Totally geodesic submanifolds of a trans-Sasakian manifold

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Abstract. We consider invariant submanifolds of a trans-Sasakian manifold and obtain the conditions under which the submanifolds are totally geodesic. We also study invariant submanifolds of a trans-Sasakian manifold satisfying $Z(X,Y).h = 0$, where $Z$ is the concircular curvature tensor.

Key words: invariant submanifold, trans-Sasakian manifold, totally geodesic, semi-parallel, recurrent, pseudo-parallel, Ricci generalized pseudo-parallel.

1. INTRODUCTION

Invariant submanifolds of a contact manifold have been a major area of research for a long time since the concept was borrowed from complex geometry. It helps us to understand several important topics of applied mathematics; for example, in studying non-linear autonomous systems the idea of invariant submanifolds plays an important role [9]. A submanifold of a contact manifold is said to be totally geodesic if every geodesic in that submanifold is also geodesic in the ambient manifold. In 1985, Oubina [14] introduced a new class of almost contact manifolds, namely, trans-Sasakian manifold of type $(\alpha, \beta)$, which can be considered as a generalization of Sasakian, Kenmotsu, and cosymplectic manifolds. Trans-Sasakian structures of type $(0,0)$, $(0,\beta)$, and $(\alpha,0)$ are cosymplectic [2], $\beta$-Kenmotsu [10], and $\alpha$-Sasakian [10], respectively. Kon [12] proved that invariant submanifolds of a Sasakian manifold are totally geodesic if the second fundamental form of the immersion is covariantly constant. On the other hand, any submanifold $M$ of a Kenmotsu manifold is totally geodesic if and only if the second fundamental form of the immersion is covariantly constant, provided $\xi \in TM$ [11]. Recently, Sular and Öztür [16] proved some equivalent conditions regarding the submanifolds of a Kenmotsu manifold to be totally geodesic. Several studies ([5,17]) have been done on invariant submanifolds of trans-Sasakian manifolds. Recently, Sarkar and Sen [15] proved some equivalent conditions of an invariant submanifold of trans-Sasakian manifolds to be totally geodesic. In the present paper we rectify proofs of most of the major theorems of [15] and [17], show some theorems of [15] as corollary of our present results, and also introduce some new equivalent conditions for an invariant submanifold of a trans-Sasakian manifold to be totally geodesic.

2. PRELIMINARIES

Let $M$ be a connected almost contact metric manifold with an almost contact metric structure $(\phi, \xi, \eta, g)$, that is, $\phi$ is a $(1,1)$-tensor field, $\xi$ is a vector field, $\eta$ is a one-form, and $g$ is the compatible Riemannian...
metric such that
\[ \phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi \xi = 0, \quad \eta \circ \phi = 0, \] (2.1)
\[ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \] (2.2)
\[ g(X, \phi Y) = -g(\phi X, Y), \quad g(\xi, \xi) = \eta(X), \] (2.3)
for all \( X, Y \in TM \) ([2,18]). The fundamental two-form \( \Phi \) of the manifold is defined by
\[ \Phi(X, Y) = g(X, \phi Y), \] (2.4)
for \( X, Y \in TM \).

An almost contact metric structure \( (\phi, \xi, \eta, g) \) on a connected manifold \( M \) is called a trans-Sasakian structure [14] if \( (M \times \mathbb{R}, J, G) \) belongs to the class \( W_4 \) [8], where \( J \) is the almost complex structure on \( M \times \mathbb{R} \) defined by
\[ J(X, f d/dt) = (\phi X - f \xi, \eta(X) d/dt), \]
for all vector fields \( X \) on \( M \) and smooth functions \( f \) on \( M \times \mathbb{R} \), and \( G \) is the product metric on \( M \times \mathbb{R} \). This may be expressed by the condition [3]
\[ (\overline{\nabla}_X \phi) Y = \alpha (g(X, Y)\xi - \eta(Y)X) + \beta (g(\phi X, Y)\xi - \eta(Y)\phi X) \] (2.5)
for smooth functions \( \alpha \) and \( \beta \) on \( M \). Here we say that the trans-Sasakian structure is of type \( (\alpha, \beta) \). From the formula (2.5) it follows that
\[ \overline{\nabla}_X \xi = -\alpha \phi X + \beta (X - \eta(X)\xi), \] (2.6)
\[ (\overline{\nabla}_X \eta) Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \] (2.7)

In a \((2n + 1)\)-dimensional trans-Sasakian manifold we also have the following:
\[ S(X, \xi) = 2n(\alpha^2 - \beta^2)\eta(X) - (2n - 1)X\beta - \eta(X)\xi\beta - (\phi X)\alpha, \] (2.8)
\[ R(X, Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha \beta (\eta(Y)\phi X - \eta(X)\phi Y) \]
\[ - (X\alpha)\phi Y + (Y\alpha)\phi X - (X\beta)\phi^2 X + Y\beta \phi^2 X, \] (2.9)
\[ R(X, \xi)\xi = (\alpha^2 - \beta^2)(X - \eta(X)\xi) + 2\alpha \beta \phi X + (\xi \alpha)\phi X + (\xi \beta)\phi^2 X, \] (2.10)
where \( S \) is the Ricci tensor of type \((0, 2)\) and \( R \) is the curvature tensor of type \((1, 3)\).

Let \( M \) be a submanifold of a contact manifold \( \overline{M} \). We denote by \( \nabla \) and \( \overline{\nabla} \) the Levi-Civita connections of \( M \) and \( \overline{M} \), respectively, and by \( T^\perp(M) \) the normal bundle of \( M \). Then for vector fields \( X, Y \in TM \), the second fundamental form \( h \) is given by the formula
\[ h(X, Y) = \overline{\nabla}_X Y - \nabla_X Y. \] (2.11)
Furthermore, for \( N \in T^\perp M \)
\[ A_N X = \nabla^\perp_X N - \overline{\nabla}_X N, \] (2.12)
where \( \nabla^\perp \) denotes the normal connection of \( M \). The second fundamental form \( h \) and \( A_N \) are related by \( g(h(X, Y), N) = g(A_N X, Y) \) [4].

The submanifold \( M \) is totally geodesic if and only if \( h = 0 \).

An immersion is said to be parallel and semi-parallel [6] if for all \( X, Y \in TM \) we get \( \nabla h = 0 \) and \( R(X, Y) \cdot h = 0 \), respectively.

It is said to be pseudo-parallel [7] if for all \( X, Y \in TM \) we get
\[ R(X, Y) \cdot h = f Q(g, h), \] (2.13)
where $f$ denotes a real function on $M$ and $Q(E,T)$ is defined by
\[
Q(E,T)(X,Y,Z,W) = -T((X \wedge E)Z,W) - T(Z,(X \wedge E)W),
\]
where $X \wedge EY$ is defined by
\[
(X \wedge EY)Z = E(Y,Z)X - E(X,Z)Y.
\]

If $f = 0$, the immersion is semi-parallel.

Similarly, an immersion is said to be 2-pseudo-parallel if for all $X,Y \in \mathcal{T}M$ we get
\[
R(X,Y).\nabla h = fQ(g,\nabla h),
\]
and Ricci generalized pseudo-parallel [13] if $R(X,Y).h = fQ(S,h)$, for all $X,Y \in \mathcal{T}M$.

The second fundamental form $h$ satisfying
\[
(\nabla_Z h)(X,Y) = A(Z)h(X,Y),
\]
where $A$ is a nonzero one-form, is said to be recurrent. It is said to be 2-recurrent if $h$ satisfies
\[
(\nabla_X \nabla_Y h - \nabla_{[X,Y]} h)(Z,W) = B(X,Y)h(Z,W),
\]
where $B$ is a nonzero two-form.

**Proposition 2.1.** [5] An invariant submanifold of a trans-Sasakian manifold is also trans-Sasakian.

**Proposition 2.2.** [5] Let $M$ be an invariant submanifold of a trans-Sasakian manifold $\bar{M}$. Then we have
\[
\begin{align*}
    h(X,\phi Y) &= \phi(h(X,Y)), \\
    h(\phi X,\phi Y) &= -(h(X,Y)), \\
    h(X,\xi) &= 0,
\end{align*}
\]
for any vector fields $X$ and $Y$ on $M$.

For a Riemannian manifold, the concircular curvature tensor $Z$ is defined by
\[
Z(X,Y)V = R(X,Y)V - \frac{\tau}{n(n-1)}[g(Y,V)X - g(X,V)Y],
\]
for vectors $X,Y,V \in \mathcal{T}M$, where $\tau$ is the scalar curvature of $M$. We also have
\[
(Z(X,Y).h)(U,V) = R^+(X,Y)h(U,V) - h(Z(X,Y)U,V) - h(U,Z(X,Y)V).
\]
In the next section we consider the submanifold $M$ to be tangent to $\xi$.

### 3. INVARIANT SUBMANIFOLDS OF A TRANS-SASAKIAN MANIFOLD WITH $\alpha,\beta = \text{CONSTANT}$

**Lemma 3.1.** If a non-flat Riemannian manifold has a recurrent second fundamental form, then it is semi-parallel.

**Proof.** The second fundamental form $h$ is said to be recurrent if
\[
\nabla h = A \otimes h,
\]
where $A$ is an everywhere nonzero one-form.

We define a function $e$ on $M$ by
\[
e^2 = g(h,h).
\]

\[\text{(3.1)}\]
Then we have \( e(Ye) = e^2A(Y) \). So we obtain \( Ye = eA(Y) \), since \( f \) is nonzero. This implies that

\[
X(Ye) - Y(Xe) = (XA(Y) - YA(X))e.
\]

Therefore we get

\[
[\tilde{\nabla}_X \tilde{\nabla}_Y - \tilde{\nabla}_Y \tilde{\nabla}_X - \tilde{\nabla}_{[X,Y]}]e = [XA(Y) - YA(X) - A([X,Y])]e.
\]

Since the left-hand side of the above equation is identically zero and \( e \) is nonzero on \( M \) by our assumption, we obtain

\[
dA(X,Y) = 0,
\]

that is, the one-form \( A \) is closed.

Now from \( (\nabla_X h)(U,V) = A(X)h(U,V) \) we get

\[
(\tilde{\nabla}_U \tilde{\nabla}_V h)(X,Y) - (\tilde{\nabla}_V \tilde{\nabla}_U h)(X,Y) = [(\tilde{\nabla}_U A)V + A(U)A(V)]h(X,Y) = 0.
\]

Using (3.2) we get

\[
(R(X,Y),h)(U,V) = [2dA(X,Y)]h(X,Y) = 0.
\]

Therefore, for a recurrent second fundamental form we have

\[
R(X,Y),h = 0
\]

for any vectors \( X,Y \) on \( M \).

If \( e = 0 \), then from (3.1) we get \( h = 0 \) and thus \( R(X,Y),h = 0 \). Hence the lemma.

**Theorem 3.1.** An invariant submanifold of a non-cosymplectic trans-Sasakian manifold is totally geodesic if and only if its second fundamental form is parallel.

**Proof.** Since \( h \) is parallel, we have

\[
(\nabla_X h)(Y,Z) = 0,
\]

which implies

\[
\nabla_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z) = 0.
\]

Putting \( Z = \xi \) in the above equation and applying (2.19) we obtain

\[
h(Y,\nabla_X \xi) = 0.
\]

(3.3)

So from (2.6) and the above equation (3.3) we obtain

\[
\alpha h(X,Y) = \beta \phi h(X,Y).
\]

(3.4)

Applying \( \phi \) to both sides of (3.4) we get

\[
\alpha \phi h(X,Y) = -\beta h(X,Y).
\]

(3.5)

From (3.4) and (3.5) we conclude that

\[
(\alpha^2 + \beta^2)h(X,Y) = 0.
\]

Hence for a non-cosymplectic trans-Sasakian manifold \( h(X,Y) = 0 \), for all \( X,Y \in TM \).

The converse part is trivial. Hence the result.
Remark 3.1. In Theorem 3.1 [15] the authors proved the same result, but they actually proved $h(Y, \nabla_X \xi) = 0$, and $h(Y, \xi) = 0$, $\forall X, Y \in TM$. Since $\nabla_X \xi$ is not an arbitrary vector of $TM$, hence from this we can not conclude that the submanifold is totally geodesic.

Remark 3.2. Again in the proof of Theorem 4.8 [17] the authors assumed $\phi(h(X, Y)) = 0$, $\forall X, Y \in TM$, which is not true in general because this condition directly implies that the submanifold is totally geodesic.

Theorem 3.2. An invariant submanifold of a non-cosymplectic trans-Sasakian manifold is totally geodesic if and only if its second fundamental form is semi-parallel.

Proof. Since $h$ is semi-parallel, we have

$$(R(X, Y).h)(U, V) = 0,$$  \hspace{1cm} (3.6)

which implies

$$R^1(X, Y)h(U, V) + h(R(X, Y)U, V) - h(U, R(X, Y)V) = 0.$$  \hspace{1cm} (3.7)

Putting $V = \xi = Y$ and applying (2.19) we get from Eq. (3.7)

$$h(U, R(X, \xi))\xi) = 0.$$  

So from (2.10) and (2.19) we get

$$(\alpha^2 - \beta^2)h(U, X) = 2\alpha\beta \phi h(U, X).$$  \hspace{1cm} (3.8)

Applying $\phi$ to both sides of Eq. (3.8) we obtain

$$(\alpha^2 - \beta^2)\phi h(U, X) = -2\alpha\beta h(U, X).$$  \hspace{1cm} (3.9)

So from (3.8) and (3.9) we conclude that

$$(\alpha^2 + \beta^2)^2h(U, X) = 0.$$  

Hence as in the previous case, for non-cosymplectic trans-Sasakian manifolds the invariant submanifold is totally geodesic. The converse part follows trivially.$\Box$

Now, by Lemma 3.1 we get that if a second fundamental form is recurrent, then it is semi-parallel. Also, the second fundamental form of a totally geodesic submanifold is trivially recurrent, so from Theorem 3.2 we obtain the following:

Corollary 3.1. An invariant submanifold of a non-cosymplectic trans-Sasakian manifold is totally geodesic if and only if its second fundamental form is recurrent.

Remark 3.3. In Theorem 3.2 [15] the authors proved the above corollary, but they just showed that $h(Y, \nabla_X \xi) = 0$, and $h(Y, \xi) = 0$, $\forall X, Y \in TM$. Since $\nabla_X \xi$ is not an arbitrary vector of $TM$, we can not conclude from this that the submanifold is totally geodesic.

In [1] Aikawa and Matsuyama proved that if a tensor field $T$ is 2-recurrent, then $R(X, Y).T = 0$. Also it can be easily seen that in a totally geodesic submanifold the second fundamental form is 2-recurrent. Therefore by Theorem 3.2 we also obtain the following:

Corollary 3.2. An invariant submanifold of a non-cosymplectic trans-Sasakian manifold is totally geodesic if and only if its second fundamental form is 2-recurrent.

Remark 3.4. In Theorem 3.4 [15] the authors proved the above corollary, but they considered $\nabla_X \xi$ as an arbitrary vector of $TM$, and actually proved $h(Y, \nabla_X \xi) = 0$, $\forall X, Y \in TM$, hence the proof of Theorem 3.4 [15] is incorrect.
**Theorem 3.3.** An invariant submanifold of a trans-Sasakian manifold is totally geodesic if and only if its second fundamental form is 2-semi-parallel, provided \( \alpha^2(\alpha^2 - 3\beta^2)^2 + \beta^2(\beta^2 - 3\alpha^2)^2 \neq 0 \).

**Proof.** Since the second fundamental form is 2-semi-parallel, we have

\[
(R(X,Y)(\nabla_U h))(Z,W) = 0,
\]

which implies

\[
(R^\perp(X,Y)(\nabla_U h))(Z,W) - (\nabla_U h)(R(X,Y)Z,W) - (\nabla_U h)(Z,R(X,Y)W) = 0.
\]

Now,

\[
(R^\perp(X,\xi)(\nabla_U h))(\xi,\xi) = 0,
\]

\[
(\nabla_U h)(R(X,\xi)\xi,\xi) = (\nabla_U h)((\alpha^2 - \beta^2)(X - \eta(X)\xi) + 2\alpha\beta \phi X,\xi) = -h((\alpha^2 - \beta^2)(X - \eta(X)\xi) + 2\alpha\beta \phi X, -\alpha\phi U - \beta\phi^2 U) = \alpha(\alpha^2 - \beta^2)h(X,\phi U) + 2\alpha^2\beta h(\phi X,\phi U) + \beta(\alpha^2 - \beta^2)h(X,\phi^2 U) + 2\alpha^2\beta^2 h(\phi X,\phi^2 U) = \alpha(\alpha^2 - 3\beta^2)\phi h(X,U) + \beta(\beta^2 - 3\alpha^2)h(X,U).
\]

Similarly,

\[
(\nabla_U h)((\xi,R(X,\xi)\xi)) = \alpha(\alpha^2 - 3\beta^2)\phi h(X,U) + \beta(\beta^2 - 3\alpha^2)h(X,U).
\]

So putting \( Y = Z = W = \xi \) in (3.10) we obtain

\[
\alpha(\alpha^2 - 3\beta^2)\phi h(X,U) + \beta(\beta^2 - 3\alpha^2)h(X,U) = 0.
\]

Applying \( \phi \) on both sides of (3.11) we get

\[
\alpha(\alpha^2 - 3\beta^2)h(X,U) = \beta(\beta^2 - 3\alpha^2)\phi h(X,U).
\]

From (3.11) and (3.12) we conclude that

\[
[\alpha^2(\alpha^2 - 3\beta^2)^2 + \beta^2(\beta^2 - 3\alpha^2)^2]h(X,U) = 0.
\]

Hence the submanifold is totally geodesic. The converse holds trivially. \( \square \)

**Theorem 3.4.** An invariant submanifold of a trans-Sasakian manifold is totally geodesic if and only if its second fundamental form is pseudo-parallel, provided \( [(\alpha^2 - \beta^2)^2 + 4\alpha^2\beta^2] \neq 0 \).

**Proof.** Since the second fundamental form is pseudo-parallel, we have

\[
(R(X,Y).h)(U,V) = fQ(g,h)(X,Y,U,V),
\]

which implies

\[
(R^\perp(X,Y)).h(U,V) - h(R(X,Y)U,V) - h(U,R(X,Y)V) = f(-g(V,X)h(U,Y) + g(U,Y)h(V,X) - g(V,Y)h(U,X) + g(U,Y)h(V,X)).
\]

Putting \( V = \xi = Y \) in Eq. (3.13) and applying (2.19) and (2.10) we obtain

\[
-h(U,(\alpha^2 - \beta^2)X + 2\alpha\beta \phi X) = f(-h(U,X)).
\]
Applying $\phi$ to both sides of (3.14) we obtain

$$\left(\alpha^2 - \beta^2 - f\right)\phi h(U, X) = 2\alpha\beta h(U, X).$$  \hfill (3.15)$$

From (3.14) and (3.15) we conclude that

$$\left[(\alpha^2 - \beta^2 - f)^2 + 4\alpha^2\beta^2\right]h(U, X) = 0.$$

Hence the submanifold is totally geodesic. The converse holds trivially.

**Theorem 3.5.** An invariant submanifold of a trans-Sasakian manifold is totally geodesic if and only if its second fundamental form is 2-pseudo-parallel.

**Proof.** Since, the second fundamental form is 2-pseudo-parallel, we have

$$(R(X, Y)\nabla h)(U, V) = f Q(g, \nabla h)(X, Y, U, V).$$  \hfill (3.16)$$

Now,

$$(R(X, Y)\nabla h)(U, V) = R^\perp(X, Y)(\nabla h)(U, V) - (\nabla h)(R(X, Y)U, V) - (\nabla h)(U, R(X, Y)V).$$  \hfill (3.17)$$

From (2.10) and (2.19) we have

$$(\nabla h)(\xi, \xi) = 0$$  \hfill (3.18)$$

and

$$(\nabla h)(R(X, \xi)\xi, \xi) = -h(R(X, \xi)\xi, \nabla Z\xi)$$

$$= \alpha(\alpha^2 - \beta^2)h(X, \phi Z) + \beta(\alpha^2 - \beta^2)h(X, \phi^2 Z) - 2\alpha\beta h(\phi X, \phi Z) - 2\alpha\beta h(\phi X, \phi^2 Z)$$

$$= (\alpha^2 + \beta^2)(\alpha h(X, Z) + \beta h(X, Z)).$$  \hfill (3.19)$$

So, putting $Y = U = V = \xi$ in (3.16) we obtain

$$2(\alpha^2 + \beta^2)(\alpha h(X, Z) + \beta h(X, Z)) = 0,$$  \hfill (3.20)$$

which implies

$$\alpha h(X, Z) + \beta h(X, Z) = 0.$$  \hfill (3.21)$$

Applying $\phi$ on both sides of Eq. (3.21) we get

$$\alpha h(X, Z) = \beta \phi h(X, Z).$$  \hfill (3.22)$$

Combining (3.21) and (3.22) we conclude that

$$[\alpha^2 + \beta^2]h(X, Z) = 0.$$  \hfill (3.23)$$

Hence the submanifold is totally geodesic. The converse holds trivially.

**Theorem 3.6.** An invariant submanifold of a trans-Sasakian manifold is totally geodesic if and only if its second fundamental form is Ricci generalized pseudo-parallel, provided

$$[(\alpha^2 - \beta^2)^2(1 - nf)^2 + 4\alpha^2\beta^2] \neq 0.$$
Proof. Since the submanifold is Ricci generalized pseudo-parallel, we have
\[(R(X,Y),h)(U,V) = fQ(S,h)(X,Y,U,V).\]  
(3.24)
So,
\[R(X,Y)h(U,V) - h(R(X,Y)U,V) - h(U,R(X,Y)V) = f(-S(V,X)h(U,Y) + S(U,X)h(V,Y) - S(V,Y)h(X,U) + S(U,Y)h(X,V)).\]  
(3.25)
Putting \(Y = V = \xi\) and applying (2.19) we obtain
\[-h(U,R(X,\xi,\xi)) = -fS(\xi,\xi)h(X,U).\]
Since \(\alpha\) and \(\beta\) are constants, from (2.19), (2.10), and (2.8) we can write
\[(\alpha^2 - \beta^2)(1 - 2nf)h(X,U) = 2\alpha\beta \phi h(X,U).\]  
(3.26)
Applying \(\phi\) on both sides of (3.26) we obtain
\[(\alpha^2 - \beta^2)(1 - 2nf)\phi h(X,U) = -2\alpha\beta h(X,U).\]  
(3.27)
From (3.26) and (3.27) we conclude that
\[
[(\alpha^2 - \beta^2)^2(1 - 2nf)^2 + 4\alpha^2\beta^2]h(X,U) = 0.  
\]
Hence the submanifold is totally geodesic. The converse holds trivially. \(\square\)

**Theorem 3.7.** An invariant submanifold of a trans-Sasakian manifold is totally geodesic if and only if it satisfies \(Z(X,Y),h = 0\), provided \((\alpha^2 - \beta^2 - \frac{\tau}{2n(2n+1)})^2 + 4\alpha^2\beta^2 \neq 0.\)

**Proof.** We have
\[(Z(X,Y),h)(U,V) = 0.\]
So from (2.21) we can write
\[R^\perp(X,Y)h(UV) - h(Z(X,Y)U,V) - h(Z(X,Y)V,U) = 0.\]
Putting \(Y = V = \xi\) in the above equation and applying (2.19) we obtain
\[h(U,Z(X,\xi,\xi)) = 0,\]
which implies that
\[h\left(U,(\alpha^2 - \beta^2)X + 2\alpha\beta \phi X - \frac{\tau}{2n(2n+1)}X\right) = 0, \text{ since } h(X,\xi) = 0.\]
Simplifying we get
\[\left[(\alpha^2 - \beta^2) - \frac{\tau}{2n(2n+1)}\right]h(U,X) + 2\alpha\beta \phi h(U,X) = 0.\]  
(3.28)
Applying \(\phi\) on both sides of the above equation we get
\[\left[(\alpha^2 - \beta^2) - \frac{\tau}{2n(2n+1)}\right]\phi h(U,X) = 2\alpha\beta h(U,X).\]  
(3.29)
From (3.28) and (3.29) we conclude
\[
\left[(\alpha^2 - \beta^2 - \frac{\tau}{2n(2n+1)})^2 + 4\alpha^2\beta^2\right]h(U,X) = 0.  
\]
The converse part follows trivially. Hence the result. \(\square\)
4. CONCLUSION

A trans-Sasakian manifold can be regarded as a generalization of Sasakian, Kenmotsu, and cosymplectic structures. For an invariant submanifold of a trans-Sasakian manifold with constant coefficients the following conditions are equivalent under certain conditions:

- the submanifold is totally geodesic,
- the second fundamental form of the submanifold is parallel,
- the second fundamental form of the submanifold is semi-parallel,
- the second fundamental form of the submanifold is recurrent,
- the second fundamental form of the submanifold is 2-recurrent,
- the second fundamental form of the submanifold is 2-semi-parallel,
- the second fundamental form of the submanifold is pseudo-parallel,
- the second fundamental form of the submanifold is 2-pseudo-parallel,
- the second fundamental form of the submanifold is Ricci generalized pseudo-parallel,
- the second fundamental form of the submanifold satisfies $Z(X, Y).h = 0$.

REFERENCES


Täielikult geodeetilised trans-Sasaki muutkonna alammuutkonad

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On vaadeldud invariantseid trans-Sasakian muutkonna alammuutkondi ja täienda tarbe tääneli geodeetilise tingimusi. Ühtlasi on uuritud trans-Sasakian muutkonna alammuutkondi, mille puhul $Z(X, Y).h = 0$, kus $Z$ on kontsirkulaarne kõverustensor.