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MECHANICS

Tone bursts in exponentially graded materials characterized by parametric plots

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Abstract. The propagation and reflection of the ultrasonic tone burst in the strongly inhomogeneous exponentially graded material are studied. Deformations of a specimen with two parallel boundaries are described by the five constant nonlinear theory of elasticity. The one-dimensional problem is considered. The influence of the variation in material properties on the profile of boundary oscillations is clarified by parametric plots. The obtained results may be useful in the ultrasonic nondestructive material characterization.

Key words: bursts propagation, nonlinear elasticity, material inhomogeneity, parametric plots, nondestructive testing.

1. INTRODUCTION

Composites with continuously varying volume fractions of their constituents are known as functionally graded materials (FGMs) [1–4]. Most of the research in the area of designing new FGMs involves the development of graded coatings and interfacial regions for the purpose of improving the resistance of material to the different external effects (extreme temperature, intensive wastage, etc.) [5–7].

The main goal of this work is to study the possibility of characterizing FGMs with strongly variable properties close to their boundaries on the basis of the recorded data about the propagation and reflection of ultrasonic harmonic bursts in the material. A physically nonlinear inhomogeneous FGM specimen with two parallel boundaries is considered. The material's properties vary only in the thickness direction. Intensive variation of these properties close to the boundaries has an exponential functionality.

The ultrasonic tone burst (harmonic wave with the finite length) is excited on one of the boundaries of the specimen in terms of stress and the evoked boundary oscillations are recorded on both boundaries in terms of displacement or particles velocity. The one-dimensional wave propagation is described on the basis of the nonlinear theory of elasticity [8]. The governing equation of motion is solved numerically using the symbolic manipulation software Maple.

Analyses of the results of numerical simulations led to the conclusion that the variation of material properties was reverberated in boundary oscillation profiles. This phenomenon is studied using the parametric plots composed on the basis of different profiles of boundary oscillations.

The properties of the nonlinear elastic FGMs are defined by the density and by combinations of the second- and the third-order elastic coefficients. It was shown that changes of boundary oscillation profiles caused by the variable density and the linear elasticity were of the same order while changes caused by the nonlinear elasticity were small phenomena of higher order. The influence of the sign and the symmetric and asymmetric exponential variation of material properties on the modulation of boundary oscillations was studied in detail.

Comparisons between the composed parametric plots enable one to determine the sign of material properties variation and to distinguish materials with (i) homogeneous properties, (ii) symmetrically distributed properties, (iii) asymmetrically distributed properties, and also to distinguish the most relevant property of the material responsible for inhomogeneity. These results may be used as the basic principles of the method for qualitative nondestructive characterization of FGMs with essentially changing continuous properties.

2. EXPONENTIALLY GRADED MATERIALS

Functionally graded materials (FGMs) produced mostly by powder-based processes [2,4] give designers full flexibility to propose different effective material gradings for optimized performance.

In this paper the specimens of FGMs with strongly changing properties close to the boundaries are studied. Such materials may be employed in many important areas such as coatings and interfacial regions for the purpose of reducing residual and thermal stresses and increasing bounding strength.

The purpose is to clarify the influence of the variation of the properties of the exponentially graded FGMs on the propagation and reflection of ultrasonic harmonic bursts in the material. A specimen of an FGM with two parallel boundaries is considered. Deformations of the specimen with continuously variable properties are described by the five-constant nonlinear theory of elasticity [8] in Lagrangian rectangular coordinates X. Material properties vary only in the thickness direction of the specimen. The variable material properties are the density $\rho(X)$, the second-order elastic (Lamé) coefficients $\lambda(X)$, $\mu(X)$, and the third-order elastic coefficients $\nu_1(X)$, $\nu_2(X)$, and $\nu_3(X)$. One-dimensional problem is examined. In this case elastic properties of the physically nonlinear material are defined by the elastic coefficients that are grouped to the linear elastic coefficient $\alpha(X)$ and to the nonlinear elastic coefficient $\beta(X)$ [9]:

$$\begin{aligned}
&\alpha(X) = \lambda(X) + 2\,\mu(X),\\
&\beta(X) = 2\,[\,\nu_1(X) + \nu_2(X) + \nu_3(X)]. \qquad (1)
\end{aligned}$$

Exponential variation of material properties close to the boundaries of the specimen of the exponentially graded FGM is described by the expression

$$\begin{aligned}
&\gamma(X) = \gamma_0\,[1 + \gamma_{11}\exp(-\gamma_{12}X) + \gamma_{21}\exp(-\gamma_{22}(X-h))],\\
&\gamma(X) = \rho(X), \alpha(X), \beta(X), \qquad (2)
\end{aligned}$$

where h is the thickness of the specimen, γ_0 denotes the main constant part of the material properties, and constants $\gamma_{ij}(X)$, $i,j = 1,2$, characterize the variation of these properties.

The one-dimensional motion of the physically nonlinear elastic FGM is governed by the equation of motion [9], which is solved here in the form of a set of three equations

$$\begin{aligned}
&f(X,t) = V_{,X}(X,t),\\
&g(X,t) = V_{,t}(X,t),\\
&[1 + k_1(X)\,f(X,t)]\,f_{,X}(X,t) + k_2(X)\,f(X,t)\\
&\quad + k_3(X)\,f(X,t)^2 = k_4(X)\,g_{,t}(X,t), \qquad (3)
\end{aligned}$$

where V denotes the displacement, t the time, and the indices after the comma indicate differentiation with respect to coordinate X or time t. The solution to the set of equations (3) determines directly the function V and its derivatives $V_{,X}$ and $V_{,t}$. The coefficients of the set of equations (3)

$$\begin{aligned}
&k_1(X) = 3\,[\,1 + k_0(X)\,\beta(X)]\,,\\
&k_2(X) = k_0(X)\,\alpha_{,X}(X),\\
&k_3(X) = \frac{3}{2}\,k_0(X)\,[\alpha_{,X}(X) + \beta_{,X}(X)]\,,\\
&k_4(X) = \rho(X)\,k_0(X) \qquad (4)
\end{aligned}$$

are functions of the variable in space material properties. Here $k_0(X) = [\alpha(X)]^{-1}$.

3. TONE BURST PROPAGATION

The properties of FGMs are functions of space coordinates and therefore wave propagation problems related to FGMs are generally difficult to analyse without resorting to some numerical approaches. The analytical solution to the set of equations (3) that governs the motion of FGMs is unknown. Some authors derived analytical solutions to the linear governing equations for special cases of material inhomogeneity (see review in [10]).

The problem of wave propagation in FGMs with arbitrary smooth changes of material properties is solved mainly by approximating the smooth changes of material properties by piling up many homogeneous [11] or inhomogeneous [12] thin layers. An alternative approach is the combination of analytical and numerical approaches, which is used below. The problem is formulated analytically and the governing equations (3) are solved numerically by the finite difference method making use of the symbolic manipulation software Maple.

The tone burst is excited on the boundary of the specimen by the initial conditions

$$V(X,0) = V_{,t}(X,0) = 0, \qquad (5)$$

and by the boundary conditions

$$\begin{aligned}
&V_{,X}(0,t) = -\,\varepsilon\,\sin(\omega t)\,[H(t) - H(t-t_0)]\,,\\
&V_{,X}(h,t) = 0. \qquad (6)
\end{aligned}$$

Here ε is a constant, ω denotes the frequency, and $H(t)$ is Heaviside's unit step function.

The solution to the set of equations (3) under the initial and boundary conditions (5) and (6) describes the propagation of the tone burst in the sample of FGM with one free boundary (see Fig. 1). The problem is solved numerically. Properties of FGMs are determined resorting to Eq. (2) by the value of the main constant part (γ_0) of the density $\rho_0 = 6000$ kg/m^3. The values for the Lamé constants and the third-order elastic constants are approximated on the basis of the experimental data [13];

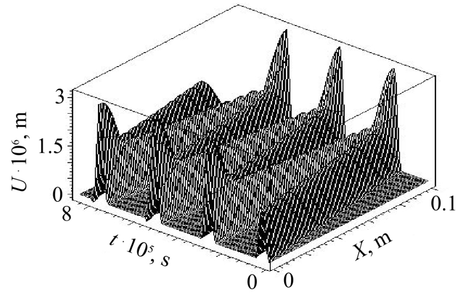

Fig. 1. Tone burst propagation in homogeneous material.

as a result the linear elastic constant is $\alpha_0 = 400$ GPa and the nonlinear elastic constant $\beta_0 = -1000$ GPa. Here the relations [14] between the Murnaghan constants [13] and the third-order elastic constants ν_i, $i = 1,2,3$ introduced by Bland [8] are taken into account. Variation of material properties is studied on the basis of Eq. (2) by the values of constants $\gamma_{i1} = \pm 0.5$, $\gamma_{i2} = 150$ m^{-1}, $i = 1,2$. The thickness of the specimen with two parallel boundaries is $h = 0.1$ m. The strain that is evoked on the boundaries by excitation is characterized by the dimensionless constant $\varepsilon = 1 \times 10^{-4}$. The frequency ω is determined from the condition that the length of two periods of waves is equal to the thickness h of the homogeneous sample with the properties determined by ρ_0, α_0, and β_0. The length of the excited burst t_0 is taken equal to one period of harmonic oscillations.

The following different cases of material inhomogeneity are considered. Case A is a symmetric case with material properties that are changing strongly and exponentially close to both boundaries. Cases B and C are asymmetric cases with the exponential change of material properties close to one of the boundaries, where X = 0 or X = h, respectively. In all the cases the positive ($\gamma_{i1} = +0.5$) and the negative ($\gamma_{i1} = -0.5$) change of material properties to the half of their basic value close to the boundaries is studied. The notation of the cases is regulated to A+, B+, C+ and A–, B–, C– (Fig. 2), respectively. It is necessary to pay attention to the dissimilar variation of the nonlinear part of elasticity $\beta(X)$ in all the cases due to its initial negative value.

The tone burst is excited on the boundary of the specimen in terms of $V_{,X}$ (see boundary conditions (6)), and the evoked oscillations may be recorded on both boundaries of the specimen in terms of particle displacement V or its derivative $V_{,t}$. Boundary oscillations in terms of $V_{,X}$ are defined by the boundary conditions (6). The program package Maple enables to determine the oscillation field on the whole $X - t$ plane (see Fig. 1). From the practical point of view it is easier to study oscillations evoked by the excited burst on the boundaries of the specimen (Fig. 3) or in some cross-section of it.

The influence of nonlinear effects on the burst propagation in the physically homogeneous FGM is characterized by the difference of the absolute values of the amplitude V of the boundary oscillations evoked by the nonlinear propagation of the burst in the physically nonlinear homogeneous material and the amplitude U evoked by the linear propagation of the burst in the physically linear homogeneous material (Fig. 4). The parametric plots with respect to time (Fig. 5) illustrate these effects more expressively.

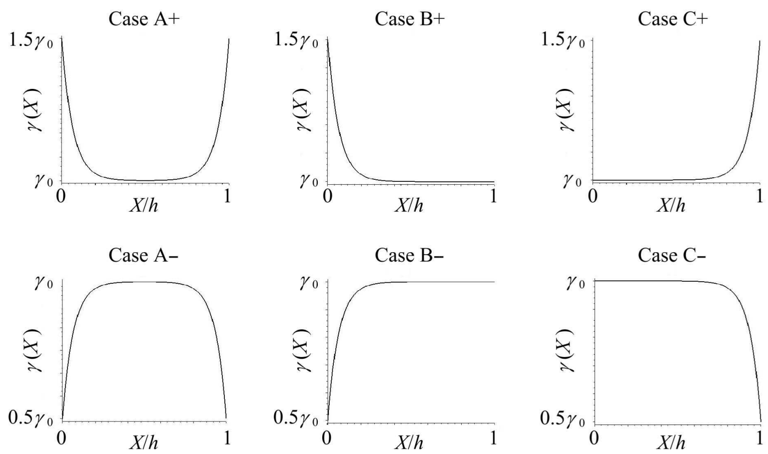

Fig. 2. Cases of material inhomogeneity.

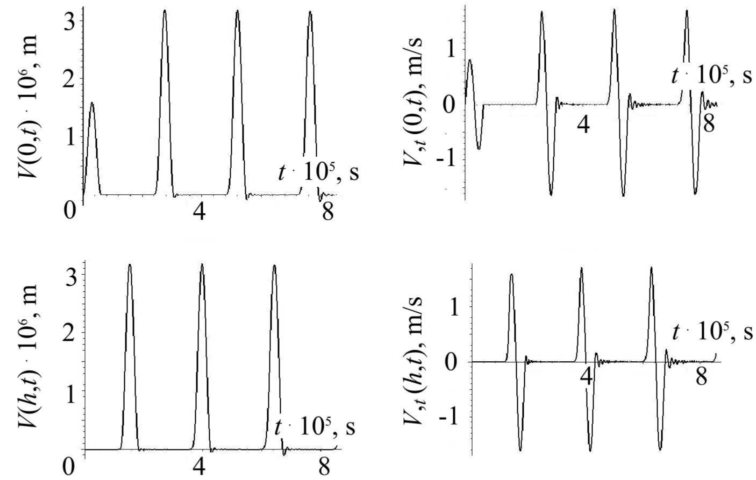

Fig. 3. Oscillations on boundaries of homogeneous material.

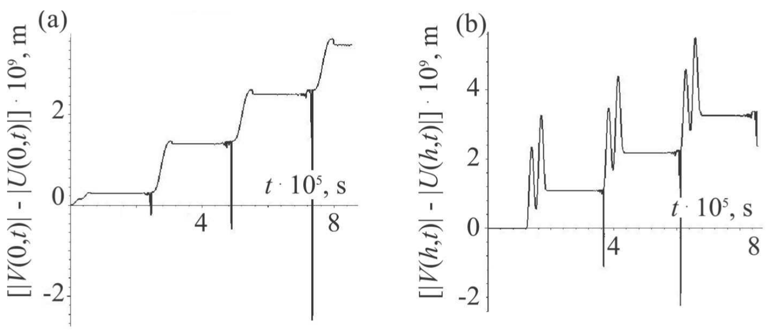

Fig. 4. Nonlinear part of oscillations on the boundaries of homogeneous material: (a) $X = 0$, (b) $X = h$.

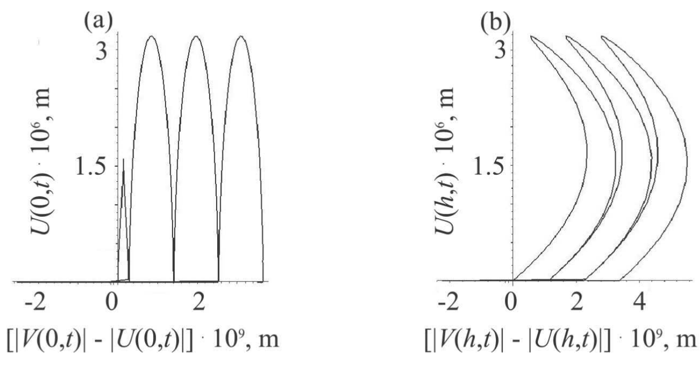

Fig. 5. Characterization of the nonlinear part of oscillations on the boundaries of homogeneous material by parametric plots: (a) $X = 0$, (b) $X = h$.

4. MATERIAL PROPERTIES VERSUS BOUNDARY OSCILLATIONS

The considered FGM has quadratic nonlinear elastic properties. The one-dimensional problem defines the properties of this FGM by the density $\rho(X)$, the linear elastic coefficient $\alpha(X)$, and the nonlinear elastic coefficient $\beta(X)$ [9]. The elastic coefficients $\alpha(X)$ and $\beta(X)$ are determined on the basis of the elastic coefficients of the well-known five-constant theory of elasticity [8] by Eq. (1).

The influence of the exponential variation of material properties $\rho(X)$, $\alpha(X)$, and $\beta(X)$ on the boundary oscillation profiles is studied numerically. The variation of these properties corresponds to the six cases (Fig. 2). Numerical values of material properties and the excited burst are described above. Here, in all cases the displacements evoked on the boundaries of the physically nonlinear inhomogeneous material by the nonlinear propagation and reflection of the burst V are compared with the displacements U evoked on the boundaries of the physically linear homogeneous material by the linear propagation and reflection of the burst. From the practical point of view it is relatively easy to find the analytical expression for the displacement U as a solution to the linear hyperbolic second-order partial differential equation with constant coefficients [15].

The results of numerical simulations confirm the fact that the variation of material properties is reverberated in boundary oscillation profiles. The relative distortion of oscillations on the boundary $X = 0$ of the specimen with the symmetric exponential variation of all material properties (case A) is plotted in Fig. 6. It is essential that the profiles of boundary oscillations caused by the variation of material properties according to the cases A+ and A– can be easily distinguished, i.e., these oscillations are sensitive to the sign of the variation of material properties. The dependence of boundary oscillations on the scheme of the variation of material properties is studied here in greater detail resorting to the parametric plots composed on the basis of different profiles of boundary oscillations.

The parametric plots in Fig. 7 illustrate the relative distortion of oscillations on the boundary $X = 0$ versus oscillations evoked by the linear propagation of the burst in the physically linear homogeneous material on the same boundary. The distortions in Fig. 7 are caused by the summary impact of the nonlinearity and the material inhomogeneity on oscillations in the physically nonlinear inhomogeneous material. The parametric plots in Fig. 5 illustrate the influence of the nonlinearity on boundary oscillations in the homogeneous linear material. Comparison of the plots in Fig. 5 and Fig. 7 enables easy determination of the presence of inhomogeneity in material properties and it verifies the fact that distortions caused by nonlinearity manifest themselves in higher order small phenomena that are measurable in practice [16].

Analysis of the plots in Fig. 7 leads to the conclusion that parametric plots are sensitive to the variation of material properties. Simultaneous variation of material properties transfigures the plots according to the case and the sign of the variation of material properties. Plots for the cases A, B, and C are qualitatively different. Cases A and B are identified by the smaller loop compared to the plots for the case C. The inclination of this loop from the vertical direction is dependent on the sign of the variation of material properties. In all cases plots that correspond to the negative sign of $\gamma_{i1}, i = 1,2$ in the expression for material properties (2) are more compact in comparison with the other plots. The conclusion is that the parametric plots presented in Fig. 7 enable to distinguish the cases of the variation of material properties.

With the view to clarify the most relevant property of the material responsible for inhomogeneity, the variation of only one material property is considered at constant values of two others. Parametric plots in Figs 8–10 illustrate the situation when only one of the material properties $\rho(X)$, $\alpha(X)$, or $\beta(X)$ is changing severally according to case A (Fig. 8), case B (Fig. 9), and case C (Fig. 10), respectively. In all cases influences of the variable density $\rho(X)$ and the linear part of elasticity $\alpha(X)$ on the burst propagation are effects of the same order whereas the variation of the nonlinear part of elasticity $\beta(X)$ induces higher-order small but very informative disturbances to the burst propagation.

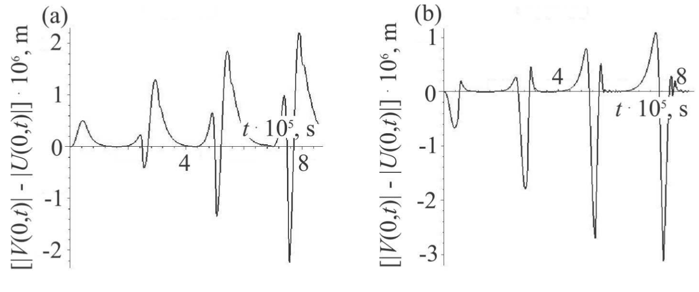

Fig. 6. Influence of simultaneous variation of all material properties on oscillations on the boundary $X = 0$: (a) case A+, (b) case A–.

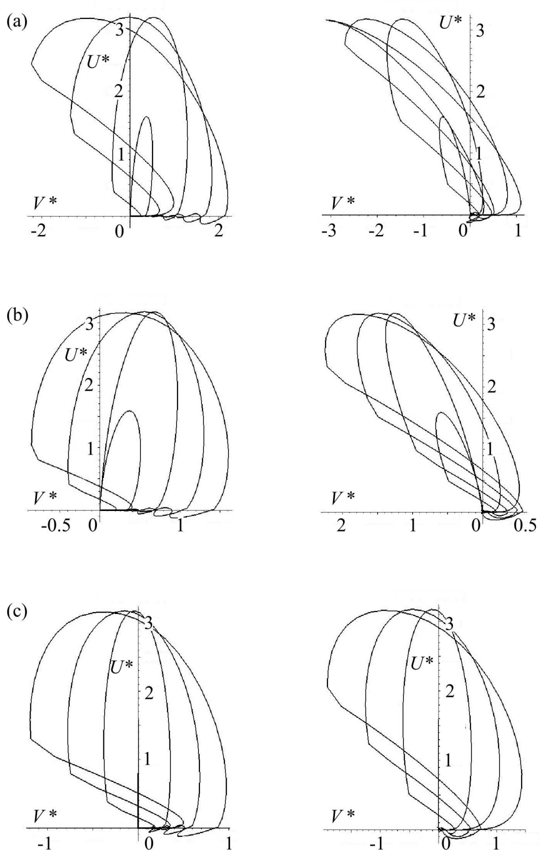

Fig. 7. Characterization of simultaneous variation of all material properties by parametric plots on the boundary $X = 0$: (a) cases A+ and A–, (b) cases B+ and B–, (c) cases C+ and C–; $U^* = U(0,t) \cdot 10^6$, m, $V^* = [| V(0,t) | - | U(0,t) |] \cdot 10^6$, m.

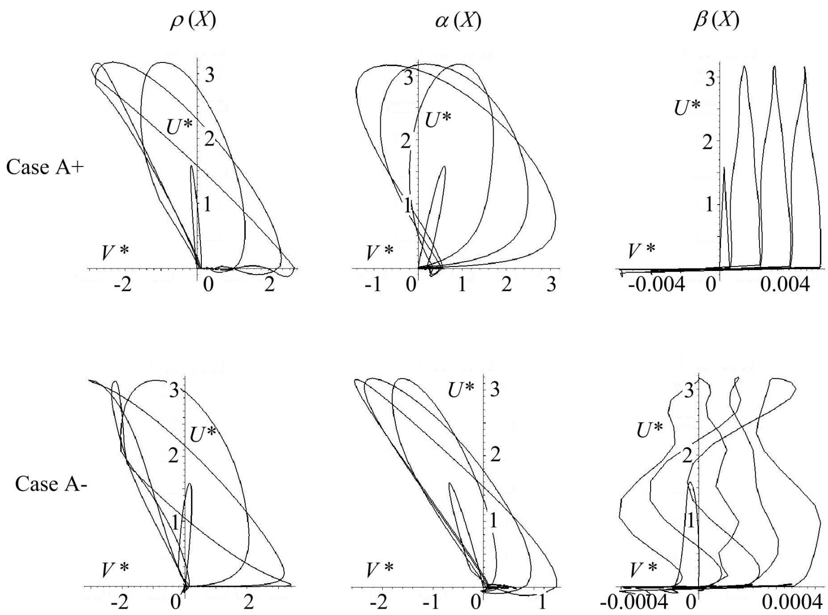

Fig. 8. Influence of variable density and elasticity on oscillations on the boundary $X = 0$ in the cases A+ and A– ($U^* = U(0,t) \cdot 10^6$, m, $V^* = [| V(0,t) | - | U(0,t) |] \cdot 10^6$, m).

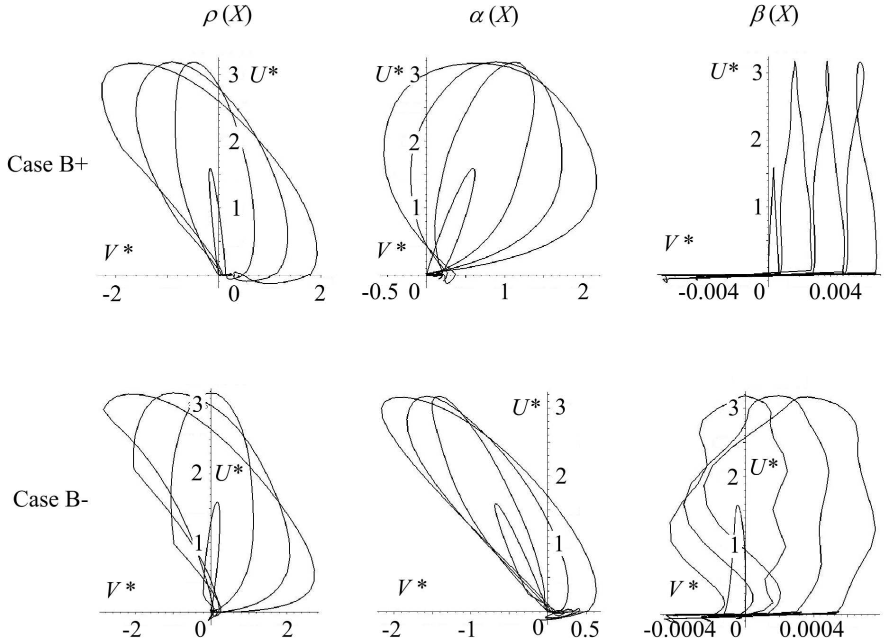

Fig. 9. Influence of variable density and elasticity on oscillations on the boundary $X = 0$ in the cases B+ and B– ($U^* = U(0,t) \cdot 10^6$, m, $V^* = [|\,V(0,t)\,| - |\,U(0,t)\,|] \cdot 10^6$, m).

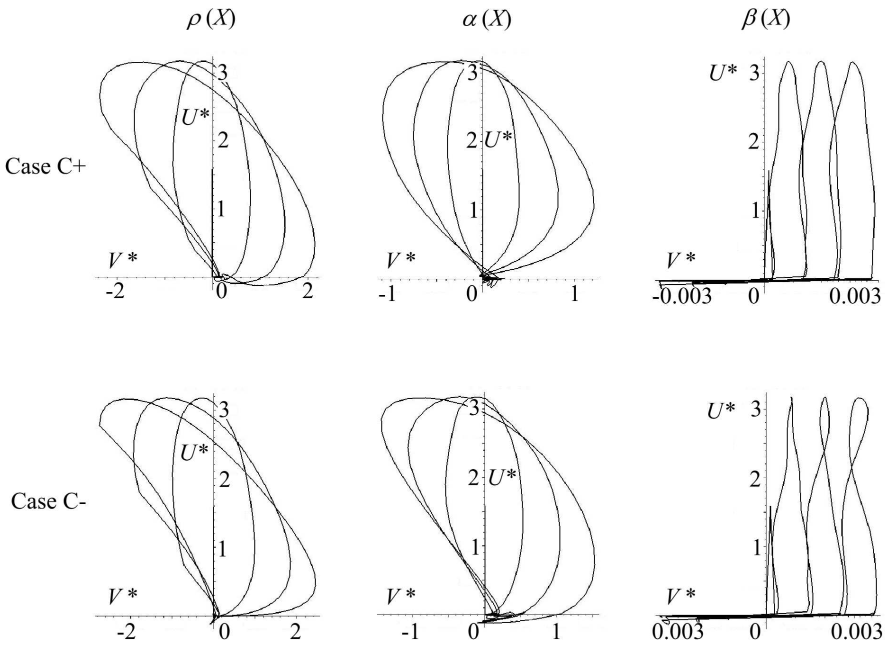

Fig. 10. Influence of variable density and elasticity on oscillations on the boundary $X = 0$ in the cases C+ and C– ($U^* = U(0,t) \cdot 10^6$, m, $V^* = [|\,V(0,t)\,| - |\,U(0,t)\,|] \cdot 10^6$, m).

The plots that characterize the variation of the density and the linear part of elasticity are qualitatively different for different cases of material inhomogeneity. Also here, the variation of the density and of the linear part of elasticity in cases A and B is characterized by the smaller loop that does not characterize the corresponding plots in case C. Comparison of plots enables to distinguish the influence of variable density and the variable linear part of elasticity on the burst propagation.

Parametric plots that depict the influence of the variation of the nonlinear part of elasticity are easily distinguishable from other plots. The essential fact is that the shape of these plots is sensitive to the case of the variation of material properties, i.e., it supplies additional information about the material properties for the problem of nondestructive material characterization.

Consequently, parametric plots are an effective source of information about the influence of the variation of material properties on the burst propagation in strongly inhomogeneous materials. Analysis of these plots makes it possible to distinguish specimens made of (i) a homogeneous material, (ii) a material with symmetrically distributed properties, (iii) a material with asymmetrically distributed properties, and to determine the most relevant property of the material responsible for the inhomogeneity.

5. CONCLUSIONS

Oscillations evoked on the boundaries of the specimen with two parallel boundaries by the propagation and reflection of the tone burst were found to be a powerful source of information about the inhomogeneous properties of the material of the specimen. The matter was studied on the basis of parametric plots with respect to time in a considerably inhomogeneous exponentially graded material. The influence of the symmetrically and asymmetrically distributed material properties on the boundary oscillation was investigated in six different cases of material inhomogeneity. It was shown that distortions of boundary oscillations caused by variable material density and linear elasticity were of the same order while changes caused by nonlinear elasticity manifested themselves in higher-order small but very informative phenomena.

The changes in material properties are reflected in composed parametric plots. The shapes of the plots are sensitive to the kind of the inhomogeneous material property and to the scheme of change of the material property and they contain information that is sufficient to qualitatively solve nondestructive material characterization problems for an exponentially graded material.

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Harmooniliste impulsside levi kirjeldamine eksponentsiaalselt skaleeritud materjalides parameetriliste joonistega

Arvi Ravasoo

On uuritud ultrahelisagedusega harmooniliste impulsside levi ja peegeldumist tugevalt muutuvate omadustega eksponentsiaalselt skaleeritud materjalides. Materjali deformeeruvust on kirjeldatud viiekonstantse mittelineaarse elastsusteooria abil. Ühemõõtmelises käsitluses on selgitatud materjali omaduste mõju katsekeha kahel paralleelsel äärepinnal häiritatud võnkumiste profiilidele, kasutades parameetrilisi jooniseid. Saadud tulemused on kasutatavad vaadeldud materjalide mittepurustaval katsetusel.