a^{-2}b,\ c^{-1}bc = ba^4, c^{-1}ac = a^{-1}\rangle$
$= \langle a, c \mid a^8 = 1,\ a^4 = c^4,\ c^{-1}ac = a^{-1}\rangle$.

Since each finite Abelian group is determined by its endomorphism semigroup in the class of all groups ([7], Theorem 4.2), so are the groups $\mathscr{G}_4, \mathscr{G}_5$, and $\mathscr{G}_6$. In [11], it was proved that the semidirect product $G = C_{p^n} \leftthreetimes C_m$, where p is a prime number, n and m are some positive integers, is determined by its endomorphism semigroup in the class of all groups. Hence the groups $\mathscr{G}_{19}, \mathscr{G}_{21}, \mathscr{G}_{22}, \mathscr{G}_{29}$, and $\mathscr{G}_{30}$ are determined by their endomorphism semigroups in the class of all groups. It was proved in [4], Theorems 14.2, 14.4, and 14.6, that the groups $\mathscr{G}_{17}, \mathscr{G}_{20}$, and $\mathscr{G}_{27}$ are determined by their endomorphism semigroups in the class of all groups. The group $\mathscr{G}_{26}$ is also determined by its endomorphism semigroup ([13], Theorem 13.3). To prove Theorem 1.1, we have to prove in addition that the groups $\mathscr{G}_{28}$ and $\mathscr{G}_{32}$ are determined by their endomorphism semigroups in the class of all groups. It is done in Theorems 4.2 and 5.2.

3. PRELIMINARY LEMMAS

For convenience of reference, let us recall some known facts that will be used in the proofs of our main results.

Lemma 3.1. *If $x \in I(G)$, then $G = \mathrm{Ker}x \leftthreetimes \mathrm{Im}x$ and $\mathrm{Im}x = \{g \in G \mid gx = g\}$.*

Lemma 3.2. *If $x \in I(G)$, then*

$$K(x) = \{\, y \in \mathrm{End}(G) \mid (\mathrm{Im}x)y \subset \mathrm{Im}x,\ (\mathrm{Ker}x)y = \langle 1\rangle \,\}$$

and $K(x)$ is a subsemigroup with the unity x of $\mathrm{End}(G)$ which is canonically isomorphic to $\mathrm{End}(\mathrm{Im}x)$. In this isomorphism element y of $K(x)$ corresponds to its restriction on the subgroup $\mathrm{Im}x$ of G.

Lemma 3.3. *If $x \in I(G)$, then*

$$J(x) = \{z \in \mathrm{End}(G) \mid (\mathrm{Im}x)z = \langle 1\rangle,\ (\mathrm{Ker}x)z \subset \mathrm{Ker}x\}.$$

Lemma 3.4. *If $x \in I(G)$, then*

$$H(x) = \{\, y \in \mathrm{End}(G) \mid (\mathrm{Im}x)y \subset \mathrm{Ker}x,\ (\mathrm{Ker}x)y = \langle 1\rangle \,\}.$$

We omit the proofs of these lemmas, because these are straightforward corollaries from the definitions.

Lemma 3.5. ([12], Theorem). *Each 2-group G such that $|G| < 32$ is determined by its endomorphism semigroup in the class of all groups.*

Lemma 3.6. *If $\mathscr{G}$ is a finite non-Abelian 2-group, then $|\mathrm{End}(\mathscr{G})| \geq 20$.*

Proof. Assume that $\mathscr{G}$ is a finite non-Abelian 2-group. Then the factor-group $\mathscr{G}/Z(\mathscr{G})$ is non-cyclic, i.e., $|\mathscr{G}/Z(\mathscr{G})| \geq 4$ and $\mathscr{G}$ has at least 4 inner automorphisms. Therefore, $|\mathrm{Aut}(\mathscr{G})| \geq 4$. By [15], Theorem 5.3.1, there exists $N \lhd \mathscr{G}$ such that $\mathscr{G}/N$ is Abelian and non-cyclic. This implies that there exists $M \lhd \mathscr{G}$ such that $N \subset M$ and $\mathscr{G}/M \cong C_2 \times C_2$. Choose $a_1, a_2 \in \mathscr{G}$ such that

$$\mathscr{G}/M = \langle a_1 M\rangle \times \langle a_2 M\rangle.$$

If $\mathscr{G}$ has only one element of order two, then $\mathscr{G}$ is isomorphic to a generalized quaternion group ([15], Theorem 5.3.6) and $|\mathrm{End}(\mathscr{G})| \geq 28$ ([9], Lemmas 2 and 3), i.e., the statement of the lemma is true. Assume that $\mathscr{G}$ has at least two elements of order two, for example, b and d. We can assume that $bd = db$. Then there exist 16 proper endomorphisms $z_{ijkl} = \pi y_{ijkl}$ of G, where $\pi : \mathscr{G} \longrightarrow \mathscr{G}/M$ is the canonical homomorphism and

$$\mathscr{G} \xrightarrow{\pi} \mathscr{G}/M \xrightarrow{y_{ijkl}} G,\ (a_1M)y_{ijkl} = b^i d^j,\ \ (a_2M)y_{ijkl} = b^k d^l$$

($i, j, k, l \in \mathbb{Z}_2$). Since $|\mathrm{Aut}(\mathscr{G})| \geq 4$, we have $|\mathrm{End}(\mathscr{G})| \geq 4+16 = 20$. The lemma is proved.

4. GROUP $\mathscr{G}_{28}$

In this section, we shall characterize the group

$$\mathscr{G}_{28} = \langle a, b, c \mid a^8 = b^2 = 1,\ ab = ba,\ c^2 = a^4,\ bc = cb,\ c^{-1}ac = a^{-1}b\rangle$$

by its endomorphism semigroup. Clearly, $\mathscr{G}_{28} = \langle ba^6, c, ac^{-1}\rangle$. Denote next the elements ba^6, c and ac^{-1} by a, b and c, respectively. Then $\mathscr{G}_{28}$ is given as follows:

$$\mathscr{G}_{28} = \langle a, b, c \mid c^4 = a^4 = 1,\ b^2 = a^2,\ b^{-1}ab = a^{-1},\ c^{-1}ac = a^{-1},\ c^{-1}bc = ba\rangle.$$

The group $\mathscr{G}_{28}$ is a group of order 32 and the numbers of its elements of orders 2, 4, and 8 are 3, 20, and 8, respectively [5]. Clearly,

$$\mathscr{G}_{28} = \langle a, b\rangle \rtimes \langle c\rangle = Q \rtimes \langle c\rangle \cong Q \rtimes C_4.$$

It is easy to check that

$$\mathscr{G}_{28}' = \langle a\rangle \cong C_4,\ Z(\mathscr{G}_{28}) = \langle a^2\rangle \times \langle c^2\rangle \cong C_2 \times C_2,$$
$$\mathscr{G}_{28}/\mathscr{G}_{28}' = \langle b\mathscr{G}_{28}'\rangle \times \langle c\mathscr{G}_{28}'\rangle \cong C_2 \times C_4.$$

Each element of $\mathscr{G}_{28}$ can be presented in the canonical form $c^i b^j a^k$, where $j \in \{0, 1\}$, $i, k \in \mathbb{Z}_4$.

Our aim is to prove the following theorem.

Theorem 4.1. *A finite group G is isomorphic to $\mathscr{G}_{28}$ if and only if $|\mathrm{Aut}(G)| = 2^6 = 64$ and there exists $x \in I(G)$ such that the following properties hold*:
1^0 $K(x) \cong \mathrm{End}(C_4)$;
2^0 $J(x) \cap I(G) = \{0\}$;
3^0 $|H(x)| = 8$;
4^0 $|\{y \in \mathrm{End}(G) \mid xy = y\}| = 24$;
5^0 $|\{y \in H(x) \mid \{z \in K(x) \mid z^2 = 0\} \cdot y = \{0\}| = 2$;
6^0 $|\{y \in \mathrm{End}(G) \mid xy = y,\ \{z \in K(x) \mid z^2 = 0\} \cdot y = \{0\}\}| = 4$.

Proof. Necessity. Let $G = \mathscr{G}_{28}$. Denote by x the projection of G onto its subgroup $\langle c\rangle$. Then $\mathrm{Im}\,x = \langle c\rangle$ and $\mathrm{Ker}\,x = \langle a, b\rangle$. By Lemma 3.2, $K(x) \cong \mathrm{End}(\mathrm{Im}\,x) \cong \mathrm{End}(\langle c\rangle) \cong \mathrm{End}(C_4)$, i.e., property 1^0 holds.

Note that each endomorphism of G is uniquely determined by its images on generators a, b, and c. By Lemma 3.3, $z \in J(x)$ has the form

$$cz = 1,\ az = b^i a^j,\ bz = b^k a^l;\ i, k \in \{0, 1\},\ j, l \in \mathbb{Z}_4. \tag{4.1}$$

The map $z : G \longrightarrow G$ given by (4.1) preserves the generating relations of G and induces an endomorphism of G if and only if $i = k = j = 0$ and $l \equiv 0 \pmod 2$. Hence

$$J(x) = \{z \mid az = cz = 1,\ bz = a^{2l_0};\ l_0 \in \mathbb{Z}_2\}$$

and $z^2 = 0$ for each $z \in J(x)$. Therefore, $J(x) \cap I(G) = \{0\}$ and property 2^0 holds.

By Lemma 3.4, $H(x)$ consists of endomorphisms $y : G \longrightarrow G$ such that

$$ay = by = 1,\ cy = b^i a^j \tag{4.2}$$

for some $i \in \{0, 1\}$, $j \in \mathbb{Z}_4$. The map y given by (4.2) preserves the generating relations of G and induces an endomorphism of G for each values of i and j. Hence

$$H(x) = \{y \mid ay = by = 1,\ cy = b^i a^j,\ \ i \in \{0, 1\}, j \in \mathbb{Z}_4\} \tag{4.3}$$

and $|H(x)| = 8$, i. e., property 3^0 holds.

Since $G = \mathrm{Ker}x \leftthreetimes \mathrm{Im}x$, we have

$$\{y \in \mathrm{End}(G) \mid xy = y\} = \{y \in \mathrm{End}(G) \mid (\mathrm{Ker}x)y = \langle 1\rangle\}. \tag{4.4}$$

Therefore, $|\{y \in \mathrm{End}(G) \mid xy = y\}|$ is equal to the number of homomorphisms $\mathrm{Im}x = \langle c\rangle \longrightarrow G$, i.e., to the number of elements $g \in G$ such that $g^4 = 1$. By [5], this number is 24. Hence property 4^0 is true.

By Lemma 3.2,

$$K(x) = \{z \mid az = bz = 1,\ cz = c^i,\ i \in \mathbb{Z}_4\}.$$

Hence

$$\{z \in K(x) \mid z^2 = 0\} = \{z \mid az = bz = 1,\ cz = c^{2i_0},\ i_0 \in \mathbb{Z}_2\}. \tag{4.5}$$

By (4.3) and (4.5),

$$\{y \in H(x) \mid \{z \in K(x) \mid z^2 = 0\} \cdot y = \{0\}\} = \{y \mid ay = by = 1,\ cy = b^i a^j,\ (cy)^2 = 1,\ \ i \in \{0, 1\}, j \in \mathbb{Z}_4\},$$

$$|\{y \in H(x) \mid \{z \in K(x) \mid z^2 = 0\} \cdot y = \{0\}\}| = |\{b^i a^j \mid (b^i a^j)^2 = 1,\ i \in \{0, 1\}, j \in \mathbb{Z}_4\}| = |\{g \in Q \mid g^2 = 1\}| = 2,$$

i.e., property 5^0 is true.

By (4.4) and (4.5),

$$\{y \in \mathrm{End}(G) \mid xy = y,\ \{z \in K(x) \mid z^2 = 0\} \cdot y = \{0\}\}$$
$$= \{y \mid ay = by = 1, cy = g,\ g \in G,\ g^2 = 1\} = |\{g \in G \mid g^2 = 1\}|.$$

By [5], the last number is 4. Therefore, property 6^0 holds. The necessity is proved.

Sufficiency. Let G be a finite group such that $|\mathrm{Aut}(G)| = 2^6$ and there exists $x \in I(G)$ which satisfies properties 1^0–6^0 of the theorem. Our aim is to prove that $G \cong \mathscr{G}_{28}$.

By Lemma 3.1, we have $G = \mathrm{Ker}x \leftthreetimes \mathrm{Im}x$. Property 1^0 and Lemma 3.2 imply that $\mathrm{End}(\mathrm{Im}x) \cong \mathrm{End}(C_4)$. We have

$$\mathrm{Im}x = \langle c\rangle \cong C_4,\ G = \mathrm{Ker}x \leftthreetimes \langle c\rangle,\ c \in G,$$

because each finite Abelian group is determined by its endomorphism semigroup in the class of all groups ([7], Theorem 4.2).

Since $\mathrm{Aut}(G)$ is a 2-group, we have $\widehat{g} = 1$ for each $2'$-element $g \in G$. Therefore, each $2'$-element of G belongs into the centre of G. Hence G splits into the direct product $G = G_2 \times G_{2'}$ of its Sylow 2-subgroup G_2 and Hall $2'$-subgroup $G_{2'}$. Each $2'$-element of G belongs into $\mathrm{Ker}x$, i.e., $G_{2'} \subset \mathrm{Ker}x$. Denote by z the projection of G onto its subgroup $G_{2'}$. Then $z \in J(x) \cap I(G)$. By property 2^0, $z = 0$, i.e., $G_{2'} = \langle 1 \rangle$ and G is a 2-group. Clearly, $\mathrm{Ker}\,x \neq \langle 1 \rangle$.

Choose an element $d \in \mathrm{Ker}\,x$ such that $d^2 = 1$ and define an endomorphism $y = \pi\tau$ of G, where π is the projection of G onto its subgroup $\mathrm{Im}\,x = \langle c \rangle$ and

$$G \xrightarrow{\pi} \mathrm{Im}\,x = \langle c \rangle \xrightarrow{\tau} \mathrm{Ker}\,x, \ c\tau = d.$$

Then $xy = y$, $yx = 0$, i.e., $y \in H(x)$. By Lemma 3.2, $K(x)$ consists of maps z, where $(\mathrm{Ker}\,x)z = \langle 1 \rangle$, $cz = c^i$, $i \in \mathbb{Z}_4$. This z satisfies $z^2 = 0$ if and only if $i = 2i_0$, $i_0 \in \mathbb{Z}_2$, and for such z the equality $zy = 0$ is true. Therefore, by property 5^0, the subgroup $\mathrm{Ker}\,x$ of G has only one element of order two. By [15], Theorem 5.3.6, $\mathrm{Ker}\,x$ is a generalized quaternion group or cyclic.

Assume that $\mathrm{Ker}\,x$ is cyclic: $\mathrm{Ker}\,x = \langle a \rangle \cong C_{2^n}$. Then $n \geq 2$, because otherwise $G = \langle a \rangle \times \langle c \rangle$ and the projection of G onto $\langle a \rangle$ belongs into $J(x) \cap I(G)$ and is non-zero. This contradicts property 2^0. By Lemma 3.4,

$$H(x) = \{y \mid ay = 1, \ cy = a^{i_0 2^{n-2}}, \ i_0 \in \mathbb{Z}_4\}$$

and hence $|H(x)| = 4$. This contradicts property 3^0. The obtained contradiction implies that $\mathrm{Ker}\,x$ cannot be cyclic. Therefore, $\mathrm{Ker}\,x$ is a generalized quaternion group Q_n, $n \geq 2$:

$$\mathrm{Ker}\,x = Q_n = \langle a, b \mid a^{2^n} = 1, \ b^2 = a^{2^{n-1}}, \ b^{-1}ab = a^{-1} \rangle.$$

By Lemma 3.4,

$$H(x) = \{y \mid ay = 1, \ cy = g, \ g \in \mathrm{Ker}\,x = Q_n, \ g^4 = 1\}.$$

Since

$$\{g \in Q_n \mid g^4 = 1\} = \{ba^i, \ a^{j2^{n-2}} \mid i \in \mathbb{Z}_{2^n}, \ j \in \mathbb{Z}_4\},$$

we have $|H(x)| = 2^n + 4$. Property 3^0 implies that $2^n + 4 = 8$, i.e., $n = 2$ and $\mathrm{Ker}\,x$ is the quaternion group $Q = Q_2$. It follows also that $|G| = |\mathrm{Ker}\,x| \cdot |\mathrm{Im}\,x| = 8 \cdot 4 = 32$, i.e., G is a non-Abelian group of order 32.

Let us find the numbers of elements $g \in G$ such that $g^4 = 1$ or $g^2 = 1$. Each homomorphism $y_0 : \mathrm{Im}\,x = \langle c \rangle \longrightarrow G$ can be uniquely extended to an endomorphism y of G such that $xy = y$ by setting $(\mathrm{Ker}\,x)y = \langle 1 \rangle$. Denote this y by $\widetilde{y_0}$. Conversely, each $y \in \mathrm{End}(G)$ such that $xy = y$ is obtained in this way. Therefore,

$$\begin{gathered}\{y \in \mathrm{End}(G) \mid xy = y\} = \{\widetilde{y_0} \mid y_0 \in \mathrm{Hom}(\langle c \rangle, G)\}, \\ |\{y \in \mathrm{End}(G) \mid xy = y\}| = |\mathrm{Hom}(\mathrm{Im}\,x, G)||\mathrm{Hom}(\langle c \rangle, G)| = |\{g \in G \mid g^4 = 1\}|,\end{gathered} \tag{4.6}$$

i.e., by property 4^0, the number of elements $g \in G$ such that $g^4 = 1$ is 24.

Denote

$$\mathscr{E} = \{y \in \mathrm{End}(G) \mid xy = y, \ \{z \in K(x) \mid z^2 = 0\} \cdot y = \{0\}\}.$$

By property 6^0, $|\mathscr{E}| = 4$. In view of (4.6),

$$\mathscr{E} = \{\widetilde{y_0} \mid y_0 \in \mathrm{Hom}(\langle c \rangle, G), \ \{z \in K(x) \mid z^2 = 0\} \cdot \widetilde{y_0} = \{0\}\}.$$

By Lemma 3.2, $K(x)$ consists of maps $z : G \longrightarrow G$, where $(\mathrm{Ker}\,x)z = \langle 1 \rangle$ and $cz = c^i$, $i \in \mathbb{Z}_4$. For those z, $z^2 = 0$ if and only if $i = 2i_0$, $i_0 \in \mathbb{Z}_2$. In this case, $c(zy) = (cy)^{2i_0} = (cy_0)^{2i_0}$ for each $y = \widetilde{y_0}$. Therefore, $\widetilde{y_0} \in \mathscr{E}$ if and only if $(cy_0)^2 = 1$. Hence

$$|\mathscr{E}| = |\{y_0 \in \mathrm{Hom}(\langle c \rangle, G) \mid (cy_0)^2 = 1\}| = |\{g \in G \mid g^2 = 1\}| = 4.$$

It follows that the group G has 3 elements of order two. Since the number of elements g such that $g^4 = 1$ is 24, the group G has 20 elements of order 4. By [5], there exists only one non-Abelian group of order 32 such that $|\mathrm{Aut}(G| = 2^6$ and which has 3 elements of order two and 20 elements of order 4. This group is $\mathscr{G}_{28}$. Therefore, $G \cong \mathscr{G}_{28}$. The sufficiency is proved.

The theorem is proved.

Theorem 4.2. *The group $\mathscr{G}_{28}$ is determined by its endomorphism semigroup in the class of all groups.*

Proof. Let G^* be a group such that the endomorphism semigroups of G^* and $\mathscr{G}_{28}$ are isomorphic:

$$\mathrm{End}(G^*) \cong \mathrm{End}(\mathscr{G}_{28}). \tag{4.7}$$

Denote by z^* the image of $z \in \mathrm{End}(\mathscr{G}_{28})$ in isomorphism (4.7). Since $\mathrm{End}(G^*)$ is finite, so is G^* ([1], Theorem 2). By Theorem 4.1, $|\mathrm{Aut}(\mathscr{G}_{28})| = 2^6 = 64$ and there exists $x \in I(\mathscr{G}_{28})$, satisfying properties 1^0–6^0 of Theorem 4.1. These properties are formulated so that they preserve in isomorphism (4.7). Therefore, $|\mathrm{Aut}(G^*)| = 2^6 = 64$ and the idempotent x^* of $\mathrm{End}(G^*)$ satisfies properties similar to properties 1^0–6^0 (it is necessary to change everywhere $z \in \mathrm{End}(\mathscr{G}_{28})$ by $z^* \in \mathrm{End}(G^*)$). Using now Theorem 4.1 for G^*, it follows that G^* and $\mathscr{G}_{28}$ are isomorphic. The theorem is proved.

5. GROUP $\mathscr{G}_{32}$

In this section, we shall characterize the group

$$\mathscr{G}_{32} = \langle a, c \mid a^8 = 1,\ a^4 = c^4,\ c^{-1}ac = a^{-1} \rangle$$

by its endomorphism semigroup. The group $\mathscr{G}_{32}$ is a group of order 32 and the numbers of its elements of orders 2, 4, and 8 are 3, 4, and 24, respectively [5]. It is easy to check that

$$\mathscr{G}_{32}' = \langle a^2 \rangle \cong C_4,\ Z(\mathscr{G}_{32}) = \langle c^2 \rangle \cong C_4,$$

$$\mathscr{G}_{32}/\mathscr{G}_{32}' = \langle c\mathscr{G}_{32}' \rangle \times \langle a\mathscr{G}_{32}' \rangle \cong C_4 \times C_2.$$

Each element of $\mathscr{G}_{32}$ can be presented in the canonical form $c^i a^j$, where $i \in \{0, 1, 2, 3\}$ and $j \in \mathbb{Z}_8$.

Our aim is to prove the following theorem.

Theorem 5.1. *A finite group G of order ≥ 32 is isomorphic to $\mathscr{G}_{32}$ if and only if it satisfies the following properties*:

1^0 $|\mathrm{Aut}(G)| = 64$;

2^0 $|\mathrm{End}(G) \setminus \mathrm{Aut}(G)| = 32$;

3^0 *the only idempotents of* $\mathrm{End}(G)$ *are* 0 *and* 1;

4^0 $|\{z \in \mathrm{End}(G) \setminus \mathrm{Aut}(G) \mid z^2 = 0\}| = 20$;

5^0 *there exist* $z, w \in \mathrm{End}(G) \setminus \mathrm{Aut}(G)$ *such that* $z^2 = w^2 = 0$ *and* $wz \neq 0$;

6^0 $|\{y \in \mathrm{Aut}(G) \mid x \in \mathrm{End}(G) \setminus \mathrm{Aut}(G) \Longrightarrow yx = x\}| = 16$.

Proof. Necessity. Let $G = \mathscr{G}_{32}$. To prove properties 1^0–6^0 for G, we have to find the endomorphisms of G. An endomorphism of G is fully determined by its action on the generators c and a. Choose $z \in \mathrm{End}(G)$. Then

$$az = c^i a^j,\ cz = c^k a^l \tag{5.1}$$

for some i, j, k, l. The map z given by (5.1) induces an endomorphism of G if and only if it preserves the defining relations of G. After easy calculations, we obtain that the map z given by (5.1) is an endomorphism of G only in the following three cases:

$$cz = c^{2k_0}a^{2l_0},\ az = a^{4j_0};\ \ j_0, k_0 \in \{0, 1\},\ l_0 \in \mathbb{Z}_4, \tag{5.2}$$

$$cz = c^{2k_0}a^{2l_0},\ az = c^2a^{2+4j_0};\ \ j_0, k_0 \in \{0, 1\},\ l_0 \in \mathbb{Z}_4, \tag{5.3}$$

$$cz = c^{2k_0+1}a^l,\ az = a^j;\ \ k_0 \in \{0, 1\};\ j, l \in \mathbb{Z}_8,\ j \equiv 1 \pmod 2. \tag{5.4}$$

The endomorphisms given by (5.2) and (5.3) are proper endomorphisms and the endomorphisms given by (5.4) are automorphisms. The numbers of endomorphisms of these three types are 16, 16, and 64, respectively. Hence G satisfies properties 1^0 and 2^0.

Immediate calculations show that 0 and 1 are only idempotents of $\mathrm{End}(G)$. Similarly, immediate calculations show that $z^2 = 0$ only for the following proper endomorphisms z:

$$cz = a^{2l_0},\ az = a^{4j_0},\ l_0 \in \mathbb{Z}_4,\ j_0 \in \mathbb{Z}_2,$$

$$cz = c^2a^2,\ az = a^{4j_0},\ j_0 \in \mathbb{Z}_2,$$
$$cz = c^2a^6,\ az = a^{4j_0},\ j_0 \in \mathbb{Z}_2,$$
$$cz = 1,\ az = c^2a^{2+4j_0},\ j_0 \in \mathbb{Z}_2,$$
$$cz = a^4,\ az = c^2a^{2+4j_0},\ j_0 \in \mathbb{Z}_2,$$
$$cz = c^2a^2,\ az = c^2a^{2+4j_0},\ j_0 \in \mathbb{Z}_2,$$
$$cz = c^2a^6,\ az = c^2a^{2+4j_0},\ j_0 \in \mathbb{Z}_2.$$

The number of such endomorphisms is $8+6\cdot 2 = 20$. Hence G satisfies properties 3^0 and 4^0.

Property 5^0 is satisfied for the proper endomorphisms z and w, where

$$cz = a^2,\ az = 1,\ cw = 1,\ aw = c^2a^2.$$

Let us prove property 6^0. Choose an arbitrary $y \in \mathrm{Aut}(G)$:

$$cy = c^{2k+1}a^l,\ ay = a^j;\ k \in \{0, 1\};\ j, l \in \mathbb{Z}_8,\ j \equiv 1 \,(\mathrm{mod}\, 2).$$

We have to find k, l, j so that $yx = x$ for each $x \in \mathrm{End}(G) \setminus \mathrm{Aut}(G)$. If x is given by (5.2), then

$$a(yx) = a^jx = a^{4jj_0} = a^{4j_0} = ax,$$

$$c(yx) = (c^{2k+1}a^l)x = (c^{2k_0}a^{2l_0})^{2k+1}a^{4j_0l} = c^{2k_0(2k+1)}a^{2l_0(2k+1)}a^{4j_0l} = c^{2k_0}a^{4k_0k+4l_0k+2l_0+4j_0l},$$
$$cx = c^{2k_0}a^{2l_0},$$

and $yx = x$ if and only if
$$4k_0k + 4l_0k + 4j_0l \equiv 0 \,(\mathrm{mod}\, 8).$$

The last congruence is satisfied for each j_0, k_0, l_0 if and only if

$$k = 0,\ l \equiv 0 \,(\mathrm{mod}\, 2). \tag{5.5}$$

Assume that x is given by (5.3). In view of (5.5) and $c^{2j}a^{2j} = c^2a^2$ for odd j, we have

$$a(yx) = a^jx = (c^2a^{2+4j_0})^j = c^{2j}a^{2j+4j_0} = c^2a^{2+4j_0} = ax,$$

$$c(yx) = (ca^l)x = c^{2k_0}a^{2l_0}(c^2a^{2+4j_0})^l = c^{2k_0}\cdot c^{2l}\cdot a^{2l_0+l(2+4j_0)} = c^{2k_0}\cdot a^{2l}\cdot a^{2l_0+2l} = c^{2k_0}a^{4l+2l_0} = c^{2k_0}a^{2l_0} = cx.$$

Therefore, if x is given by (5.3), then always $yx = x$. It follows that $yx = x$ for each $x \in \mathrm{End}(G) \setminus \mathrm{Aut}(G)$ if and only if k and l satisfy (5.5). Since $j \equiv 1 \,(\mathrm{mod}\, 2)$, the number of such automorphisms y is $4\cdot 4 = 16$. Property 6^0 is proved.

The necessity is proved.

Sufficiency. Let G be a finite group of order ≥ 32 which satisfies properties 1^0–6^0. Our aim is to prove that $G \cong \mathscr{G}_{32}$. We shall now introduce a series of lemmas which give the proof.

Lemma 5.1. *The group G is a non-Abelian 2-group and it is not a generalized quaternion group. The group has at least two elements of order two.*

Proof. By property 1^0, $\widehat{g} = 1$ for each $2'$-element g of G. Hence all $2'$-elements of G belong into its centre $Z(G)$. Therefore, the group G splits into the direct product $G = G_{2'} \times G_2$ of its Hall $2'$-subgroup $G_{2'}$ and Sylow 2-subgroup G_2. Denote by z the projection of G onto its subgroup $G_{2'}$. By property 3^0, $z = 0$ or $z = 1$. Assume that $z = 1$. Then $G = G_{2'}$ is Abelian and, again by property 3^0, G is cyclic, i.e., $G = C_n$ for an odd integer n. By properties 1^0 and 2^0, we have $|\mathrm{End}(G)| = n = 64 + 32 = 96$. This contradicts the fact that n is odd. Hence $z = 0$ and G is a 2-group. The group G is non-Abelian, because otherwise, by property 3^0, G would be cyclic and $|G| = |\mathrm{End}(G)| = 2^m = 96$ for an integer m, which is impossible. Since a generalized quaternion group has only four proper endomorphisms ([8], Lemma 2), the group G cannot be a generalized quaternion group. The last statement of the lemma follows from [15], Theorem 5.3.6. The lemma is proved.

The factor-group G/G' splits into a direct product

$$G/G' = \langle a_1 G' \rangle \times \ldots \times \langle a_n G' \rangle;\ a_1, \ldots, a_n \in G \setminus G'.$$

Denote further by ε the canonical homomorphism $\varepsilon : G \longrightarrow G/G'$.

Lemma 5.2. $n = 2$.

Proof. Remark that G/G' is not cyclic ([15], Theorem 5.3.1). Hence $n \geq 2$. Assume that $n \geq 3$. By Lemma 5.1, G has at least two elements of order two, for example b and d. We can assume that $bd = db$, and, therefore, G has at least 64 proper endomorphisms z_{ijklst}:

$$z_{ijklst} = \varepsilon \pi_{ijklst} : G \xrightarrow{\varepsilon} G/G' \xrightarrow{\pi_{ijklst}} \langle b, d \rangle,$$

$$(a_1 G')\pi_{ijklst} = b^i d^j,\ (a_2 G')\pi_{ijklst} = b^k d^l,\ (a_3 G')\pi_{ijklst} = b^s d^t,$$

$$(a_u G')\pi_{ijklst} = 1,\ u \geq 3;\ \ i, j, k, l, s, t \in \mathbb{Z}_2.$$

This contradicts property 2^0. Therefore, $n < 3$ and $n = 2$. The lemma is proved.

By Lemma 5.2, we can fix $c, a \in G \setminus G'$ in the following way:

$$G/G' = \langle cG' \rangle \times \langle aG' \rangle \cong C_{2^s} \times C_{2^t} \tag{5.6}$$

and

$$o(cG') = 2^s,\ \ o(aG') = 2^t,\ \ s \geq t \geq 1.$$

By property 3^0, G does not split into a nontrivial semidirect product. Hence

$$c^{2^s} \neq 1,\ a^{2^t} \neq 1. \tag{5.7}$$

If $g, h \in G$ such that $gh = hg$ and $o(g) \leq 2^s$, $o(h) \leq 2^t$, then there exists a proper endomorphism $y(g; h) = \varepsilon \cdot \pi$ of G defined as follows:

$$G \xrightarrow{\varepsilon} G/G' = \langle cG' \rangle \times \langle aG' \rangle \xrightarrow{\pi} G,\ \ (cG')\pi = g,\ \ (aG')\pi = h.$$

In the further proofs, we shall use proper endomorphisms of this kind a number of times.

Next we shall prove that G' is cyclic and determine the values of s and t.

Lemma 5.3. *The derived group G' of G has an element of order four.*

Proof. On the contrary, assume that $g^2 = 1$ for each $g \in G^{'}$. Then $G^{'}$ is Abelian and splits into a direct product

$$G^{'} = \langle b_1 \rangle \times \ldots \times \langle b_n \rangle \cong C_2 \times \ldots \times C_2,\ n \geq 1,$$

and there exist proper endomorphisms $y(b_1^{i_1} \ldots b_n^{i_n};\, b_1^{j_1} \ldots b_n^{j_n})$, where $i_1, \ldots, i_n,\ j_1, \ldots,\ j_n \in \mathbb{Z}_2$. The number of such endomorphisms is $2^n \cdot 2^n = 2^{2n}$, and, by property 2^0, $2^{2n} \leq 32 = 2^5$, $2n \leq 5$, i.e. $n \leq 2$. Let us consider the cases $n = 1$ and $n = 2$ separately.

Assume that $n = 1$. By (5.7), we have

$$G^{'} = \langle c^{2^s} \rangle = \langle a^{2^t} \rangle \cong C_2,\ c^{2^s} = a^{2^t}, \tag{5.8}$$

$$[c, a] = c^{-1}a^{-1}ca \in Z(G),\ c^2 \cdot a = a \cdot c^2,\ \ a^2 \cdot c = c \cdot a^2.$$

Therefore, there exist proper endomorphisms $y(c^{2i_1}a^{2i_2};\, c^{2^{s-t+1}j_1}a^{2j_2})$. We get all elements of the form $c^{2i_1}a^{2i_2}$ if we take $i_1 \in \mathbb{Z}_{2^s}$ and $i_2 = 0, 1, 2, \ldots, 2^{t-1} - 1$. Similarly, we get all elements of the form $c^{2^{s-t+1}j_1}a^{2j_2}$ if we take $j_1 \in \mathbb{Z}_{2^t}$ and $j_2 = 0, 1, 2, \ldots, 2^{t-1} - 1$. It follows that the number of endomorphisms $y(c^{2i_1}a^{2i_2};\, c^{2^{s-t+1}j_1}a^{2j_2})$ is $2^s \cdot 2^{t-1} \cdot 2^t \cdot 2^{t-1} = 2^{s+3t-2}$. By property 2^0,

$$2^{s+3t-2} \leq 32 = 2^5,\ 4t \leq s + 3t \leq 7,\ t = 1.$$

In view of (5.6), (5.8), and $t = 1$, each element $g \in G$ of order two is c^{2^s} or has a form $g = ac^i$ for a suitable integer i. If $g = ac^i$, then $G = \langle c \rangle \leftthreetimes \langle g \rangle$ and the projection of G onto $\langle g \rangle$ is non-zero and non-identity idempotent of $\mathrm{End}(G)$. This contradicts property 3^0. Hence G has only one element of order two (it is c^{2^s}) and, therefore, is cyclic or a generalized quaternion group. This contradicts Lemma 5.1. It follows that $n \neq 1$.

Assume that $n = 2$. Then

$$G^{'} \cong C_2 \times C_2,\ \ |G| = 2^{s+t+2}.$$

The case $s = t = 1$ is impossible, because $|G| \geq 32$. Hence

$$s \geq t,\ \ s \geq 2.$$

Assume that $s > t$. Choose $g \in \langle c^2, a, G^{'} \rangle$. Then $g^{2^s} = 1$ and there exists the proper endomorphism $y(g; 1)$ of G. The number of such endomorphisms is $|G|/2 = 2^{s+t+1}$. Choose $g \in \langle c^{2^{s-t+1}}, a^2, G^{'} \rangle$. Then $g^{2^t} = 1$ and there exists the proper endomorphism $y(1; g)$ of G. The number of such endomorphisms is $2^{2(t-1)+2} = 2^{2t}$. Among endomorphisms $y(g; 1)$ and $y(1; g)$ only zero is a common endomorphism. Therefore, the number of such endomorphisms is $2^{s+t+1} + 2^{2t} - 1$, and, by property 2^0, $2^{s+t+1} + 2^{2t} - 1 \leq 32$. The only solution of this inequality under conditions $s > t$, $s \geq 2$ is

$$t = 1,\ s = 2. \tag{5.9}$$

Choose $g, d, b \in G$ such that

$$g \in \langle c^2, a, G^{'} \rangle,\ d \in Z(G),\ o(d) = 2,\ b \in G,\ o(b) = 2,\ b \neq d.$$

By (5.9), we have $g^4 = 1$ and there exist proper endomorphisms $y(g; d^i)$, $i \in \mathbb{Z}_2$, and $y(1; b)$ of G. The number of those endomorphisms is $2 \cdot |\langle c^2, a, G^{'} \rangle| + 1 = 33$. This contradicts property 2^0. Therefore, the inequality $s > t$ is false and

$$s = t \geq 2,\ \ |G| = 2^{2s+2}. \tag{5.10}$$

Choose $g_1, g_2, g_3, g_4 \in G$ such that

$$g_1, g_2 \in \langle c^2, a^2, G^{'} \rangle,\ o(g_3) = o(g_4) = 2,\ g_3 \neq g_4.$$

Then $g_1^{2^s} = g_2^{2^s} = 1$ and there exist proper endomorphisms $y(g_1; 1)$, $y(1; g_2)$, $y(g_3; g_3)$, and $y(g_4; g_4)$ of G. Among these endomorphisms only zero appears twice, and, therefore, their number is

$$2 \cdot |\langle c^2, a^2, G^{'} \rangle| - 1 + 2 = 2 \cdot 2^{2s} + 1.$$

By property 2^0,

$$2 \cdot 2^{2s} + 1 \leq 32,\ 2 \cdot 2^{2s} \leq 31,\ 2 \cdot 2^{2s} \leq 30,\ 2^{2s} \leq 15,$$
$$2^{2s} \leq 8 = 2^3,\ 2s \leq 3,\ s = 1.$$

This contradicts (5.10). The obtained contradiction implies that the derived group $G^{'}$ contains an element of order four. The lemma is proved.

Lemma 5.4. *The derived group $G^{'}$ has only one element of order two or $s = t = 1$.*

Proof. Assume that $s \geq 2$ and $G^{'}$ has at least two different elements b and d of order two. We can assume that $bd = db$. In view of Lemma 5.3, there exists an element $h \in G^{'}$ of order four. It is possible to choose so that $h^2 = b$. There exist proper endomorphisms $z_{ijkl} = y(b^i d^j; b^k d^l)$ and $w_{\alpha\beta} = y(h^\alpha; b^\beta) = y(h^\alpha; h^{2\beta})$ of G, where $i, j, k, l, \beta \in \mathbb{Z}_2$; $\alpha \in \{1, 3\}$. By the construction, $z_{ijkl}^2 = w_{\alpha\beta}^2 = 0$. The number of endomorphisms z_{ijkl} and $w_{\alpha\beta}$ is $2^4 + 2^2 = 20$. By property 4^0, each proper endomorphism z of G which satisfies $z^2 = 0$ has one of these two forms.

Let us consider the cases $dh \neq hd$ and $dh = hd$ separately. Assume that $dh = hd$. There exist proper endomorphisms $\tau_{\alpha\beta} = y(h^\alpha; d^\beta)$, where $\alpha \in \{1, 3\}$, $\beta \in \mathbb{Z}_2$. By the construction, $\tau_{\alpha\beta}^2 = 0$ and $\tau_{\alpha 1}$ differs from z_{ijkl} and $w_{\alpha\beta}$. This contradicts the previous part of the proof. Hence $dh \neq hd$. Since $(w_{10} \cdot \widehat{d})^2 = 0$, we have $w_{10} \cdot \widehat{d} = w_{\alpha 0}$ for some $\alpha \in \{1, 3\}$. Therefore,

$$c(w_{10} \cdot \widehat{d}\,) = h\widehat{d} = d^{-1}hd = cw_{\alpha\beta} = h^\alpha,$$

i.e., $d^{-1}hd = h^{-1}$. It follows from here that dh is an element of order two and there exists a proper endomorphism $z = y(1; dh)$. By the construction, $z^2 = 0$ and z differs from z_{ijkl} and $w_{\alpha\beta}$. This contradicts the first part of the proof. Therefore, the case $dh \neq hd$ is also impossible. Consequently, the assumption that $s \geq 2$ and $G^{'}$ has at least two different elements b and d of order two is false. This implies that $s = 1$ (i.e., $s = t = 1$, because $s \geq t$) or $G^{'}$ has only one element of order two. The lemma is proved.

Lemma 5.5. *The case $s = t = 1$ is impossible, i.e., $s \geq 2$.*

Proof. On the contrary, assume that $s = t = 1$. Then all elements of order two of G belong to $G^{'}$, because otherwise the group G would split into a non-trivial semidirect product, which contradicts property 3^0. By Lemma 5.1, $G^{'}$ has at least two different elements b and d of order two. We can assume that $b \in Z(G)$. We get 16 proper endomorphisms $z_{ijkl} = y(b^i d^j; b^k d^l)$ of G $(i, j, k, l \in \mathbb{Z}_2)$. By the construction, $z_{ijkl}^2 = 0$.

If G has an element h of order two such that $h \notin \langle b, d \rangle$, then $h \in G^{'}$ and we get six additional proper endomorphisms $y(b^i h; b^j h)$, $y(h; 1)$, and $y(1; h)$ of G such that $y(b^i h; b^j h)^2 = y(h; 1)^2 = y(1; h)^2 = 0$ $(i, j \in \mathbb{Z}_2)$. Hence we have already $16 + 6 = 22$ proper endomorphisms z of G such that $z^2 = 0$. This contradicts property 4^0. Therefore, all elements of order two of G belong to $\langle b, d \rangle = \langle b \rangle \times \langle d \rangle \subset G^{'}$.

Let z be a proper endomorphism of G such that $\mathrm{Im}\, z$ is Abelian. Then $G/\mathrm{Ker}\, z$ is Abelian, $G^{'} \subset \mathrm{Ker}\, z$ and $z = \varepsilon y$ for a homomorphism $y : G/G^{'} \longrightarrow \langle b \rangle \times \langle d \rangle$. Hence z is equal to z_{ijkl} for some i, j, k, l. By property 4^0, there exists a proper endomorphism w of G such that $w^2 = 0$ and $\mathrm{Im}\, w$ is non-Abelian. Fix w of this kind. Define $w_\alpha = w \cdot \alpha \in \mathrm{End}(G)$ for each $\alpha \in \mathrm{End}(\mathrm{Im}\, w)$. By the construction,

$$w_\alpha w_\beta = w_\alpha w \cdot \beta = 0 \cdot \beta = 0,\ w_\alpha^2 = 0 \tag{5.11}$$

$(\alpha, \beta \in \mathrm{End}(\mathrm{Im}\, w))$. Lemma 3.6 implies that the number of such endomorphisms w_α is at least 20. Therefore, by property 4^0, each proper endomorphism z of G for which $z^2 = 0$ can be presented in the form w_α for some α. In view of (5.11), $z_1 z_2 = 0$ for each z_1 and z_2 such that $z_1^2 = z_2^2 = 0$. This contradicts property 5^0. It follows that the case $s = t = 1$ is impossible and $s \geq 2$. The lemma is proved.

Lemma 5.6. *The derived group $G^{'}$ is cyclic.*

Proof. Lemmas 5.4 and 5.5 imply that $G^{'}$ has only one element of order two. By [15], Theorem 5.3.6, $G^{'}$ is cyclic or a generalized quaternion group. Assume that $G^{'}$ is a generalized quaternion group $Q_n,\ n \geq 2$:

$$G^{'} = Q_n = \langle a_0, b_0 \mid a_0^{2^n} = 1,\ a_0^{2^{n-1}} = b_0^2,\ b_0^{-1} a_0 b_0 = a_0^{-1} \rangle.$$

The elements of order four of Q_n are

$$a_0^{2^{n-2}},\ a_0^{-2^{n-2}},\ b_0 a_0^i,\ i \in \mathbb{Z}_{2^n}.$$

The number of them is 2^n+2. Choose an element $h \in G \setminus G^{'}$ of order two. By Lemma 5.1, it is possible. Then $h = c^{u2^{s-1}} a^{v2^{t-1}} d$ for some $u, v \in \mathbb{Z}_2$ and $d \in G^{'}$. If $t = 1$, then $v = 0$, because otherwise $h = c^{u2^{s-1}} ad$, $G = \langle c, G^{'} \rangle \leftthreetimes \langle h \rangle$ and the projection of G onto $\langle g \rangle$ would be non-zero and non-identity idempotent of $\mathrm{End}(G)$, which contradicts property 3^0. Let $g \in G^{'}$ be an element of order four. Since $a_0^{2^{n-1}} \in Z(G)$ and $s \geq 2$, there exist proper endomorphisms $z_{gj} = y(g; a_0^{j2^{n-1}})$ and $z_{ijkl} = y(h^i a_0^{j2^{n-1}}; h^k a_0^{l2^{n-1}})$ $(i, j, k, l \in \mathbb{Z}_2)$. Clearly, $z_{gj}^2 = 0$. The following calculations show that $z_{ijkl}^2 = 0$:

$$\begin{aligned} cz_{ijkl}^2 &= (h^i a_0^{j2^{n-1}}) z_{ijkl} = (hz_{ijkl})^i = ((c^{u2^{s-1}} a^{v2^{t-1}} d) z_{ijkl})^i \\ &= ((cz_{ijkl})^{u2^{s-1}} (az_{ijkl})^{v2^{t-1}})^i = ((h^i a_0^{j2^{n-1}})^{u2^{s-1}} (h^k a_0^{l2^{n-1}})^{v2^{t-1}})^i = 1. \end{aligned}$$

The number of endomorphisms z_{gj} and z_{ijkl} is $2(2^n+2)+2^4 = 2^{n+1}+20$, which contradicts property 4^0. Therefore, $G^{'}$ cannot be a generalized quaternion group. The lemma is proved.

It is now possible to find the values of s and t.

Lemma 5.7. $t = 1$.

Proof. Assume that $t \geq 2$. Choose $g \in G^{'}$ of order four and $h \in G \setminus G^{'}$ of order two. By Lemmas 5.1 and 5.3, it is possible. Then $h = c^{u2^{s-1}} a^{v2^{t-1}} d$ for some $u, v \in \mathbb{Z}_2$ and $d \in G^{'}$. Clearly, $g^2 \in Z(G)$. Therefore, there exist proper endomorphisms $z_{ij} = y(g^i; g^j)$, $w_{kl} = y(hg^{2k}; hg^{2l})$, and $z = y(h; 1)$ $(i, j \in \mathbb{Z}_4;\ k, l \in \mathbb{Z}_2)$. Clearly, $z_{ij}^2 = z^2 = 0$. Similarly to the proof of Lemma 5.6, one can prove that $w_{kl}^2 = 0$. The number of endomorphisms z_{ij}, w_{kl} and z is $16+4+1 = 21$. This contradicts property 4^0. Therefore, $t = 1$. The lemma is proved.

Lemma 5.8. *Let $b \in G^{'}$ be an element of order four and $h \in G \setminus G^{'}$ be an element of order two. Then $h = c^{2^{s-1}} d$ for some $d \in G^{'}$ and*

$$\{z \in \mathrm{End}(G) \setminus \mathrm{Aut}(G) \mid z^2 = 0\} = \{z_{ijkl}, w_{\alpha\beta} \mid \beta, i, j, k, l \in \mathbb{Z}_2;\ \alpha = \pm 1\},$$

where $z_{ijkl} = y(h^i b^{2j}; h^k b^{2l})$ and $w_{\alpha\beta} = y(b^\alpha; b^{2\beta})$.

Proof. Let $b \in G^{'}$ be an element of order four and $h \in G \setminus G^{'}$ be an element of order two. By Lemmas 5.1, 5.4, and 5.6, there exist those elements. Then $h = c^{u2^{s-1}} a^{v2^{t-1}} d = c^{u2^{s-1}} a^v d$ for some $u, v \in \mathbb{Z}_2$ and $d \in G^{'}$. If $v = 1$, then $G = \langle c, G^{'} \rangle \leftthreetimes \langle h \rangle$, which contradicts property 3^0. Therefore, $v = 0$, $u = 1$, and $h = c^{2^{s-1}} d$.

Since $b^2 \in Z(G)$, the maps z_{ijkl} and $w_{\alpha\beta}$ given in the lemma are the proper endomorphisms of G. Evidently, $w_{\alpha\beta}^2 = 0$. Since $G^{'} z_{ijkl} = \langle 1 \rangle$ and

$$cz_{ijkl}^2 = (h^i b^{2j}) z_{ijkl} = (hz_{ijkl})^i = ((c^{2^{s-1}} d) z_{ijkl})^i = (cz_{ijkl})^{i2^{s-1}} = (h^i b^{2j})^{i2^{s-1}} = 1,$$

$$az_{ijkl}^2 = (h^k b^{2l}) z_{ijkl} = (hz_{ijkl})^k = ((c^{2^{s-1}} d) z_{ijkl})^k = (cz_{ijkl})^{k2^{s-1}} = (h^i b^{2j})^{k2^{s-1}} = 1,$$

we have $z_{ijkl}^2 = 0$. The number of endomorphisms z_{ijkl} and $w_{\alpha\beta}$ is $2^4 + 2 \cdot 2 = 20$. Property 4^0 implies the second statement of the lemma. The lemma is proved.

Lemma 5.9. *The group $H = \langle a, G^{'} \rangle$ is cyclic.*

Proof. By Lemmas 5.6 and 5.8, H has only one element of order two. Therefore, H is cyclic or a generalized quaternion group

$$Q_n = \langle a_0, b_0 \mid a_0^{2^n} = 1,\ b_0^2 = a_0^{2^{n-1}},\ b_0^{-1} a_0 b_0 = a_0^{-1} \rangle,\ \ n \geq 2,$$

([15], Theorem 5.3.6). Assume that $H = Q_n$. The elements $b_0 a_0^{\gamma}$ ($\gamma \in \mathbb{Z}_{2^n}$) are the elements of order four and there exist proper endomorphisms $x_\gamma = y(b_0 a_0^{\gamma}; 1)$ of G. By the construction, $x_\gamma^2 = 0$. Since the number of such endomorphisms is $2^n \geq 4$, some of them are different from the endomorphisms given in Lemma 5.8. This contradiction implies that $H \neq Q_n$. Hence H is cyclic. The lemma is proved.

Lemma 5.10. $s = 2$.

Proof. On the contrary, assume that $s \geq 3$. If the order of c^{2^s} is two, then the order of $c^{2^{s-1}}$ is four and there exist a proper endomorphism $z = y(c^{2^{s-1}}; 1)$. We have $z^2 = 0$:

$$cz^2 = c^{2^{s-1}} z = (c^{2^{s-1}})^{2^{s-1}} = c^{2^{s+s-2}} = 1,\ az^2 = 1.$$

The endomorphism z differs from the endomorphisms which were given in Lemma 5.8. Therefore, $o(c^{2^s}) \neq 2$ and there exists an integer k such that the order of $c^{2^s k}$ is four. By Lemma 5.8, there exist an element $h = c^{2^{s-1}} d$ of order two, where $d \in G^{'}$. Since $c^{2^s k} \in G^{'}$, the elements $c^{2^s k}$ and h commute. Hence there exists the proper endomorphism $z = y(c^{2^s k}; h)$. In this case also $z^2 = 0$:

$$cz^2 = c^{2^s k} z = (c^{2^s k})^{2^s k} = 1,$$

$$az^2 = hz = (c^{2^{s-1}} d) z = (cz)^{2^{s-1}} = (c^{2^s k})^{2^{s-1}} = 1.$$

The endomorphism z differs from the endomorphisms which were given in Lemma 5.8. Therefore, the case $s \geq 3$ is impossible. The lemma is proved.

By Lemma 5.9, the group $\langle a, G^{'} \rangle$ is cyclic. Therefore, we can assume that $\langle a, G^{'} \rangle = \langle a \rangle \cong C_{2^m}$ for some m. By (5.6) and Lemmas 5.3, 5.7, and 5.10, $m \geq 3$ and $c^4 = a^{2u}$ for some $u \in \mathbb{Z}_{2^{m-1}}$. Let us present u in the form

$$u = u_0 2^n,\ u_0 \equiv 1 \,(\mathrm{mod}\, 2),\ 0 \leq n \leq m-2$$

(by (5.7), $n \neq m-1$). We can assume that $u_0 = 1$, because we can replace a with suitable a^i, where $i \equiv 1 \,(\mathrm{mod}\, 2)$. By (5.6), $c^{-1} a c = a^r$ for some $r \in \mathbb{Z}_{2^m}$, $r \equiv 1 \,(\mathrm{mod}\, 2)$. Since $c^{-i} a c^i = a^{r^i}$ and $c^4 = a^{2u}$, we have $r^4 \equiv 1 \,(\mathrm{mod}\, 2^m)$.

Let us summarize the obtained results on the group G:

$$G = \langle c, a \mid a^{2^m} = 1,\ c^4 = a^{2^{n+1}},\ c^{-1} a c = a^r \rangle, \tag{5.12}$$

where

$$r^4 \equiv 1 \,(\mathrm{mod}\, 2^m),\ m \geq 3,\ 0 \leq n \leq m-2.$$

Since $r = 1 + 2v$ for an integer v, we have

$$r^2 = 1 + 4v + 4v^2,\ r^2 \equiv 1 \,(\mathrm{mod}\, 4), \tag{5.13}$$

$$1 + r^2 = 2(1 + 2v + 2v^2) = 2(1 + 2v_0).$$

Our next aim is to find all elements $g \in G$ such that $g^4 = 1$ and to prove that these elements commute with each other. Using the obtained results, we find the proper endomorphisms of G and certain automorphisms of G. It allows us to prove that $m = 3$ and $n = 1$. Finally, this implies the order structure of elements of G and the isomorphism $G \cong \mathscr{G}_{32}$.

Lemma 5.11. *If $g \in G$ and $g^4 = 1$, then*

$$g = a^{s2^{m-2}} \quad or \quad g = c^2 a^{-2^n(1+2v_0)^{-1}+s2^{m-2}},$$

where $s \in \mathbb{Z}_4$.

Proof. Clearly, if $g = a^i$, $i \in \mathbb{Z}_{2^m}$, then $g^4 = 1$ if and only if $g = a^{s2^{m-2}}$ for some $s \in \mathbb{Z}_4$. Assume $g \notin \langle a \rangle$ and $g^4 = 1$. Then $g = c^k a^i$ for some $k \in \{1, 2, 3\}$ and $i \in \mathbb{Z}_{2^m}$. If $k = 1$ or $k = 3$, then $G = \langle a \rangle \leftthreetimes \langle g \rangle$ and the projection of G onto $\langle g \rangle$ contradicts property 3^0. Hence $g = c^2 a^i$ for some $i \in \mathbb{Z}_{2^m}$ and

$$\begin{gathered} g^2 = c^2 a^i \cdot c^2 a^i = c^4 \cdot c^{-2} a^i c^2 \cdot a^i = a^{2^{n+1}} \cdot a^{ir^2} \cdot a^i = a^{2^{n+1}+i(1+r^2)}, \\ g^4 = (c^2 a^i)^4 = a^{2^{n+2}+2i(1+r^2)}. \end{gathered} \tag{5.14}$$

Therefore, $g^4 = 1$ if and only if

$$2^{n+2} + 2i(1+r^2) \equiv 0 \, (\mathrm{mod}\, 2^m),$$

$$\begin{gathered} 2^{n+2} + 4i(1+2v_0) \equiv 0 \, (\mathrm{mod}\, 2^m), \\ i \equiv -2^n(1+2v_0)^{-1} \, (\mathrm{mod}\, 2^{m-2}), \\ i = -2^n(1+2v_0)^{-1} + s2^{m-2} \end{gathered}$$

for some $s \in \mathbb{Z}_4$. The lemma is proved.

Similarly, using (5.14), we get

Lemma 5.12. *If $g \in G$ and $g^2 = 1$, then*

$$g = a^{w2^{m-1}} \quad or \quad g = c^2 a^{-2^n(1+2v_0)^{-1}+w2^{m-1}},$$

where $w \in \mathbb{Z}_2$.

Lemma 5.13. *If $g, h \in G$ and $g^4 = h^4 = 1$, then $gh = hg$.*

Proof. By (5.12) and (5.13),

$$c^{-2} a^{2^{m-2}} c^2 = a^{r^2 2^{m-2}} = a^{2^{m-2}}, \; c^2 \cdot a^{2^{m-2}} = a^{2^{m-2}} \cdot c^2.$$

By Lemma 5.11, if $g = a^{s2^{m-2}}$ $(s \in \mathbb{Z}_4)$, then $gh = hg$ for each $h \in G$ such that $h^4 = 1$. Assume that

$$g = c^2 a^{k+s_1 2^{m-2}}, \; h = c^2 a^{k+s_2 2^{m-2}},$$

where $k = -2^n(1+2v_0)^{-1}$. Then

$$\begin{aligned} gh &= c^2 a^{k+s_1 2^{m-2}} \cdot c^2 a^{k+s_2 2^{m-2}} = c^4 \cdot c^{-2} a^{k+s_1 2^{m-2}} c^2 \cdot a^{k+s_2 2^{m-2}} \\ &= a^{2^{n+1}} \cdot a^{r^2(k+s_1 2^{m-2})} \cdot a^{k+s_2 2^{m-2}} = a^{2^{n+1}+k(1+r^2)+(s_1+s_2)2^{m-2}}, \end{aligned}$$

because

$$r^2(k+s_1 2^{m-2}) \equiv r^2 k + (1+4v+4v^2)s_1 2^{m-2} \equiv r^2 k + s_1 2^{m-2} \, (\mathrm{mod}\, 2^m).$$

It follows from here that $gh = hg$. The lemma is proved.

Lemma 5.14. *We have*

$$\mathrm{End}(G)\setminus\mathrm{Aut}(G)=\{y(g;h)\mid g,h\in G;\ g^4=h^2=1\}. \tag{5.15}$$

If $x\in\mathrm{End}(G)\setminus\mathrm{Aut}(G)$, *then*

$$G^{'}\subset \mathrm{Ker}x. \tag{5.16}$$

Proof. In view of Lemma 5.13 and $\langle cG^{'}\rangle\times\langle aG^{'}\rangle\cong C_4\times C_2$, the proper endomorphism $y(g;h)$ exists for each $g,h\in G$ such that $g^4=h^2=1$. By Lemmas 5.11 and 5.12, the numbers of elements g and h, satisfying $g^4=h^2=1$, are 8 and 4, respectively. Therefore, the number of proper endomorphisms $y(g;h)$ of G is $8\cdot 4=32$. Property 2^0 implies that each proper endomorphism of G has this form. Hence (5.15) holds. It follows also that $\mathrm{Im}x$ is Abelian and hence $G^{'}\subset\mathrm{Ker}x$ for each proper endomorphism x of G. The lemma is proved.

Lemma 5.15. *The map* $y:G\longrightarrow G$ *given by*

$$cy=c^{1+4i}=ca^{i2^{n+1}},\ ay=a^j,\ i\in\mathbb{Z}_{2^{m-n-1}},\ j\in\mathbb{Z}_{2^m}, \tag{5.17}$$

on the generators of G, *induces an endomorphism of* G *if and only if*

$$j\equiv 1+4i\,(mod\,2^{m-n-1}). \tag{5.18}$$

The number of such endomorphisms is 2^m *and those endomorphisms are automorphisms of* G.

Proof. The map y given by (5.17) induces an endomorphism of G if and only if it preserves the generating relations (5.12) of G, i.e.,

$$(ay)^{2^m}=1,\ (cy)^4=(ay)^{2^{n+1}},\ (cy)^{-1}(ay)(cy)=(ay)^r. \tag{5.19}$$

Clearly, $(ay)^{2^m}=1$ for all values of i and j. The last equation of (5.19) holds also for all values of i and j:

$$(cy)^{-1}(ay)(cy)=c^{-(1+4i)}a^jc^{1+4i}=a^{jr^{1+4i}}=a^{jr}=(ay)^r.$$

Since

$$(cy)^4=c^{4(1+4i)}=a^{2^{n+1}(1+4i)},\ (ay)^{2^{n+1}}=a^{j2^{n+1}},$$

the second equation of (5.19) holds if and only if equivalence (5.18) is true. The first part of the lemma is proved.

In view of (5.18) and (5.19), we have $j\equiv 1\,(\mathrm{mod}\,2)$, and, therefore, $G=\langle c,a\rangle=\langle cy,ay\rangle$, i.e., y is an automorphism of G. The solutions of (5.19) are

$$i\in\mathbb{Z}_{2^{m-n-1}},\ j=1+4i+s2^{m-n-1},\ s\in\mathbb{Z}_{2^{n+1}}$$

and the number of solutions is $2^{m-n-1}\cdot 2^{n+1}=2^m$. The lemma is proved.

Lemma 5.16. $m\le 4,\ n\le m-2\le 2.$

Proof. Assume that x is a proper endomorphism of G and let y be an automorphism of G given by Lemma 5.15. Then

$$c^{-1}\cdot cy,\, a^{-1}\cdot ay\in\langle a^2\rangle=G^{'}.$$

By (5.16),

$$(c^{-1}\cdot cy)x=(a^{-1}\cdot ay)x=1,\ c(yx)=cx,\ a(yx)=ax,$$

i.e., $yx=x$. Property 6^0 and Lemma 5.15 follow $m\le 4$. Hence $n\le m-2\le 2$. The lemma is proved.

Lemma 5.17. $m=3$.

Proof. Let us examine the map $y : G \longrightarrow G$ given as follows:

$$cy = ca^{2i},\ ay = a^j,\ (c^w a^t)y = (cy)^w (ay)^t,$$

where $i \in \mathbb{Z}_{2^{m-1}};\ j,t \in \mathbb{Z}_{2^m},\ w = 0,1,2,3$. It is easy to check that the map y preserves the generating relations and induces an endomorphism of G if and only if

$$2^{n+1} + 2i(1+r)(1+r^2) \equiv j2^{n+1} \pmod{2^m}.$$

The solution of the last congruence is

$$j \equiv 1 + 2^{2-n} i(1+v)(1+2v_0) \pmod{2^{m-n-1}},$$

i.e.,

$$j = 1 + 2^{2-n} i(1+v)(1+2v_0) + s2^{m-n-1},\ s \in \mathbb{Z}_{2^{n+1}}. \tag{5.20}$$

By Lemma 5.16, $m=3$ or $m=4$. Assume on the contrary that $m=4$.

If v is odd, then similarly to Lemma 5.16, the obtained endomorphism is an automorphism and $yx = x$ for each proper endomorphism of G. Therefore, by property 6^0, the number of solutions (5.20) is ≤ 16. The numbers of possible values for i and s are $2^{m-1} = 2^3$ and 2^{n+1}, respectively. Hence the number of automorphisms y is $2^3 \cdot 2^{n+1} = 2^{n+4}$. By property 6^0, $2^{n+4} \le 16$, i.e., $n=0$.

If v is even and $n<2$, then similarly to the previous segment, $n=0$. If v is even and $n=2$, then we can choose $s=0$ and i such that

$$i(1+v)(1+2v_0) \equiv 1 \pmod{16}.$$

In this case, we get an endomorphism

$$cy = ca^{2i},\ ay = a^2,$$

which is a proper endomorphism and ay is of order eight. This contradicts Lemma 5.14.

We have proved that if $m=4$, then $n=0$ and G is given by the relations

$$a^{16} = 1,\ c^4 = a^2,\ c^{-1}ac = a^r$$

or, equivalently, by the relations

$$c^{32} = 1,\ a^2 = c^4,\ a^{-1}ca = c^{\rho}$$

for some $\rho \ne 1$ such that

$$\rho^2 \equiv 1 \pmod{32}. \tag{5.21}$$

Congruence (5.21) is satisfied only for $\rho \in \{15, -15, -1\}$. If $\rho = 15$, then $[c,a] = c^{\rho-1} = c^{14}$ and $c^2 \in G'$, which is impossible. Similarly, the case $\rho = -1$ is impossible. If $\rho = -15$, then $[c,a] = c^{-16} = c^{16} \in Z(G)$ and $G' = \langle c^{16} \rangle \cong C_2$, which is also impossible. Therefore, the case $m=4$ is impossible and $m=3$. The lemma is proved.

By Lemmas 5.16 and 5.17, the group G is a group of order 32 and it is given by the relations

$$a^8 = 1,\ c^4 = a^{2^{n+1}},\ c^{-1}ac = a^r,\ 0 \le n \le 1.$$

Similarly to the last part of the proof of Lemma 5.17, it is easy to check that the case $n=0$ is impossible. Hence $n=1$ and

$$c^4 = a^4.$$

We have $r \ne 1$, because G is non-Abelian. Also $r \ne -3$, because otherwise $[a,c] = a^{-4}$, and we have $G' = \langle a^4 \rangle \cong C_2$, which is impossible. Hence $r=3$ or $r=-1$. In both cases the numbers of elements of order two is 3, of order four is 4, and of order eight is 24. By [5], there is only one non-Abelian group of order 32 which has this order structure of its elements. This group is $\mathscr{G}_{32}$. The theorem is proved.

Theorem 5.2. *The group $\mathscr{G}_{32}$ is determined by its endomorphism semigroup in the class of all groups.*

Proof. Let G^* be a group such that the endomorphism semigroups of G^* and $\mathscr{G}_{32}$ are isomorphic:

$$\mathrm{End}(G^*) \cong \mathrm{End}(\mathscr{G}_{32}). \tag{5.22}$$

Since $\mathrm{End}(G^*)$ is finite, so is G^* ([1], Theorem 2). The group $\mathscr{G}_{32}$ satisfies properties 1^0–6^0 of Theorem 5.1. In view of isomorphism (5.22), the group G^* satisfies also these properties. Properties 1^0–3^0 imply that G^* is a 2-group. By Lemma 3.5 and isomorphism (5.22), $|G^*| \geq 32$. Theorem 5.1 implies the isomorphism $G^* \cong \mathscr{G}_{32}$. The theorem is proved.

ACKNOWLEDGEMENT

This work was partially supported by the Estonian Ministry of Education and Research (project SF0140011s09).

REFERENCES

1. Alperin, J. L. Groups with finitely many automorphisms. *Pacific J. Math.*, 1962, **12**, 1–5.
2. Corner, A. L. S. Every countable reduced torsion-free ring is an endomorphism ring. *Proc. London Math. Soc.*, 1963, **13**, 687–710.
3. Gramushnjak, T. and Puusemp, P. A characterization of a class of groups of order 32 by their endomorphism semigroups. *Algebras, Groups, Geom.*, 2005, **22**, 387–412.
4. Gramushnjak, T. and Puusemp, P. A characterization of a class of 2-groups by their endomorphism semigroups. In *Generalized Lie Theory in Mathematics, Physics and Beyond* (Silvestrov, S., Paal, E., Abramov, V., and Stolin, A., eds). Springer-Verlag, Berlin, 2009, 151–159.
5. Hall, M., Jr. and Senior, J. K. *The Groups of Order* 2^n, $n \leq 6$. Macmillan, New York; Collier-Macmillan, London, 1964.
6. Krylov, P. A., Mikhalev, A. V., and Tuganbaev, A. A. *Endomorphism Rings of Abelian Groups.* Kluwer Academic Publisher, Dordrecht, 2003.
7. Puusemp, P. Idempotents of the endomorphism semigroups of groups. *Acta Comment. Univ. Tartuensis*, 1975, 366, 76–104 (in Russian).
8. Puusemp, P. Endomorphism semigroups of generalized quaternion groups. *Acta Comment. Univ. Tartuensis*, 1976, 390, 84–103 (in Russian).
9. Puusemp, P. A characterization of divisible and torsion Abelian groups by their endomorphism semigroups. *Algebras, Groups, Geom.*, 1999, **16**, 183–193.
10. Puusemp, P. On endomorphism semigroups of dihedral 2-groups and alternating group A_4. *Algebras, Groups, Geom.*, 1999, **16**, 487–500.
11. Puusemp, P. On the definability of a semidirect product of cyclic groups by its endomorphism semigroup. *Algebras, Groups, Geom.*, 2002, **19**, 195–212.
12. Puusemp, P. Groups of order less than 32 and their endomorphism semigroups. *J. Nonlinear Math. Phys.*, 2006, **13**, Supplement, 93–101.
13. Puusemp, P. Semidirect products of generalized quaternion groups by a cyclic group. In *Generalized Lie Theory in Mathematics, Physics and Beyond* (Silvestrov, S., Paal, E., Abramov, V., and Stolin, A., eds). Springer-Verlag, Berlin, Heidelberg, 2009, 141–149.
14. Puusemp, P. and Puusemp, P. On endomorphisms of groups of order 32 with maximal subgroups $C_4 \times C_2 \times C_2$. *Proc. Estonian Acad. Sci.*, 2014, **63**, 105–120.
15. Robinson, D. J. S. *A Course in the Theory of Groups*. Springer-Verlag, New York, 1996.

Maksimaalset alamrühma $C_8 \times C_2$ omavate 32. järku rühmade endomorfismidest

Piret Puusemp ja Peeter Puusemp

On tõestatud, et kõik 32. järku rühmad, mille üheks maksimaalseks alamrühmaks on $C_8 \times C_2$, on määratud oma endomorfismipoolrühmadega kõigi rühmade klassis. Ühtlasi on antud mainitud rühmade kirjeldused nende endomorfismipoolrühmade kaudu.