Proceedings of the Estonian Academy of Sciences,
2014, **63**, 2, 105–120
doi: 10.3176/proc.2014.2.01
Available online at www.eap.ee/proceedings

MATHEMATICS

On endomorphisms of groups of order 32 with maximal subgroups $C_4 \times C_2 \times C_2$

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Received 5 July 2013, revised 11 September 2013, accepted 12 September 2013, available online 20 May 2014

Abstract. It is proved that each group of order 32, which has a maximal subgroup isomorphic to $C_4 \times C_2 \times C_2$, is determined by its endomorphism semigroup in the class of all groups.

Key words: group, semigroup, endomorphism semigroup.

1. INTRODUCTION

It is well known that all endomorphisms of an Abelian group form a ring and many of its properties can be characterized by this ring. An excellent overview of the present situation in the theory of endomorphism rings of groups is given by Krylov, Mikhalev, and Tuganbaev [6]. All endomorphisms of an arbitrary group form only a semigroup. The theory of endomorphism semigroups of groups is quite modestly developed. In a number of our papers we have made efforts to describe some properties of groups by the properties of their endomorphism semigroups. For example, we have proved that many well-known classes of groups are determined by their endomorphism semigroups in the class of all groups. Note that if G is a fixed group and an isomorphism of semigroups $\mathrm{End}(G)$ and $\mathrm{End}(H)$, where H is an arbitrary group, always implies an isomorphism of G and H, we say that the group G is determined by its endomorphism semigroup in the class of all groups. Some of such groups are finite Abelian groups ([7], Theorem 4.2), generalized quaternion groups ([8], Corollary 1), torsion-free divisible Abelian groups ([10], Theorem 1), etc. On the other hand, there exist many examples of groups that are not determined by their endomorphism semigroups in the class of all groups. For example, the following result of Corner [2] is well known: any countable, reduced, torsion-free, associative ring with unity is an endomorphism ring for a continual number of countable, reduced, torsion-free Abelian groups. An example of non-Abelian groups that are not determined by their endomorphism semigroups in the class of all groups is the following: the groups

$$G = \langle a, b \mid b^3 = a^{91} = 1,\ b^{-1}ab = a^{16} \rangle = \langle a \rangle \leftthreetimes \langle b \rangle$$

and

$$H = \langle c, d \mid d^3 = c^{91} = 1,\ d^{-1}cd = c^{9} \rangle = \langle c \rangle \leftthreetimes \langle d \rangle$$

are non-isomorphic but their endomorphism semigroups are isomorphic [9].

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We know a complete answer to this problem for finite groups of order less than 32. It was proved in [13] that among the finite groups of order less than 32 only the alternating group A_4 (also called the tetrahedral group) and the binary tetrahedral group $\langle a, b \mid b^3 = 1,\ aba = bab \rangle$ are not determined by their endomorphism semigroups in the class of all groups. These two groups are non-isomorphic but their endomorphism semigroups are isomorphic. It was natural to consider the groups of order 32. All groups of order 32 were described by Hall and Senior [5]. There exist exactly 51 non-isomorphic groups of order 32. In [5], these groups are numbered by $1, 2, \ldots, 51$. We shall mark these groups by $\mathscr{G}_1, \mathscr{G}_2, \ldots, \mathscr{G}_{51}$, respectively. The groups $\mathscr{G}_1 - \mathscr{G}_7$ are Abelian, and, therefore, are determined by their endomorphism semigroups in the class of all groups ([7], Theorem 4.2). In [3], it was proved that the groups of order 32, presentable in the form $(C_4 \times C_4) \rightthreetimes C_2$ (C_k – the cyclic group of order k), are determined by their endomorphism semigroups in the class of all groups. The groups of this type are $\mathscr{G}_3, \mathscr{G}_{14}, \mathscr{G}_{16}, \mathscr{G}_{31}, \mathscr{G}_{34}, \mathscr{G}_{39}, \mathscr{G}_{41}$. In [4], it was proved that the groups of order 32 presentable in the form $(C_8 \times C_2) \rightthreetimes C_2$ are determined by their endomorphism semigroups in the class of all groups. The groups of this type are $\mathscr{G}_4, \mathscr{G}_{17}, \mathscr{G}_{20}, \mathscr{G}_{26}, \mathscr{G}_{27}$.

In this paper, we consider the groups of order 32 that have a maximal subgroup isomorphic to $C_4 \times C_2 \times C_2$ and prove the following theorem:

Theorem 1.1. *Each group of order* 32, *which has a maximal subgroup isomorphic to* $C_4 \times C_2 \times C_2$, *is determined by its endomorphism semigroup in the class of all groups.*

The groups of order 32 which have a maximal subgroup isomorphic to $C_4 \times C_2 \times C_2$ are:

$$\mathscr{G}_2, \mathscr{G}_3, \mathscr{G}_4, \mathscr{G}_8, \mathscr{G}_9, \mathscr{G}_{10}, \mathscr{G}_{11}, \mathscr{G}_{12}, \mathscr{G}_{13}, \mathscr{G}_{14}, \mathscr{G}_{16}, \mathscr{G}_{18}, \mathscr{G}_{20}, \mathscr{G}_{36}, \mathscr{G}_{37}, \mathscr{G}_{38}.$$

To prove the theorem, the characterization of these groups by their endomorphism semigroups will be given. These characterization properties, which are preserved by isomorphisms of endomorphism semigroups, will then be used in the proofs.

We shall use the following notations:
G – a group;
$\mathrm{End}(G)$ – the endomorphism semigroup of G;
C_k – the cyclic group of order k;
$\mathbb{Z}_k$ – the ring of residual classes modulo k;
$\langle K, \ldots, g, \ldots \rangle$ – the subgroup generated by subsets $K, \ldots$ and elements $g, \ldots$;
$[a, b] = a^{-1}b^{-1}ab$ $(a, b \in G)$;
G' – the commutator-group of G;
$\widehat{g}$ – the inner automorphism of G, generated by an element $g \in G$;
$I(G)$ – the set of all idempotents of $\mathrm{End}(G)$;
$K(x) = \{z \in \mathrm{End}(G) \mid zx = xz = z\}$;
$P(x) = \{z \in \mathrm{End}(G) \mid zx = xz = x\}$;
$J(x) = \{z \in \mathrm{End}(G) \mid zx = xz = 0\}$;
$V(x) = \{z \in \mathrm{Aut}(G) \mid zx = x\}$;
$H(x) = \{z \in \mathrm{End}(G) \mid xz = z,\ zx = 0\}$;
$[x] = \{z \in I(G) \mid xz = z,\ zx = x\},\ x \in I(G)$.

The sets $K(x), V(x), P(x)$, and $J(x)$ are subsemigroups of $\mathrm{End}(G)$, however, $V(x)$ is a subgroup of $\mathrm{Aut}(G)$. We shall write the mapping right from the element on which it acts.

2. GROUPS THAT HAVE A MAXIMAL SUBGROUP $C_4 \times C_2 \times C_2$

In this section, using results obtained by Hall and Senior [5], the list of all groups of order 32 that have a maximal subgroup $C_4 \times C_2 \times C_2$ is given. To this end, denote:

• $Q = \langle a, b \mid a^4 = 1,\ b^2 = a^2,\ b^{-1}ab = a^{-1}\rangle$ – the quaternion group;
• $D_4 = \langle a, b \mid a^4 = b^2 = 1,\ b^{-1}ab = a^{-1}\rangle$ – the dihedral group of order 8;
• $\mathscr{G}_{16,1} = \langle a, b, c \mid a^4 = b^2 = c^2 = 1,\ ab = ba,\ c^{-1}bc = ba^2,\ c^{-1}ac = a^{-1}\rangle$;
• $\mathscr{G}_{16,2} = \langle a, b, c \mid a^4 = b^2 = c^2 = 1,\ ab = ba,\ bc = cb,\ c^{-1}ac = a^{-1}b\rangle$;
• $\mathscr{G}_{16,3} = \langle a, b \mid a^4 = b^4 = 1,\ b^{-1}ab = a^{-1}\rangle$;
• $\mathscr{G}_{16,4} = \langle a, b \mid a^8 = b^2 = 1,\ b^{-1}ab = a^5\rangle$.

The groups $\mathscr{G}_{16,1} - \mathscr{G}_{16,4}$ are groups of order 16.

The groups of order 32 that have a maximal subgroup isomorphic to $C_4 \times C_2 \times C_2$ are:
• $\mathscr{G}_2 = C_4 \times C_2 \times C_2 \times C_2$, $\mathscr{G}_3 = C_4 \times C_4 \times C_2$, $\mathscr{G}_4 = C_2 \times C_2 \times C_8$,
• $\mathscr{G}_8 = C_2 \times C_2 \times D_4$, $\mathscr{G}_9 = C_2 \times C_2 \times Q$, $\mathscr{G}_{10} = C_2 \times \mathscr{G}_{16,1}$,
• $\mathscr{G}_{11} = C_2 \times \mathscr{G}_{16,2}$, $\mathscr{G}_{12} = C_2 \times \mathscr{G}_{16,3}$, $\mathscr{G}_{13} = C_2 \times \mathscr{G}_{16,4}$, $\mathscr{G}_{14} = C_4 \times D_4$,
• $\mathscr{G}_{16} = \langle a, b, c \mid a^4 = b^4 = c^2 = 1,\ ab = ba,\ bc = cb,\ c^{-1}ac = ab^2\rangle = (\langle a\rangle \times \langle b\rangle) \leftthreetimes \langle c\rangle = (C_4 \times C_4) \leftthreetimes C_2$,
• $\mathscr{G}_{18} = \langle a, b, c \mid a^4 = b^2 = c^4 = 1,\ ab = ba,\ bc = cb,\ c^{-1}ac = ab\rangle$,
• $\mathscr{G}_{20} = \langle a, b, c \mid a^8 = b^2 = c^2 = 1,\ ab = ba,\ bc = cb,\ c^{-1}ac = ab\rangle = (\langle a\rangle \times \langle b\rangle) \leftthreetimes \langle c\rangle = (C_8 \times C_2) \leftthreetimes C_2$,
• $\mathscr{G}_{36} = \langle a, b, c, d \mid a^4 = b^2 = c^2 = d^2 = 1,\ ab = ba,\ ac = ca,\ bc = cb,\ dc = cd,\ d^{-1}ad = a^{-1},\ d^{-1}bd = bc\rangle$,
• $\mathscr{G}_{37} = \langle a, b, c, d \mid a^4 = b^2 = c^2 = d^4 = 1,\ ab = ba,\ ac = ca,\ bc = cb,\ dc = cd,\ d^2 = a^2,\ d^{-1}ad = a^{-1},$
$d^{-1}bd = bc\rangle$,
• $\mathscr{G}_{38} = \langle a, b, c, d \mid a^4 = b^2 = c^2 = d^2 = 1,\ ab = ba,\ ac = ca,\ bc = cb,\ dc = cd,\ d^{-1}ad = ac,\ d^{-1}bd = ba^2\rangle$.

It is known that the following groups are determined by their endomorphism semigroups in the class of all groups: finite Abelian groups ([7], Theorem 4.2), dihedral 2-groups ([9], Theorem 3.1), generalized quaternion groups [8], finite groups of order 16 [12]. On the other hand, if the groups $G_1, G_2, \ldots, G_n$ are determined by their endomorphism semigroups in the class of all groups, then so is their direct product $G_1 \times G_2 \times \ldots \times G_n$ ([7], Theorem 1.13). Therefore, the groups $\mathscr{G}_2 - \mathscr{G}_4$ and $\mathscr{G}_8 - \mathscr{G}_{14}$ are determined by their endomorphism semigroups in the class of all groups. The groups $\mathscr{G}_{16}$ and $\mathscr{G}_{20}$ are also determined by their endomorphism semigroups in the class of all groups [4,12]. To prove Theorem 1.1, we have to prove in addition that the groups $\mathscr{G}_{18}, \mathscr{G}_{36}, \mathscr{G}_{37}$, and $\mathscr{G}_{38}$ are determined by their endomorphism semigroups in the class of all groups. It is done in Theorems 4.2, 5.2, 6.2, and 7.2.

3. PRELIMINARY LEMMAS

For convenience of reference, let us recall some known facts that will be used in the proofs of our main results. We omit the proofs, because these are straightforward corollaries from the definitions.

Lemma 3.1. *If $x \in I(G)$, then $G = \mathrm{Ker}\,x \leftthreetimes \mathrm{Im}\,x$ and $\mathrm{Im}\,x = \{g \in G \mid gx = g\}$.*

Lemma 3.2. *If $x \in I(G)$, then*

$$K(x) = \{\, y \in \mathrm{End}(G) \mid (\mathrm{Im}\,x)y \subset \mathrm{Im}\,x,\ (\mathrm{Ker}\,x)y = \langle 1\rangle \,\}$$

and $K(x)$ is a subsemigroup with the unity x of $\mathrm{End}(G)$ which is canonically isomorphic to $\mathrm{End}(\mathrm{Im}\,x)$. In this isomorphism element y of $K(x)$ corresponds to its restriction on the subgroup $\mathrm{Im}\,x$ of G.

Lemma 3.3. *If $x \in I(G)$, then*

$$J(x) = \{z \in \mathrm{End}(G) \mid (\mathrm{Im}\,x)z = \langle 1\rangle,\ (\mathrm{Ker}\,x)z \subset \mathrm{Ker}\,x\}.$$

Lemma 3.4. *If $x, y \in I(G)$ and $xy = yx = 0$, then*

$$G = ((\mathrm{Ker}\,x \cap \mathrm{Ker}\,y) \leftthreetimes \mathrm{Im}\,x) \leftthreetimes \mathrm{Im}\,y = ((\mathrm{Ker}\,x \cap \mathrm{Ker}\,y) \leftthreetimes \mathrm{Im}\,y) \leftthreetimes \mathrm{Im}\,x,$$

$$\mathrm{Ker}\,x = (\mathrm{Ker}\,x \cap \mathrm{Ker}\,y) \leftthreetimes \mathrm{Im}\,y,\ \mathrm{Ker}\,y = (\mathrm{Ker}\,x \cap \mathrm{Ker}\,y) \leftthreetimes \mathrm{Im}\,x.$$

Lemma 3.5. *If* $x \in \mathrm{End}(G)$ *and* $\mathrm{Im}x$ *is Abelian, then* $\widehat{g} \in V(x)$ *for each* $g \in G$.

Lemma 3.6. *If* $x \in I(G)$, *then*

$$H(x) = \{\, y \in \mathrm{End}(G) \mid (\mathrm{Im}x)y \subset \mathrm{Ker}x,\ (\mathrm{Ker}x)y = \langle 1 \rangle \,\}.$$

Lemma 3.7. *If* $x \in I(G)$, *then*

$$P(x) = \{y \in \mathrm{End}(G) \mid y|_{\mathrm{Im}x} = 1|_{\mathrm{Im}x},\ (\mathrm{Ker}x)y \subset \mathrm{Ker}x\}.$$

Lemma 3.8. *If* $x \in I(G)$, *then* $[x] = \{\, y \in I(G) \mid \mathrm{Ker}x = \mathrm{Ker}y\}$.

4. GROUP $\mathscr{G}_{18}$

In this section, we shall characterize the group

$$\begin{aligned}\mathscr{G}_{18} &= \langle a, b, c \mid a^4 = b^2 = c^4 = 1,\ ab = ba,\ bc = cb,\ c^{-1}ac = ab\rangle \\ &= (\langle a\rangle \times \langle b\rangle) \rightthreetimes \langle c\rangle = (\langle c\rangle \times \langle b\rangle) \rightthreetimes \langle a\rangle \cong (C_4 \times C_2) \rightthreetimes C_4\end{aligned}$$

by its endomorphism semigroup.

Theorem 4.1. *A finite group* G *is isomorphic to* $\mathscr{G}_{18}$ *if and only if there exist* $x, y \in I(G)$ *such that the following properties hold*:
1^0 $K(x) \cong K(y) \cong \mathrm{End}(C_4)$;
2^0 $xy = yx = 0$;
3^0 $J(x) \cap J(y) = \{0\}$;
4^0 $V(x)$ *is a* 2*-group*;
5^0 $|\{u \in \mathrm{End}(G) \mid xu = u,\ ux = uy = 0\}| = 2$.

Proof. Necessity. Let $G = \mathscr{G}_{18}$. Denote by x and y the projections of G onto its subgroups $\langle c\rangle$ and $\langle a\rangle$, respectively. Then $x, y \in I(G)$. We shall prove that x and y satisfy properties 1^0–5^0.

By Lemma 3.2 and the definition of x and y, properties 1^0 and 2^0 hold. By Lemma 3.3, $J(x) \cap J(y)$ consists of $z \in \mathrm{End}(G)$ such that

$$cz = az = 1,\ bz = b^i,\ i \in \mathbb{Z}_2. \tag{4.1}$$

Map (4.1) preserves the generating relations of G if and only if $i = 0$, i.e., $z = 0$. Therefore, $J(x) \cap J(y) = \{0\}$ and property 3^0 is true. The subgroup $V(x)$ of $\mathrm{Aut}(G)$ consists of $z \in \mathrm{Aut}(G)$ such that $g^{-1} \cdot gz \in \mathrm{Ker}x$ for each $g \in G$. Therefore, $z \in V(x)$ maps on generators of G as follows:

$$cz = ca^ib^j,\ az = a^kb^l,\ bz = a^sb^t;\ i, k, s \in \mathbb{Z}_4;\ j, l, t \in \mathbb{Z}_2. \tag{4.2}$$

Map (4.2) is an automorphism of G if and only if

$$s = 0,\ t = 1,\ k \equiv 1 \,(\mathrm{mod}\, 2).$$

It follows that $|V(x)| = 4 \cdot 2 \cdot 2 \cdot 2 = 2^5$, i.e., $V(x)$ is a 2-group and property 4^0 is true.

Assume that $u \in \mathrm{End}(G)$ and $xu = u,\ ux = uy = 0$. Then

$$au = bu = 1,\ cu = b^i,\ i \in \mathbb{Z}_2. \tag{4.3}$$

Map (4.3) is an endomorphism of G for each $i \in \mathbb{Z}_2$. It follows from here that property 5^0 holds. The necessity is proved.

Sufficiency. Let G be a finite group and let there exist $x, y \in I(G)$ which satisfy properties 1^0–5^0 of the theorem. Our aim is to prove that $G \cong \mathscr{G}_{18}$.

Lemma 3.2 and property 1^0 imply that

$$\mathrm{End}(\mathrm{Im}x) \cong \mathrm{End}(\mathrm{Im}y) \cong \mathrm{End}(C_4).$$

Since each finite Abelian group is determined by its endomorphism semigroup in the class of all groups ([7], Theorem 4.2), we have

$$\mathrm{Im}x = \langle c \rangle \cong C_4, \ \mathrm{Im}y = \langle a \rangle \cong C_4$$

for some $c, a \in G$. By Lemma 3.4,

$$G = (N \leftthreetimes \langle a \rangle) \leftthreetimes \langle c \rangle = (N \leftthreetimes \langle a \rangle) \leftthreetimes \langle c \rangle,$$

where

$$N = \mathrm{Ker}x \cap \mathrm{Ker}y, \ \mathrm{Ker}x = N \leftthreetimes \langle a \rangle, \ \mathrm{Ker}y = N \leftthreetimes \langle c \rangle.$$

In view of Lemma 3.5 and property 4^0, $\widehat{g} = 1$ for each $2'$-element g of G. Hence all $2'$-elements of G belong into its centre $Z(G)$. Therefore, the group G splits into the direct product $G = G_{2'} \times G_2$ of its Hall $2'$-subgroup $G_{2'}$ and Sylow 2-subgroup G_2. Denote by z the projection of G onto its subgroup $G_{2'}$. Then $z \in J(x) \cap J(y)$, and, by property 3^0, $z = 0$, i.e. $G_{2'} = \langle 1 \rangle$ and G is a 2-group.

Each homomorphism $v : \mathrm{Im}x = \langle c \rangle \longrightarrow N$ induces an endomorphism u of G by setting $gu = 1, \ g \in N \leftthreetimes \langle a \rangle, \ cu = cv$. This endomorphism u satisfies equalities $xu = u, \ ux = uy = 0$. By 5^0, we have two homomorphisms v of such kind. Therefore, the subgroup N of G contains only one element of order 2 and does not have any element of order 4. By [14], Theorem 5.46, N is a cyclic group of order 2:

$$N = \langle b \rangle \cong C_2, \ b \in G.$$

Since N is an invariant subgroup of G, we have

$$ab = ba, \ cb = bc.$$

Elements a and c do not commute, because otherwise $G = N \times \langle a \rangle \times \langle c \rangle$ and the projection z of G onto N is a non-zero element of $J(x) \cap J(y)$, which contradicts property 3^0. In view of (2.5), $a^{-1}c^{-1}ac \in N$. Hence $a^{-1}c^{-1}ac = b$ and $c^{-1}ac = ab$. Consequently,

$$G = \langle a, b, c \mid a^4 = b^2 = c^4 = 1, \ ab = ba, \ bc = cb, \ c^{-1}ac = ab \rangle$$

and the groups G and $\mathscr{G}_{18}$ are isomorphic. The sufficiency is proved and so is the theorem.

Theorem 4.2. *The group $\mathscr{G}_{18}$ is determined by its endomorphism semigroup in the class of all groups.*

Proof. Let G^* be a group such that the endomorphism semigroups of G^* and $\mathscr{G}_{18}$ are isomorphic:

$$\mathrm{End}(G^*) \cong \mathrm{End}(\mathscr{G}_{18}). \tag{4.4}$$

Denote by z^* the image of $z \in \mathrm{End}(\mathscr{G}_{18})$ in isomorphism (4.4). Since $\mathrm{End}(G^*)$ is finite, so is G^* ([1], Theorem 2). By Theorem 4.1, there exist $x, y \in I(\mathscr{G}_{18})$, satisfying properties 1^0–5^0 of Theorem 4.1. These properties are formulated so that they are preserved in isomorphism (4.4). Therefore, the idempotents x^* and y^* of $\mathrm{End}(G^*)$ satisfy properties, similar to properties 1^0–5^0 (it is necessary to change everywhere $z \in \mathrm{End}(\mathscr{G}_{18})$ by $z^* \in \mathrm{End}(G^*)$). Using now Theorem 4.1 for G^*, it follows that G^* and $\mathscr{G}_{18}$ are isomorphic. The theorem is proved.

5. GROUP $\mathcal{G}_{36}$

In this section, we shall characterize the group

$$\mathcal{G}_{36} = \langle a, b, c, d \mid a^4 = b^2 = c^2 = d^2 = 1,\ ab = ba,\ ac = ca,\ bc = cb,\ dc = cd,\ d^{-1}ad = a^{-1},\ d^{-1}bd = bc\rangle$$

by its endomorphism semigroup. The group $\mathcal{G}_{36}$ splits into the following semidirect products:

$$\mathcal{G}_{36} = (\langle a\rangle \times \langle b\rangle \times \langle c\rangle) \rtimes \langle d\rangle \cong (C_4 \times C_2 \times C_2) \rtimes C_2,$$

$$\mathcal{G}_{36} = (\langle b\rangle \times \langle c\rangle) \rtimes (\langle a\rangle \rtimes \langle d\rangle) \cong (C_2 \times C_2) \rtimes (C_4 \rtimes C_2),$$

$$\mathcal{G}_{36} = \langle a\rangle \rtimes ((\langle b\rangle \times \langle c\rangle) \rtimes \langle d\rangle) \cong C_4 \rtimes ((C_2 \times C_2) \rtimes C_2).$$

We will prove that the isomorphism $\mathrm{End}(G) \cong \mathrm{End}(\mathcal{G}_{36})$, where G is another group, implies the isomorphism $G \cong \mathcal{G}_{36}$.

We need the following fact on endomorphisms of an arbitrary group G. Let $x, x_1, x_2 \in I(G)$. In [11], Theorems 2.1 and 3.1–3.3, the necessary and sufficient conditions were given for x, x_1, x_2 under which the group G decomposes into the following semidirect products:

$$G = (G_1 \times G_2) \rtimes K = G_1 \rtimes (G_2 \rtimes K) = G_2 \rtimes (G_1 \rtimes K), \tag{5.1}$$

where

$$\mathrm{Im}x = K,\ \mathrm{Im}x_1 = G_1 \rtimes K,\ \mathrm{Im}x_2 = G_2 \rtimes K, \tag{5.2}$$

$$\mathrm{Ker}x = G_1 \times G_2,\ \mathrm{Ker}x_1 = G_2,\ \mathrm{Ker}x_2 = G_1. \tag{5.3}$$

Denote these conditions $C(x, x_1, x_2)$. Assume that G^* is another group such that the endomorphism semigroups of G and G^* are isomorphic and x^*, x_1^*, x_2^* correspond to x, x_1, x_2 in this isomorphism. Then x^*, x_1^*, x_2^* satisfy conditions $C(x^*, x_1^*, x_2^*)$ in $\mathrm{End}(G^*)$ and the group G^* decomposes similarly to (5.1)–(5.3).

Theorem 5.1. *A finite group G is isomorphic to $\mathcal{G}_{36}$ if and only if there exist $x, x_1, x_2 \in I(G)$ such that the following properties hold*:
1^0 x, x_1, *and* x_2 *satisfy* $C(x, x_1, x_2)$;
2^0 $K(x_1) \cong K(x_2) \cong \mathrm{End}(D_4)$;
3^0 $K(x) \cong \mathrm{End}(C_2)$;
4^0 $|\{z \in K(x_2) \mid xz = z,\ zx_1 = 0\}| = 4$;
5^0 $|\{z \in K(x_1) \mid xz = z,\ zx_2 = 0\}| = 2$.

Proof. Necessity. Let $G = \mathcal{G}_{36}$. Denote by x, x_1, and x_2 the projections of G onto its subgroups $\langle d\rangle$, $\langle a\rangle \rtimes \langle d\rangle$, and $(\langle b\rangle \times \langle c\rangle) \rtimes \langle d\rangle$, respectively. Then $x, x_1, x_2 \in I(G)$. We shall prove that x, x_1, and x_2 satisfy properties 1^0–5^0.

By the definition, G decomposes into semidirect products (5.1), where

$$K = \mathrm{Im}x,\ G_1 = \mathrm{Ker}x_2 = \langle a\rangle,\ G_2 = \mathrm{Ker}x_1 = \langle b\rangle \times \langle c\rangle,$$

$$\mathrm{Im}x_1 = G_1 \rtimes K = \langle a\rangle \rtimes \langle d\rangle \cong D_4,$$

$$\mathrm{Im}x_2 = G_2 \rtimes K = (\langle b\rangle \times \langle c\rangle) \rtimes \langle d\rangle,$$

$$\mathrm{Ker}x = G_1 \times G_2 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle.$$

Hence x, x_1, and x_2 satisfy property 1^0.

By Lemma 3.2,

$$K(x) \cong \mathrm{End}(\langle d\rangle) \cong \mathrm{End}(C_2),\ \ K(x_1) \cong \mathrm{End}(D_4).$$

Since $(db)^2 = c,\ (db)^4 = 1,\ b^{-1} \cdot db \cdot b = (db)^{-1}$, we have

$$\mathrm{Im}x_2 = (\langle b\rangle \times \langle c\rangle) \rightthreetimes \langle d\rangle = \langle db\rangle \rightthreetimes \langle b\rangle \cong D_4,$$

and, by Lemma 3.2, $K(x_1) \cong \mathrm{End}(D_4)$. Therefore, properties 2^0 and 3^0 hold.

In view of Lemma 3.2, the set $\{z \in K(x_2) \mid xz = z,\ zx_1 = 0\}$ consists of endomorphisms z such that

$$(\mathrm{Ker}x_2)z = \langle 1\rangle,\ (\mathrm{Im}x_2)z \subset \mathrm{Im}x_2,$$

$$(\mathrm{Im}x_2 \cap \mathrm{Ker}x)z = \langle 1\rangle,\ (\mathrm{Im}x)z \subset \mathrm{Ker}x_1 \cap \mathrm{Im}x_2,$$

i.e., each such z is uniquely induced by a homomorphism

$$\mathrm{Im}x = \langle d\rangle \xrightarrow{z} \mathrm{Ker}x_1 \cap \mathrm{Im}x_2 = \langle b\rangle \times \langle c\rangle \cong C_2 \times C_2.$$

The number of such homomorphisms is 4. Property 4^0 is proved.

Similarly to the previous case, the set $\{z \in K(x_1) \mid xz = z,\ zx_2 = 0\}$ consists of endomorphisms z which are induced by a homomorphism

$$\mathrm{Im}x = \langle d\rangle \xrightarrow{z} \mathrm{Ker}x_2 \cap \mathrm{Im}x_1 = \langle a\rangle \cong C_4.$$

The number of such homomorphisms is 2. Property 5^0 is proved. The necessity is proved.

Sufficiency. Let G be a finite group and let there exist $x, x_1, x_2 \in I(G)$ which satisfy properties 1^0–5^0 of the theorem. Our aim is to prove that $G \cong \mathscr{G}_{36}$.

By property 1^0, G splits into semidirect products (5.1), where equalities (5.2) and (5.3) hold. In view of Lemma 3.2, properties 2^0 and 3^0 imply

$$\mathrm{End}(\mathrm{Im}x) \cong \mathrm{End}(C_2),\ \mathrm{End}(\mathrm{Im}x_1) \cong \mathrm{End}(\mathrm{Im}x_2) \cong \mathrm{End}(D_4).$$

Since each finite Abelian group and the group D_4 are determined by their endomorphism semigroups in the class of all groups ([7], Theorem 4.2 and [9], Corollary 3.7), we have

$$K = \mathrm{Im}x = \langle d\rangle \cong C_2,\ \mathrm{Im}x_1 \cong \mathrm{Im}x_2 \cong D_4 \tag{5.4}$$

for an element $d \in G$.

In view of (5.2) and (5.3),

$$\mathrm{Im}x_1 = G_1 \rightthreetimes K = (\mathrm{Ker}x \cap \mathrm{Im}x_1) \rightthreetimes \mathrm{Im}x. \tag{5.5}$$

Similarly to the proof of the necessity of property 5^0, each $z \in K(x_1)$ for which $xz = z,\ zx_2 = 0$ satisfies the conditions

$$(\mathrm{Ker}x)z = \langle 1\rangle,\ (\mathrm{Im}x)z \subset \mathrm{Ker}x \cap \mathrm{Im}x_1, \tag{5.6}$$

and is uniquely induced by a homomorphism $\mathrm{Im}x \longrightarrow \mathrm{Ker}x \cap \mathrm{Im}x_1$. Since $\mathrm{Im}x = \langle d\rangle \cong C_2$ and, by property 5^0, the number of such homomorphisms is two, the subgroup $G_1 = \mathrm{Ker}x \cap \mathrm{Im}x_1$ of $\mathrm{Im}x_1$ is cyclic ([14], Theorem 5.46). Therefore, $\mathrm{Ker}x \cap \mathrm{Im}x_1 = \langle a\rangle$ for some $a \in G$. It follows from (5.4)–(5.6) that

$$\mathrm{Im}x_1 = G_1 \rightthreetimes K = \langle a\rangle \rightthreetimes \langle d\rangle \cong D_4,\ d^2 = a^4 = 1,\ d^{-1}ad = a^{-1}. \tag{5.7}$$

In view of (5.2) and (5.3),

$$\mathrm{Im}x_2 = G_2 \rightthreetimes K = (\mathrm{Ker}x \cap \mathrm{Im}x_2) \rightthreetimes \mathrm{Im}x.$$

Similarly to the previous case, each $z \in K(x_2)$ for which $xz = z$, $zx_1 = 0$, is uniquely induced by a homomorphism $\mathrm{Im}\,x \longrightarrow \mathrm{Ker}\,x \cap \mathrm{Im}\,x_2$. Since $\mathrm{Im}\,x = \langle d \rangle \cong C_2$ and $\mathrm{Im}\,x_2 \cong D_4$, property 4^0 implies that

$$G_2 = \mathrm{Ker}\,x \cap \mathrm{Im}\,x_2 \cong C_2 \times C_2.$$

Therefore,

$$\mathrm{Im}\,x_2 = (\langle b \rangle \times \langle c \rangle) \leftthreetimes \langle d \rangle \cong D_4$$

for some $b, c \in \mathrm{Ker}\,x \cap \mathrm{Im}\,x_2$. By the properties of D_4, b and c can be chosen so that $dc = cd$ and $d^{-1}bd = bc$. Hence

$$\mathrm{Im}\,x_2 = \langle b, c, d \mid b^2 = c^2 = d^2 = 1,\ bc = cb,\ cd = dc,\ d^{-1}bd = bc \rangle. \tag{5.8}$$

It follows from (5.1), (5.7), and (5.8) that

$$G = \langle a, b, c, d \mid a^4 = b^2 = c^2 = d^2 = 1,\ ab = ba,\ ac = ca,\ bc = cb,\ dc = cd,\ d^{-1}ad = a^{-1},\ d^{-1}bd = bc \rangle,$$

i.e., the groups G and $\mathscr{G}_{36}$ are isomorphic. The sufficiency is proved and the theorem is also proved.

Theorem 5.2. *The group $\mathscr{G}_{36}$ is determined by its endomorphism semigroup in the class of all groups.*

The proof of Theorem 5.2 is similar to the proof of Theorem 4.2.

6. GROUP $\mathscr{G}_{37}$

In this section, we shall characterize the group

$$\begin{aligned}\mathscr{G}_{37} =& \langle a, b, c, d \mid a^4 = b^2 = c^2 = d^4 = 1,\ ab = ba,\ ac = ca,\\ & bc = cb,\ dc = cd,\ d^2 = a^2,\ d^{-1}ad = a^{-1},\ d^{-1}bd = bc \rangle\end{aligned} \tag{6.1}$$

by its endomorphism semigroup. We will prove that the isomorphism $\mathrm{End}(G) \cong \mathrm{End}(\mathscr{G}_{37})$, where G is another group, implies the isomorphism $G \cong \mathscr{G}_{37}$.

Elements a and d in (6.1) generate a subgroup isomorphic to Q:

$$Q = \langle a, d \mid a^4 = 1,\ d^2 = a^2,\ d^{-1}ad = a^{-1} \rangle.$$

The group $\mathscr{G}_{37}$ splits into the following semidirect products:

$$\mathscr{G}_{37} = (\langle b \rangle \times \langle c \rangle) \leftthreetimes \langle a, d \rangle = (\langle b \rangle \times \langle c \rangle) \leftthreetimes Q = \langle a, d, c \rangle \leftthreetimes \langle b \rangle. \tag{6.2}$$

Theorem 6.1. *A finite group G is isomorphic to $\mathscr{G}_{37}$ if and only if* $\mathrm{Aut}(G)$ *is a* 2*-group and there exist $x, y \in I(G)$ such that the following properties hold*:
1^0 $K(x) \cong \mathrm{End}(Q)$;
2^0 $K(y) \cong \mathrm{End}(C_2)$;
3^0 $yx = xy = 0$;
4^0 *if* $z \in \mathrm{End}(G)$ *and* $xz = yz = 0$, *then* $z = 0$;
5^0 $|J(x) \cap H(y)| = 2$;
6^0 $|\{z \in H(x) \mid zy = 0\}| = 4$;
7^0 $|\{z \in \mathrm{End}(G) \mid xz = z,\ zx = x,\ zy = 0\}| = 4$;
8^0 $|\{z \in \mathrm{End}(G) \mid zy = y,\ yz = z,\ zx = 0\}| = 2$.

Proof. Necessity. Let $G = \mathscr{G}_{37}$ and G be given by (6.2). It was proved in [5] that $|\mathrm{Aut}(G)| = 2^7$. Denote by x and y the projections of G onto its subgroups $Q = \langle a, d\rangle$ and $\langle b\rangle$, respectively. Then $x, y \in I(G)$ and

$$\mathrm{Im}\,x = Q = \langle a, d\rangle,\ \mathrm{Ker}\,x = \langle b\rangle \times \langle c\rangle,\ \mathrm{Im}\,y = \langle b\rangle \cong C_2,\ \mathrm{Ker}\,y = \langle d, a, c\rangle.$$

We shall prove that x and y satisfy properties 1^0–8^0.

By Lemma 3.2, properties 1^0 and 2^0 hold. Since $\mathrm{Im}\,x \subset \mathrm{Ker}\,y$ and $\mathrm{Im}\,y \subset \mathrm{Ker}\,x$, property 3^0 is true. Property 4^0 also holds, because $z \in \mathrm{End}(G)$ and $xz = yz = 0$ imply $az = bz = cz = dz = 0$, i.e., $z = 0$.

In view of Lemmas 3.3 and 3.6, each $z \in J(x) \cap H(y)$ acts on the generators of G as follows:

$$az = dz = cz = 1,\ bz = c^i;\ \ i \in \mathbb{Z}_2. \tag{6.3}$$

The map z, given by (6.3), preserves the generating relations of G, and, therefore, induces an endomorphism of G for each $i \in \mathbb{Z}_2$. Hence $|J(x) \cap H(y)| = 2$ and property 5^0 holds.

By Lemma 3.6, each $z \in H(x)$, where $zy = 0$, acts on the generators of G as follows:

$$az = c^i,\ bz = cz = 1,\ dz = c^j;\ \ i, j \in \mathbb{Z}_2. \tag{6.4}$$

The map z, given by (6.4), preserves the generating relations of G, and, therefore, induces an endomorphism of G for each $i, j \in \mathbb{Z}_2$. Hence $|\{z \in H(x) \mid zy = 0\}| = 4$ and property 6^0 holds.

An endomorphism z of G satisfies the equalities $xz = z$, $zx = x$, and $zy = 0$ if and only if $\mathrm{Ker}\,x \subset \mathrm{Ker}\,z$, $\mathrm{Im}\,z \subset \mathrm{Ker}\,y$, $g^{-1} \cdot gz \in \mathrm{Ker}\,x$, $g \in G$, i.e.,

$$az = ac^i,\ bz = cz = 1,\ dz = dc^j \tag{6.5}$$

for some $i, j \in \mathbb{Z}_2$. The map z, given by (6.5), preserves the generating relations of G, and, therefore, induces an endomorphism of G for each $i, j \in \mathbb{Z}_2$. The number of such endomorphisms z is 4, i.e. property 7^0 holds. The proof of property 8^0 is similar. The necessity is proved.

Sufficiency. Let G be a finite group such that $\mathrm{Aut}(G)$ is a 2-group and there exist $x, y \in I(G)$ which satisfy properties 1^0–8^0 of the theorem. Our aim is to prove that $G \cong \mathscr{G}_{37}$.

In view of Lemma 3.2, properties 1^0 and 2^0 imply

$$\mathrm{End}(\mathrm{Im}\,x) \cong \mathrm{End}(Q),\ \mathrm{End}(\mathrm{Im}\,y) \cong \mathrm{End}(C_2).$$

Since each finite Abelian group and the quaternion group Q are determined by their endomorphism semigroups in the class of all groups ([7], Theorem 4.2 and [8], Corollary 1), we have

$$\mathrm{Im}\,x = \langle a, d \mid a^4 = 1,\ a^2 = d^2,\ d^{-1}ad = a^{-1}\rangle \cong Q,$$

$$\mathrm{Im}\,y = \langle b\rangle \cong C_2$$

for some $a, b, d \in G$.

By Lemma 3.4 and property 3^0, G decomposes into semidirect products as follows:

$$G = (N \leftthreetimes \mathrm{Im}\,x) \leftthreetimes \mathrm{Im}\,y = (N \leftthreetimes \mathrm{Im}\,y) \leftthreetimes \mathrm{Im}\,x,$$

$$\mathrm{Ker}\,x = N \leftthreetimes \mathrm{Im}\,y,\ \mathrm{Ker}\,y = N \leftthreetimes \mathrm{Im}\,x,$$

where

$$N = \mathrm{Ker}\,x \cap \mathrm{Ker}\,y.$$

Since $\mathrm{Aut}(G)$ is a 2-group, $\widehat{g} = 1$ for each $2'$-element g of G. Hence all $2'$-elements of G belong into its centre $Z(G)$. Therefore, the group G splits into the direct product $G = G_{2'} \times G_2$ of its Hall $2'$-subgroup $G_{2'}$

and Sylow 2-subgroup G_2. Denote by z the projection of G onto its subgroup $G_{2'}$. Clearly, $zx = zy = 0$, and, by property 4^0, $z = 0$, i.e. $G_{2'} = \langle 1\rangle$ and G is a 2-group.

In view of Lemmas 3.3 and 3.6, each $z \in J(x) \cap H(y)$ is uniquely induced by a homomorphism $\mathrm{Im}\, y = \langle b\rangle \longrightarrow N$. By property 5^0, the number of such homomorphisms is 2. Therefore, the subgroup N of G has only one element of order 2. Hence N is cyclic or a generalized quaternion group ([14], Theorem 5.46). Assume that N is a generalized quaternion group Q_m for some $m \geq 2$. By Lemma 3.6, each $z \in H(x)$, $zy = 0$, is uniquely induced by a homomorphism $\mathrm{Im}\, x = Q \longrightarrow N$. Since Q is a subgroup of Q_m and $|\mathrm{Aut}(Q)| = 24$, the number of such homomorphisms is ≥ 24. This contradicts property 6^0. Hence N is cyclic, i.e.,

$$N = \langle c\rangle \cong C_{2^n}$$

for some $c \in N$ and $n \geq 1$. Note that the element $c^{2^{n-1}}$ belongs into the centre $Z(G)$ of G.

Let us consider the map

$$z_{ij} = xu_{ij} : G \xrightarrow{x} Q = \langle a, d\rangle \xrightarrow{u_{ij}} G,$$

$$du_{ij} = dc^{i2^{n-1}},\ au_{ij} = ac^{j2^{n-1}};\ i, j \in \mathbb{Z}_2.$$

It is easy to check that u_{ij} preserves the generating relations of Q, and, therefore, it is a homomorphism. Hence $z_{ij} \in \mathrm{End}(G)$. The number of such endomorphisms is 4 and these endomorphisms satisfy equalities

$$xz_{ij} = z_{ij},\ z_{ij}x = x,\ z_{ij}y = 0.$$

By property 7^0,

$$\{z \in \mathrm{End}(G) \mid xz = z,\ zx = x,\ zy = 0\} = \{z_{ij} \mid i, j \in \mathbb{Z}_2\}. \tag{6.6}$$

Since

$$x(x\widehat{c}) = x\widehat{c},\ (x\widehat{c})x = x,\ (x\widehat{c})y = 0,$$

it follows from (6.6) that $x\widehat{c} = z_{ij}$ for some $i, j \in \mathbb{Z}_2$ and we have

$$c^{-1}dc = dc^{i2^{n-1}},\ c^{-1}ac = ac^{j2^{n-1}}. \tag{6.7}$$

Similarly to (6.7), looking for endomorphisms $y\widehat{c}$, $y\widehat{d}$, and $y\widehat{a}$, property 8^0 implies that

$$c^{-1}bc = bc^{s2^{n-1}},\ d^{-1}bd = bc^{t2^{n-1}},\ a^{-1}ba = bc^{v2^{n-1}} \tag{6.8}$$

for some $s, t, v \in \mathbb{Z}_2$.

Denote

$$M = \langle a, b, d, c^{2^{n-1}}\rangle.$$

In view of (6.7) and (6.8), M is an invariant subgroup of G. Clearly,

$$G/M = \langle cM\rangle \cong C_{2^{n-1}} \cong \langle c^2\rangle.$$

Define $z = \pi w$, where $\pi : G \longrightarrow G/M$ is the natural homomorphism and $w : G/M = \langle cM\rangle \longrightarrow \langle c^2\rangle$, $(cM)w = c^2$. Then $xz = yz = 0$, and, by property 4^0, $z = 0$. Hence $n = 1$, $c^2 = 1$ and (6.6)–(6.8) imply

$$cd = dc,\ ac = ca,\ bc = cb,\ d^{-1}bd = bc^t,\ a^{-1}ba = bc^v$$

for some $t, v \in \mathbb{Z}_2$ ($i = j = s = 0$, because of $c = c^{2^{n-1}} \in Z(G)$). If $t = v = 0$, then $G = \langle c\rangle \times \langle b\rangle \times \langle a, d\rangle$ and the projection z of G onto $\langle c\rangle$ satisfies equalities $xz = yz = 0$, which contradicts property 4^0. If $(t, v) = (1, 0)$, $(t, v) = (0, 1)$ or $(t, v) = (1, 1)$, then the group G is isomorphic to $\mathscr{G}_{37}$. The corresponding isomorphisms are

$$a\varphi = a,\ d\varphi = d,\ b\varphi = b,\ c\varphi = c,$$

$$a\varphi = d,\ d\varphi = a,\ b\varphi = b,\ c\varphi = c,$$

$$a\varphi = da^{-1},\ d\varphi = d,\ b\varphi = b,\ c\varphi = c,$$

respectively (on the left sides of the given equalities are the generators a, b, c, d of G and on the right sides are the generators a, b, c, d of $\mathscr{G}_{37}$). We have proved that $G \cong \mathscr{G}_{37}$. The sufficiency is proved. The theorem is proved.

Theorem 6.2. *The group $\mathscr{G}_{37}$ is determined by its endomorphism semigroup in the class of all groups.*

The proof of Theorem 6.2 is similar to the proof of Theorem 4.2.

7. GROUP $\mathscr{G}_{38}$

In this section, we shall characterize the group

$$\mathscr{G}_{38} = \langle a, b, c, d \mid a^4 = b^2 = c^2 = d^2 = 1,\ ab = ba,\ ac = ca,\ bc = cb,\ dc = cd,\ d^{-1}ad = ac,\ d^{-1}bd = ba^2\rangle \tag{7.1}$$

by its endomorphism semigroup. We will prove that the isomorphism $\mathrm{End}(G) \cong \mathrm{End}(\mathscr{G}_{38})$, where G is another group, implies the isomorphism $G \cong \mathscr{G}_{38}$.

Theorem 7.1. *A finite group G is isomorphic to $\mathscr{G}_{38}$ if and only if* $\mathrm{Aut}(G)$ *is a* 2*-group and there exist $x, y \in I(G)$ such that the following properties hold*:
1^0 $K(x) \cong K(y) \cong \mathrm{End}(C_2)$;
2^0 $yx = xy = 0$;
3^0 *if $z \in I(G)$ and $x, y \in K(z)$, then $z = 1$;*
4^0 $J(x) \cap J(y) \cap I(G) = \{0\}$;
5^0 $|J(x) \cap J(y)| = 4$;
6^0 $|\{z \in \mathrm{End}(G) \mid xz = z,\ zx = zy = 0\}| = 4$;
7^0 $|[x]| = 4$ *and if $z \in [x]$, then* $z \cdot (J(x) \cap J(y)) = \{0\}$;
8^0 $|[y]| = 4$ *and if $z \in [y]$, then* $z \cdot (J(x) \cap J(y)) = \{0\}$;
9^0 $|J(x) \cap P(y)| = |J(y) \cap P(x)| = 4$;
10^0 $(J(x) \cap P(y)) \cdot (J(x) \cap J(y)) = (J(y) \cap P(x)) \cdot (J(x) \cap J(y)) = \{0\}$;
11^0 $[x] \cdot (J(x) \cap P(y)) = \{0\},\ [y] \cdot (J(x) \cap P(y)) = \{y\}$;
12^0 $[y] \cdot (J(y) \cap P(x)) = \{0\},\ [x] \cdot (J(y) \cap P(x)) = \{x\}$;
13^0 $\{z \in \mathrm{Aut}(G) \mid xzy = xz,\ yzx = yz\} = \emptyset$.

Proof. Necessity. Let $G = \mathscr{G}_{38}$ and G be given by (7.1). It was proved in [5] that $|\mathrm{Aut}(G)| = 2^7$, i.e., $\mathrm{Aut}(G)$ is a 2-group. The group G splits into the following semidirect products:

$$G = (\langle a\rangle \times \langle b\rangle \times \langle c\rangle) \leftthreetimes \langle d\rangle = ((\langle a\rangle \times \langle c\rangle) \leftthreetimes \langle d\rangle) \leftthreetimes \langle b\rangle.$$

Denote by x and y the projections of G onto its subgroups $\langle d\rangle$ and $\langle b\rangle$, respectively. Then $x, y \in I(G)$ and

$$\mathrm{Im}\,x = \langle d\rangle,\ \mathrm{Ker}\,x = \langle a\rangle \times \langle b\rangle \times \langle c\rangle,\ \mathrm{Im}\,y = \langle b\rangle,\ \mathrm{Ker}\,y = (\langle a\rangle \times \langle c\rangle) \leftthreetimes \langle d\rangle.$$

We shall prove that x and y satisfy properties 1^0–13^0.

By Lemma 3.2 and the definition of x and y, properties 1^0 and 2^0 hold. By Lemma 3.2, each $z \in I(G)$ such that $x, y \in K(z)$ is given on the generators of G as follows:

$$dz = d,\ bz = b,\ az = a^i c^j,\ cz = a^k c^l \tag{7.2}$$

$(i, k \in \mathbb{Z}_4;\ \ j, l \in \mathbb{Z}_2)$. Map (7.2) preserves the generating relations of G and induces an idempotent endomorphism of G if and only if $z = 1$. Hence property 3^0 holds.

By Lemma 3.3, $J(x) \cap J(y)$ consists of $z \in \mathrm{End}(G)$ such that

$$dz = bz = 1,\ az = a^i c^j,\ cz = a^k c^l;\ i, k \in \mathbb{Z}_4;\ j, l \in \mathbb{Z}_2. \tag{7.3}$$

Map (7.3) preserves the generating relations of G if and only if $k = l = 0$ and $i \equiv 0\,(\mathrm{mod}\,2)$. Therefore, $|J(x) \cap J(y)| = 4$ and property 5^0 is true. An endomorphism z given by (7.3) is an idempotent if and only if $i = j = k = l = 0$, i.e., $z = 0$. Hence property 4^0 holds.

Assume that $z \in \mathrm{End}(G)$ and $xz = z$, $zx = zy = 0$. Then $\mathrm{Ker}\,x \subset \mathrm{Ker}\,z$, $\mathrm{Im}\,z \subset \mathrm{Ker}\,x \cap \mathrm{Ker}\,y$, i.e.,

$$bz = az = cz = 1,\ dz = a^i c^j;\ i \in \mathbb{Z}_4;\ j \in \mathbb{Z}_2. \tag{7.4}$$

Map (7.4) preserves the generating relations of G if and only if $i \equiv 0\,(\mathrm{mod}\,2)$. Therefore, the number of such z is 4 and property 6^0 is true.

By Lemma 3.8, $[x]$ consists of the maps w such that

$$bw = aw = cw = 1,\ dw = da_1, \tag{7.5}$$

where $a_1 \in \mathrm{Ker}\,x = \langle a, b, c\rangle$ and $(da_1)^2 = 1$. Easy calculations show that these conditions satisfy only elements $a_1 = a^{2i_0} c^{m_0};\ i_0, m_0 \in \mathbb{Z}_2$. Hence $|[x]| = 4$. Choose $w \in [x]$ and $z \in J(x) \cap J(y)$ given by (7.5) and (7.3), respectively. Then $d(wz) = b(wz) = a(wz) = c(wz) = 1$, i.e., $wz = 0$. Therefore, property 7^0 is true. Similarly, property 8^0 holds.

By Lemmas 3.3 and 3.7, $J(x) \cap P(y)$ consists of the maps u such that

$$du = 1,\ bu = b,\ au = a^m c^n,\ cu = a^s c^t;\ m, s \in \mathbb{Z}_4;\ n, t \in \mathbb{Z}_2. \tag{7.6}$$

Map (7.6) preserves the generating relations of G if and only if $s = t = 0,\ m \equiv 0\,(\mathrm{mod}\,2)$. Therefore, $|J(x) \cap P(y)| = 4$. Similarly, $|J(y) \cap P(x)| = 4$. We have obtained property 9^0.

Choose $u \in J(x) \cap P(y)$ and $z \in J(x) \cap J(y)$ given by (7.6) and (7.3), respectively. Then $k = l = s = t = 0,\ i \equiv m \equiv 0\,(\mathrm{mod}\,2)$, and $d(uz) = b(uz) = a(uz) = c(uz) = 1$, i.e., $uz = 0$. Hence $(J(x) \cap P(y)) \cdot (J(x) \cap J(y)) = \{0\}$. Similarly, $(J(y) \cap P(x)) \cdot (J(x) \cap J(y)) = \{0\}$. Therefore, property 10^0 is true.

To prove property 11^0, choose $w \in [x]$ and $u \in J(x) \cap P(y)$. Then w and u are given by (7.5) and (7.6), respectively, where $a_1 = a^{2i_0} c^{m_0},\ s = t = 0,\ m \equiv 0\,(\mathrm{mod}\,2)$. Calculating wu, we get $wu = 0$. Hence $[x] \cdot (J(x) \cap P(y)) = \{0\}$. Similarly to $[x]$, the set $[y]$ consists of maps v such that

$$dv = av = cv = 1,\ bv = ba_2,\ a_2 = a^{2j_0} c^{k_0};\ j_0, k_0 \in \mathbb{Z}_2.$$

We have

$$d(vu) = a(vu) = c(vu) = 1,\ b(vu) = (ba^{2j_0} c^{k_0})u = b(a^m c^n)^{2j_0} = b,$$

i.e., $vu = y$. Therefore, $[y] \cdot (J(x) \cap P(y)) = \{y\}$. Property 11^0 is proved. The proof of property 12^0 is similar.

Finally, let us prove property 13^0. Choose $z \in \mathrm{Aut}(G)$ such that $xzy = xz$ and $yzx = yz$. Since $d(xzy) = d(zy) = (dz)y$ and $d(xz) = dz$, we have $(dz)y = dz$, and, by Lemma 3.1, $dz = b$. Similarly, $yzx = yz$ implies $bz = d$. The equality $ab = ba$ implies $az \cdot bz = bz \cdot az$ and hence $az \cdot d = d \cdot az$. The centralizer of d in G consists of elements $d^i a^{2j} c^k$, where $i, j, k \in \mathbb{Z}_2$. Therefore, $az = d^i a^{2j} c^k$ and $(az)^2 = (d^i a^{2j} c^k)^2 = 1$. It is impossible, because a and az are elements of order 4. This contradiction proves property 13^0. The necessity is proved.

Sufficiency. Let G be a finite group such that $\mathrm{Aut}(G)$ is a 2-group and there exist $x, y \in I(G)$ which satisfy properties 1^0–13^0 of the theorem. Our aim is to prove that $G \cong \mathscr{G}_{38}$.

In view of Lemma 3.2, property 1^0 implies

$$\mathrm{End}(\mathrm{Im}\,x) \cong \mathrm{End}(\mathrm{Im}\,y) \cong \mathrm{End}(C_2).$$

Since each finite Abelian group is determined by its endomorphism semigroups in the class of all groups, we have

$$\mathrm{Im}\,x = \langle d \rangle \cong C_2, \;\; \mathrm{Im}\,y = \langle b \rangle \cong C_2$$

for some $b, d \in G$. By Lemma 3.4 and property 2^0, G decomposes into semidirect products as follows:

$$G = (N \rtimes \mathrm{Im}\,x) \rtimes \mathrm{Im}\,y = (N \rtimes \mathrm{Im}\,y) \rtimes \mathrm{Im}\,x, \tag{7.7}$$

$$\mathrm{Ker}\,x = N \rtimes \mathrm{Im}\,y, \;\; \mathrm{Ker}\,y = N \rtimes \mathrm{Im}\,x,$$

where

$$N = \mathrm{Ker}\,x \cap \mathrm{Ker}\,y.$$

Therefore,

$$G/N = \langle dN \rangle \times \langle bN \rangle \cong C_2 \times C_2, \;\; G' \subset N. \tag{7.8}$$

Since $\mathrm{Aut}(G)$ is a 2-group, $\widehat{g} = 1$ for each $2'$-element g of G. Hence all $2'$-elements of G belong into its centre $Z(G)$. Therefore, the group G splits into the direct product $G = G_{2'} \times G_2$ of its Hall $2'$-subgroup $G_{2'}$ and Sylow 2-subgroup G_2. Denote by z the projection of G onto its subgroup $G_{2'}$. Clearly, $z \in J(x) \cap J(y) \cap I(G)$, and, by property 4^0, $z = 0$, i.e. $G_{2'} = \langle 1 \rangle$ and G is a 2-group.

Each $z \in \mathrm{End}(G)$, where $xz = z$, $zx = zy = 0$, is product $z = \pi u$ of the natural homomorphism $\pi : G \longrightarrow G/\langle N, b \rangle = \langle d\langle N, b \rangle \rangle \cong C_2$ and a homomorphism $u : G/\langle N, b \rangle \longrightarrow N$. By property 6^0, the number of such homomorphisms u is 4. Hence N contains four elements g such that $g^2 = 1$. Since N is a normal subgroup of G, one of the elements of order 2 of N belongs to the centre of G. Therefore, N contains three elements of order 2 and they commute with each other. Denote these elements of order 2 by c_1, c_2, and c_3. Clearly, $c_1c_2 = c_3$, $c_1c_3 = c_2$, $c_2c_3 = c_1$.

By property 5^0, we can choose non-zero $z \in J(x) \cap J(y)$. Then $d, b \in \mathrm{Ker}\,z \neq G$ and $G = N \cdot \mathrm{Ker}\,z$. There exists a normal subgroup M of G such that $\mathrm{Ker}\,z \subset M$ and $G/M \cong C_2$, i.e., $G = N \cdot M$ and $G' \subset M$. On the other hand, $G' \subset N$. If $G' = N$, then $N \subset M$ and $G = N \cdot M = M$, which contradicts $G/M \cong C_2$. Hence G' is a proper subgroup of N and, in view of (7.7) and (7.8), the factor-group G/G' splits into a direct product

$$G/G' = \langle d_1G' \rangle \times \ldots \times \langle d_kG' \rangle \times \langle dG' \rangle \times \langle bG' \rangle,$$

where $d_1, \ldots, d_k \in N \setminus G'$, $k \geq 1$, and $\langle dG' \rangle \cong \langle bG' \rangle \cong C_2$. Define $z_{ijl} \in \mathrm{End}(G)$ as follows:

$$z_{ijl} = \pi\pi_i\tau_i : G \xrightarrow{\pi} G/G' \xrightarrow{\pi_i} \langle d_iG' \rangle \xrightarrow{\tau_i} \langle c_j^l \rangle,$$

where π is the natural homomorphism, π_i is the projection of G/G' onto $\langle d_iG' \rangle$, $(d_iG')\tau_i = c_j^l$, and $1 \leq i \leq k$, $l \in \mathbb{Z}_2$, $j = 1, 2, 3$. By the definition, $z_{ijl} \in J(x) \cap J(y)$. For a fixed i, the number of such endomorphisms z_{ijl} of G is 4. Property 5^0 implies that $k = 1$ and

$$J(x) \cap J(y) = \{z_{111}, z_{121}, z_{131}, 0\}. \tag{7.9}$$

Hence

$$G/G' = \langle aG' \rangle \times \langle dG' \rangle \times \langle bG' \rangle, \;\; G' \subset N = \mathrm{Ker}\,x \cap \mathrm{Ker}\,y \tag{7.10}$$

($a = d_1$). Note that $N/G' = \langle aG' \rangle \cong C_2$, because otherwise $J(x) \cap J(y)$ contains an element z different from 0, $z_{111}, z_{121}, z_{131}$: $z = \pi\pi_i\tau$, where

$$\langle aG' \rangle = \langle d_1G' \rangle \xrightarrow{\tau} \langle a^{m/4} \rangle, \; (d_1G')\tau = a^{m/4}$$

and m is the order of a. If $a^2 = 1$, then $G = \langle G', d, b\rangle \rightthreetimes \langle a\rangle$ and the projection u of G onto $\langle a\rangle$ belongs to $J(x) \cap J(y) \cap I(G)$, which contradicts property 4^0. Therefore, $a^2 \neq 1$ and the elements c_1, c_2, c_3 of order 2 of N belong to G'. Note that

$$bd \neq db, \tag{7.11}$$

because otherwise $G = N \rightthreetimes (\langle b\rangle \times \langle d\rangle)$, and the projection z of G onto $\langle b\rangle \times \langle d\rangle$ satisfies conditions $z \in I(G)$, $z \neq 1$; $x, y \in K(z)$, which contradicts property 3^0.

The derived subgroup G' of G does not contain any subgroup M of G such that $K = \langle d, b, M\rangle$ is a normal subgroup of G and $K \neq \langle d, b, G'\rangle$. To prove this, assume that there exist a subgroup M of G such that $K = \langle d, b, M\rangle$ is a normal subgroup of G and $K \neq \langle d, b, G'\rangle$. Then there exists a normal subgroup L of G such that $K \subset L$ and $G/L \cong C_4$ or $G/L \cong C_2 \times C_2$. Consider the endomorphism $z = \pi u$ of G, where $\pi : G \longrightarrow G/L$ is the natural homomorphism and u is an isomorphism $G/L \longrightarrow \langle a^m\rangle$ (if $G/L \cong C_4$) or an isomorphism $G/L \longrightarrow \langle c_1\rangle \times \langle c_2\rangle$ (if $G/L \cong C_2 \times C_2$) and a^m is a power of a with order 4. By the definition of z, we have $z \in J(x) \cap J(y)$ and $z \notin \{z_{111}, z_{121}, z_{131}, 0\}$. This contradicts (7.9).

Since $\operatorname{Im} x = \langle d\rangle \cong C_2$ and the set $[x]$ consists of $z \in I(G)$ such that $\operatorname{Ker} x = \operatorname{Ker} z$, we have $\operatorname{Im} z = \langle dc\rangle \cong C_2$, where $c \in \operatorname{Ker} x$, and $|[x]|$ is equal to the number of elements dc, $c \in \operatorname{Ker} x$, of order 2. By property 7^0, the number of such elements is 4 and $c \in G'$. Similarly, by property 8^0, $|[y]|$ is equal to the number of elements bc, $c \in G'$ and the number of such elements is 4. Denote

$$D = \{c \in G' \mid (dc)^2 = 1\}, \ \ B = \{c \in G' \mid (bc)^2 = 1\}.$$

Then $1 \in D$, $1 \in B$, and

$$|D| = |B| = 4. \tag{7.12}$$

Choose $c \in D$. Then $d^{-1}cd = c^{-1}$, i.e., $(dc^i)^2 = 1$, $c^i \in D$ for each integer i, and, by (7.12), $c^4 = 1$. If c is an element of order 4, then D is a cyclic subgroup of G': $D = \langle c\rangle \cong C_4$. If D does not contain any element of order 4, then (7.12) implies that $D = \{1, c_1, c_2, c_3\}$, i.e., D is also a subgroup of G': $D = \langle c_1\rangle \times \langle c_2\rangle \cong C_2 \times C_2$. Let us prove that D and $\langle d, D\rangle$ are normal subgroups of G. Assume that $c \in D$, $g \in G$. Since $g^{-1}dg = d \cdot [d, g]$ is an element of order 2 and $[d, g] \in G'$, we have $g^{-1}dg = d\tilde{c}$, $\tilde{c} = [d, g] \in D$. Similarly, $g^{-1}dcg = g^{-1}dg \cdot g^{-1}cg = d\tilde{c} \cdot g^{-1}cg$ is an element of order 2, i.e., $\tilde{c} \cdot g^{-1}cg \in D$ and $g^{-1}cg \in D$. We have proved that $g^{-1}dg \in \langle d, D\rangle$ and $g^{-1}cg \in D$. Hence D and $\langle d, D\rangle = D \rightthreetimes \langle d\rangle$ are the normal subgroups of G. Similarly, we can prove that $B \cong C_4$ or $B = \langle c_1\rangle \times \langle c_2\rangle \cong C_2 \times C_2$ and D and $\langle b, B\rangle = B \rightthreetimes \langle b\rangle$ are normal subgroups of G. Therefore, DB and $\langle d, b, DB\rangle$ are also normal subgroups of G and $\langle d, b, DB\rangle \subset \langle d, b, G'\rangle$. It was proved above that in this case $\langle d, b, DB\rangle = \langle d, b, G'\rangle$, i.e.,

$$G' = DB.$$

Let us prove now that

$$G' = D = B \cong C_2 \times C_2.$$

To do this, we consider the sets $J(x) \cap P(y)$ and $J(y) \cap P(x)$. By Lemmas 3.3, 3.8, and property 10^0, the set $J(x) \cap P(y)$ consists of endomorphisms z of G such that

$$dz = 1, \ bz = b, \ az \in G', \ G'z \subset G'.$$

Property 11^0 implies that $Dz = Bz = G'z = \{1\}$ and $G' \subset \operatorname{Ker} z$ for such z. Since $G/G' \cong C_2 \times C_2 \times C_2$, we have $\operatorname{Im} z \cong C_2 \times C_2$ or $\operatorname{Im} z \cong C_2$, and, by property 9^0, G' has three elements of order 2 and these elements commute with b. Similarly, by properties 10^0, 11^0, and 9^0, d commutes with each element of order 2 from G'. It follows from the first parts of properties 7^0 and 8^0 that $D = B = G' \cong C_2 \times C_2$. Since $a^2 \neq 1$ and $a^2 \in G'$, we have $a^4 = 1$, i.e., a is an element of order 4. Note that a commutes with each element of G'. Indeed, $\langle a, G'\rangle$ is a group of order 8. It cannot be the quaternion group, because the quaternion group has only one element of order 2. It cannot be the dihedral group either, because the dihedral of order 8 has five elements of order 2. Therefore, the group $\langle a, G'\rangle$ is Abelian and a commutes with each element from G'. It also follows that G' is contained in the centre of G and $G = \langle d, b, a\rangle$.

Denote

$$[a,d] = a_1,\ [b,d] = a_2,\ [a,b] = a_3,$$

i.e.,

$$d^{-1}ad = aa_1,\ d^{-1}bd = ba_2,\ b^{-1}ab = aa_3.$$

Clearly, $G' = \langle a_1, a_2, a_3\rangle$ and, by (7.11), $a_2 \neq 1$. Let us prove that $G \cong \mathscr{G}_{38}$. To do this, we will separate the following three possible cases: (a) $a_1 = a_3$; (b) $a_1 \neq a_3$ and $a_1 \neq 1,\ a_3 \neq 1$; (c) $a_1 \neq a_3$ and $a_1 = 1$ or $a_3 = 1$.

Assume that $a_1 = a_3$. Then the map z, given by

$$dz = b,\ bz = d,\ az = a,\ cz = c,\ c \in G',$$

can be extended to an automorphism of G. The automorphism z satisfies equalities $xzy = xz,\ yzx = yz$, which contradicts property 13^0. Hence the case $a_1 = a_3$ is impossible.

Assume that $a_1 \neq a_3$ and $a_1 \neq 1,\ a_3 \neq 1$. Since $G' \cong C_2 \times C_2$, we have $a_2 = a_1a_3,\ a_1 = a_2a_3,\ a_3 = a_1a_2$. Then the map z, given by

$$dz = b,\ bz = d,\ az = a,\ a_1z = a_3,\ a_3z = a_1,\ a_2z = a_2,$$

can be extended to an automorphism of G. The automorphism z satisfies equalities $xzy = xz,\ yzx = yz$, which contradicts property 13^0. Hence this case is also impossible.

Assume that $a_1 \neq a_3$ and $a_1 = 1$. Then $G' = \langle a_2\rangle \times \langle a_3\rangle$ and

$$ad = da,\ d^{-1}bd = ba_2,\ b^{-1}ab = aa_3.$$

There are three possible cases: $a^2 = a_3$ or $a^2 = a_2$ or $a^2 = a_2a_3$. If $a^2 = a_3$, then $G = \langle d, a_2\rangle \rightthreetimes \langle b, a\rangle$ and the projection z of G onto the subgroup $\langle b, a\rangle$ satisfies the conditions $z \in J(x) \cap P(y)$ and $z \cdot (J(x) \cap J(y)) \neq \{0\}$, which contradicts property 10^0. If $a^2 = a_2$, then G is isomorphic to $\mathscr{G}_{38}$. Assume that $a^2 = a_2a_3$. Then

$$\begin{aligned} dba \cdot dba &= d^{-1}bd \cdot aba = ba_2 \cdot aba = a_2 \cdot b^{-1}ab \cdot a \\ &= a_2 \cdot aa_3 \cdot a = a^2 \cdot a_2a_3 = a^2 \cdot a^2 = 1 \end{aligned}$$

and, therefore, $G = \mathrm{Ker}x \rightthreetimes \langle dba\rangle$. Denote by z the projection of G onto $\langle dba\rangle$. Then $z \in [x]$ and $z \cdot (J(x) \cap J(y)) \neq \{0\}$, which contradicts property 7^0. Hence the case $a^2 = a_2a_3$ is impossible.

Assume that $a_1 \neq a_3$ and $a_3 = 1$. Then $G' = \langle a_1\rangle \times \langle a_2\rangle$ and

$$ab = ba,\ d^{-1}bd = ba_2,\ d^{-1}ad = aa_1.$$

There are three possible cases: $a^2 = a_1$ or $a^2 = a_2$ or $a^2 = a_1a_2$. If $a^2 = a_1$, then $G = \langle b, a_2\rangle \rightthreetimes \langle d, a\rangle$ and the projection z of G onto the subgroup $\langle d, a\rangle$ satisfies the conditions $z \in J(y) \cap P(x)$ and $z \cdot (J(x) \cap J(y)) \neq \{0\}$, which contradicts property 10^0. If $a^2 = a_2$, then G is isomorphic to $\mathscr{G}_{38}$. Assume that $a^2 = a_1a_2$. Then

$$\begin{aligned} dba \cdot dba &= d^{-1}bd \cdot d^{-1}ad \cdot ba = ba_2 \cdot aa_1 \cdot ba \\ &= b^2a^2a_1a_2 = a^2a^2 = 1 \end{aligned}$$

and, therefore, $G = \mathrm{Ker}x \rightthreetimes \langle dba\rangle$. Denote by z the projection of G onto $\langle dba\rangle$. Then $z \in [x]$ and $z \cdot (J(x) \cap J(y)) \neq \{0\}$, which contradicts property 7^0. Hence the case $a^2 = a_2a_3$ is impossible.

We have proved that $G \cong \mathscr{G}_{38}$. The sufficiency is proved. The theorem is proved.

Theorem 7.2. *The group $\mathscr{G}_{38}$ is determined by its endomorphism semigroup in the class of all groups.*

The proof of Theorem 7.2 is similar to that of Theorem 4.2.

ACKNOWLEDGEMENT

This work was partially supported by the Estonian Ministry of Education and Research (project SF0140011s09).

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Maksimaalset alamrühma $C_4 \times C_2 \times C_2$ omavate 32. järku rühmade endomorfismidest

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On tõestatud, et kõik 32. järku rühmad, mille üheks maksimaalseks alamrühmaks on $C_4 \times C_2 \times C_2$, on määratud oma endomorfismipoolrühmadega kõigi rühmade klassis. Ühtlasi on antud mainitud rühmade kirjeldused nende endomorfismipoolrühmade kaudu.