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MATHEMATICS

On some operator equations in the space of analytic functions and related questions

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Abstract. We investigate extended eigenvalues, extended eigenvectors, and cyclicity problems for some convolution operators. By using the Duhamel product technique, we also estimate the norm of the inner derivation operator Δ_A.

Key words: extended eigenvalue, extended eigenvector, α-Duhamel product, starlike region, Frechet space, inner derivation operator.

1. INTRODUCTION AND BACKGROUND

Let $\mathscr{B}(E)$ be an algebra of all continuous linear operators acting on the topological vector space E. The operator equation

$$AX = XB \tag{1}$$

naturally arises in numerous issues of spectral theory of operators, representation theory, stability theory (Lyapunov's equation), etc. For example, if the set of solutions of Eq. (1) contains a boundedly invertible operator X_0, then A and B are similar, $B = X_0^{-1}AX_0$, and hence have many common spectral properties. In general case, it is of interest to describe the set of all solutions of Eq. (1).

If $B = \lambda A$, $\lambda \in \mathbb{C}$, then following [1], one refers to λ as an extended eigenvalue of A, and each bounded solution X of the equation

$$AX = \lambda XA,$$

i.e., Eq. (1) with $B = \lambda A$, is called an extended eigenvector of A.

In this paper we investigate the so-called extended eigenvalues and extended eigenvectors and cyclicity problems for some convolution operators acting on the space of analytic functions defined on the starlike domain $\mathscr{D}$ of the complex plane. Our investigation is motivated by the results of Nagnibida's paper [11]. By using the Duhamel product method (see [13]), we also give a lower estimate for the inner derivation operator Δ_A defined in the Banach algebra $\mathscr{B}\left(C_A^{(n)}(\mathbb{D})\right)$ by $\Delta_A(X) := AX - XA$.

The integration operator V on $L^p[0,1]$ $(1 \leq p < \infty)$ is defined by $Vf(x) = \int_0^x f(t)\,dt$. The set of intertwining operators for the pair $\left\{V^\beta, \lambda V^\beta\right\}$ with $\beta > 0$ and $\lambda \in \mathbb{C}$ was studied by Malamud in [3,9,10]. Namely, he showed that there exists a nonzero intertwining operator for the pair $\left\{V^\beta, \lambda V^\beta\right\}$ only if $\lambda > 0$. Furthermore, the paper [10] provides a description of the set $\left\{V^\beta\right\}'_\lambda$ of all intertwining operators for the pair

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$\{V^{\beta}, \lambda V^{\beta}\}$ for $\lambda > 0$. For $\beta = 1$, the latter result was reproved by another method by Biswas, Lambert, and Petrovic [1], and Karaev [6]. For more details, see [1,2,4,5,9,10].

Let α be a fixed complex number, let $\mathscr{D}$ be a simply connected region in the complex plane $\mathbb{C}$ that is starlike with respect to the point $z = \alpha$ (i.e., $\lambda z + (1-\lambda)\alpha \in \mathscr{D}$), and let $\mathscr{A}(\mathscr{D})$ be the space of all single-valued and analytic functions in $\mathscr{D}$ that have a topology of uniform convergence on compact subsets. It is well known that $\mathscr{A}(\mathscr{D})$ is a Frechet space. By $\mathscr{J}_{\alpha}$ we shall denote the integration operator in the space $A(\mathscr{D})$ defined by the formula

$$(\mathscr{J}_{\alpha} f)(z) = \int_{\alpha}^{z} f(t)\,dt \ \ (\forall f \in \mathscr{A}(\mathscr{D})),$$

where the integration is performed over straight-line segments connecting the points α and z $(z \in \mathscr{A}(\mathscr{D}))$.

Recall that for $f, g \in A(\mathscr{D})$ their α-Duhamel product is defined by

$$\left(f \underset{\alpha}{\circledast} g\right)(z) = \frac{d}{dz}\int_{\alpha}^{z} f(z+\alpha-t)\,g(t)\,dt$$
$$= \int_{\alpha}^{z} f'(z+\alpha-t)\,g(t)\,dt + f(\alpha)\,g(z), \tag{2}$$

where the integrals are taken over the segment joining the points α and z $(z \in A(\mathscr{D}))$. It is easy to see that the α-Duhamel product satisfies all the axioms of multiplication, $A(\mathscr{D})$ is an algebra with respect to $\underset{\alpha}{\circledast}$ as well, and the function $f(z) \equiv 1$ is the unit element of the algebra $\left(A(\mathscr{D}), \underset{\alpha}{\circledast}\right)$. The operator $\mathscr{D}_f$, $\mathscr{D}_f^{\alpha} g := f \underset{\alpha}{\circledast} g$, is called the α-Duhamel operator on $A(\mathscr{D})$.

2. EXTENDED EIGENVALUES AND EXTENDED EIGENVECTORS FOR SOME CONVOLUTION OPERATORS

Let $\mathscr{D} \subset \mathbb{C}$ be a starlike region with respect to the origin. For any fixed nonzero function $f \in \mathscr{A}(\mathscr{D})$, let $\mathscr{K}_f$ be the usual convolution operator acting on the space $\mathscr{A}(\mathscr{D})$ by the formula

$$(\mathscr{K}_f g)(z) = (f * g)(z) := \int_0^z f(z-t)\,g(t)\,dt.$$

It follows from the classical Titchmarsh convolution theorem and uniqueness theorem for analytic functions that $\ker(\mathscr{K}_f) = \{0\}$. This means that 0 is not an extended eigenvalue of the operator $\mathscr{K}_f$, and therefore $\mathrm{ext}(\mathscr{K}_f) \subset \mathbb{C}\setminus\{0\}$ (here $\mathrm{ext}(\mathscr{K}_f)$ denotes the set of all extended eigenvalues of the operator $\mathscr{K}_f$).

The integration operator $\mathscr{J}$ on $\mathscr{A}(\mathscr{D})$ is defined by $\mathscr{J} f(z) = \int_0^z f(t)\,dt$. Let $f^{\circledast k}$ denote the $\circledast$-product (which is clearly $\underset{0}{\circledast}$, that is the usual Duhamel product) of f with itself k times for $k \geq 0$, i.e., $f^{\circledast k} := \underbrace{f \circledast \ldots \circledast f}_{k}$, where $f^{\circledast 0}(z) \equiv 1$. If f is a function in $\mathscr{A}(\mathscr{D})$ such that $\{(\mathscr{J} f)^{\circledast n}\}_{n \geq 0}$ is a complete system in $\mathscr{A}(\mathscr{D})$, we will denote by Λ_f the set of all $\lambda \in \mathbb{C}\setminus\{0\}$ for which the diagonal operator

$$D_{\{\lambda\}}(\mathscr{J} f)^{\circledast n} = \lambda^n (\mathscr{J} f)^{\circledast n}, \ n \geq 0,$$

is continuous in $\mathscr{A}(\mathscr{D})$.

The following theorem gives necessary and sufficient conditions under which the set Λ_f lies in the set $\mathrm{ext}(\mathscr{K}_f)$. Our result is apparently the first result in the "extended theory" for more general convolution operators, which is an extension of Karaev's result [7, Theorem 2, (ii)]. The related results for the integration operator are considered in [4,7].

Theorem 1. *Let $f \in \mathscr{A}(\mathscr{D})$ be a nonzero function. Suppose that the system $\left\{(\mathscr{J} f)^{\circledast n}\right\}_{n\geq 0}$ is complete in $\mathscr{A}(\mathscr{D})$. Let $A \in \mathscr{B}(\mathscr{A}(\mathscr{D}))$ be a nonzero operator and $\lambda \in \Lambda_f$ be any number. Then*

$$A\mathscr{K}_f = \lambda \mathscr{K}_f A$$

if and only if there exists $\varphi \in \mathscr{A}(\mathscr{D})$ such that $A = \mathscr{D}_\varphi D_{\{\lambda\}}$.

Proof. By using the usual Duhamel product $\circledast$, which is defined by

$$(f_1 \circledast f_2)(z) := \frac{d}{dz}\int_0^z f_1(z-t) f_2(t)\, dt,$$

we have that any function $f_1 \in \mathscr{A}(\mathscr{D})$ defines the continuous operator (Duhamel operator) $\mathscr{D}_{f_1} f_2 := f_1 \circledast f_2$, $f_2 \in \mathscr{A}(\mathscr{D})$. Then we have

$$\begin{aligned} \mathscr{K}_f g &= \mathscr{J}\mathscr{D}_f g = z \circledast (f \circledast g) \\ &= (z \circledast f) \circledast g = \mathscr{D}_{z\circledast f} g = \mathscr{D}_{\mathscr{J} f} g \end{aligned}$$

for all $g \in \mathscr{A}(\mathscr{D})$. Thus $\mathscr{K}_f = \mathscr{A}(\mathscr{D})_{\mathscr{J} f}$.

Now, let $\lambda \in \Lambda_f$ be any number, and suppose that

$$\lambda \mathscr{K}_f A = A\mathscr{K}_f.$$

Then, obviously

$$\lambda^n \mathscr{K}_f^n A g = A\mathscr{K}_f^n g$$

for all $g \in \mathscr{A}(\mathscr{D})$ and $n \geq 0$. In particular, putting $g = 1$ in the last equality, we have

$$A\mathscr{K}_f^n 1 = \lambda^n \mathscr{K}_f^n A1$$

for all $n \geq 0$. Since $\mathscr{K}_f = \mathscr{D}_{\mathscr{J} f}$, clearly we have

$$\mathscr{K}_f^n 1 = \mathscr{D}^n_{\mathscr{J} f} 1 = (\mathscr{J} f)^{\circledast n} \circledast 1 = (\mathscr{J} f)^{\circledast n}$$

for all $n \geq 0$. This shows that

$$\begin{aligned} A(\mathscr{J} f)^{\circledast n} &= \lambda^n \left((\mathscr{J} f)^{\circledast n} \circledast A1\right) \\ &= \lambda^n (\mathscr{J} f)^{\circledast n} \circledast A1 = \mathscr{D}_{A1}\left(\lambda^n (\mathscr{J} f)^{\circledast n}\right) \\ &= \mathscr{D}_{A1} D_{\{\lambda\}} (\mathscr{J} f)^{\circledast n} \end{aligned}$$

for all $n \geq 0$. Since $\left\{(\mathscr{J} f)^{\circledast n}\right\}_{n\geq 0}$ is a complete system of the space $\mathscr{A}(\mathscr{D})$ and $D_{\{\lambda\}}$ is a continuous operator on $\mathscr{A}(\mathscr{D})$, it follows from the last equalities that

$$Ag = \mathscr{D}_{A1} D_{\{\lambda\}} g$$

for all $g \in \mathscr{A}(\mathscr{D})$, which means that $A = \mathscr{D}_\varphi D_{\{\lambda\}}$, where $\varphi = A1 \in \mathscr{A}(\mathscr{D})$, as desired.

Conversely, let us show that if A has the form $A = \mathscr{D}_\varphi D_{\{\lambda\}}$, where $\varphi \in \mathscr{A}(\mathscr{D})$, it satisfies the equation $A\mathscr{K}_f = \lambda\mathscr{K}_f A$. In fact, by considering the formula $A\mathscr{K}_f = \mathscr{D}_{\mathscr{J}f}$, and commutativity of the product $\circledast$, we obtain

$$\begin{aligned}\lambda\mathscr{K}_f A(\mathscr{J}f)^{\circledast n} &= \lambda\mathscr{K}_f\mathscr{D}_\varphi D_{\{\lambda\}}(\mathscr{J}f)^{\circledast n}\\ &= \lambda\mathscr{D}_{\mathscr{J}f}\mathscr{D}_\varphi\left(\lambda^n(\mathscr{J}f)^{\circledast n}\right) = \lambda\mathscr{D}_\varphi\mathscr{D}_{\mathscr{J}f}\left(\lambda^n(\mathscr{J}f)^{\circledast n}\right)\\ &= \lambda\mathscr{D}_\varphi\left(\mathscr{J}f\circledast\lambda^n(\mathscr{J}f)^{\circledast n}\right) = \mathscr{D}_\varphi\lambda^{n+1}\left(\mathscr{J}f\circledast(\mathscr{J}f)^{\circledast n}\right)\\ &= \mathscr{D}_\varphi\lambda^{n+1}(\mathscr{J}f)^{\circledast n+1} = \mathscr{D}_\varphi D_{\{\lambda\}}(\mathscr{J}f)^{\circledast n+1}\\ &= A\left(\mathscr{J}f\circledast(\mathscr{J}f)^{\circledast n}\right) = A\mathscr{D}_{\mathscr{J}f}(\mathscr{J}f)^{\circledast n}\\ &= A\mathscr{K}_f(\mathscr{J}f)^{\circledast n}\end{aligned}$$

for all $n \geq 0$. By considering completeness of the system $\left\{(\mathscr{J}f)^{\circledast n}\right\}_{n\geq 0}$ in $\mathscr{A}(\mathscr{D})$, from the last equalities we deduce that $A\mathscr{K}_f = \lambda\mathscr{K}_f A$. The theorem is proved. □

3. CYCLIC VECTORS OF CONVOLUTION OPERATOR $\mathscr{K}_{f,\alpha}$

Let $\mathscr{D}$ be a starlike region in the complex plane $\mathbb{C}$ with respect to $z = \alpha$. Our next result describes all cyclic vectors of some convolution operators of the form

$$(\mathscr{K}_{f,\alpha}g)(z) := \int_\alpha^z f(z+\alpha-t)\,g(t)\,dt.$$

Theorem 2. *Let $f \in \mathscr{A}(\mathscr{D})$, and assume that $\left\{(\mathscr{J}_\alpha f)^{\underset{\alpha}{\circledast} n}\right\}_{n\geq 0}$ is a complete system in $\mathscr{A}(\mathscr{D})$. If $g \in \mathscr{A}(\mathscr{D})$, then g is a cyclic vector for the convolution operator $\mathscr{K}_{f,\alpha}$ if and only if $g(\alpha) \neq 0$.*

Proof. It follows from the definition of α-Duhamel product $\underset{\alpha}{\circledast}$ that

$$\begin{aligned}\mathscr{K}_{f,\alpha}h &= \mathscr{J}_\alpha\mathscr{D}_{f,\alpha}h = (z-\alpha)\underset{\alpha}{\circledast}\left(f\underset{\alpha}{\circledast}h\right)\\ &= \left((z-\alpha)\underset{\alpha}{\circledast}f\right)\underset{\alpha}{\circledast}h = \mathscr{D}_{(z-\alpha)\underset{\alpha}{\circledast}f}h = \mathscr{D}_{\mathscr{J}_\alpha f}h\end{aligned}$$

for all $h \in \mathscr{A}(\mathscr{D})$, which means that $\mathscr{K}_{f,\alpha} = \mathscr{D}_{\mathscr{J}_\alpha f}$. Then according to the condition of the theorem, we obtain that

$$\begin{aligned}E_g &= \operatorname{span}\left\{\mathscr{K}_{f,\alpha}^n g : n\geq 0\right\} = \operatorname{span}\left\{\left(\mathscr{D}_{\mathscr{J}_\alpha f}\right)^n g : n\geq 0\right\}\\ &= \operatorname{span}\left\{(\mathscr{J}_\alpha f)^{\underset{\alpha}{\circledast} n}\underset{\alpha}{\circledast}g : n\geq 0\right\} = \operatorname{span}\left\{\mathscr{D}_{g,\alpha}(\mathscr{J}_\alpha f)^{\underset{\alpha}{\circledast} n} : n\geq 0\right\}\\ &= \operatorname{clos}\mathscr{D}_{g,\alpha}\operatorname{span}\left\{(\mathscr{J}_\alpha f)^{\underset{\alpha}{\circledast} k} : k\geq 0\right\} = \operatorname{clos}\mathscr{D}_{g,\alpha}\mathscr{A}(\mathscr{D}),\end{aligned}$$

so

$$E_g = \operatorname{clos}\mathscr{D}_{g,\alpha}\mathscr{A}(\mathscr{D}).$$

Now, if $g(\alpha) \neq 0$, then by virtue of Nagnibida's result operator $\mathscr{D}_{g,\alpha}$ is invertible in $\mathscr{A}(\mathscr{D})$, which implies that

$$\mathscr{D}_g\mathscr{A}(\mathscr{D}) = \mathscr{A}(\mathscr{D}).$$

Hence $E_g = \mathscr{A}(\mathscr{D})$, which shows that g is a cyclic vector for the convolution operator $\mathscr{K}_{f,\alpha}$.

Conversely, suppose that $g \in \mathscr{A}(\mathscr{D})$ is a cyclic vector for the operator $\mathscr{K}_{f,\alpha}$, that is $E_g = \mathscr{A}(\mathscr{D})$. If $g(\alpha) \neq 0$, it is easy to see from the equality $E_g = \operatorname{clos}\mathscr{D}_{g,\alpha}\mathscr{A}(\mathscr{D})$ that $E_g \subset \{h \in \mathscr{A}(\mathscr{D}) : h(\alpha) = 0\}$, which is impossible because $E_g = \mathscr{A}(\mathscr{D})$. Consequently, $g(\alpha) \neq 0$, which proves the theorem. □

Since $\{(z-\alpha)^n\}_{n\geq 0}$ is a complete system in $\mathscr{A}(\mathscr{D})$, the next corollary immediately follows from Theorem 2.

Corollary 1. *Let $\mathscr{J}_\alpha$ be an integration operator defined on $\mathscr{A}(\mathscr{D})$ by $(\mathscr{J}_\alpha g)(z) = \int_\alpha^z g(t)\,dt$. Then*

$$\operatorname{Cyc}(\mathscr{J}_\alpha) = \{g \in \mathscr{A}(\mathscr{D}) : g(\alpha) \neq 0\},$$

where $\operatorname{Cyc}(\mathscr{J}_\alpha)$ *denotes the set of all cyclic vectors of $\mathscr{J}_\alpha$.*

For the related results see [6–8] and Tkachenko [12]; in [7] the analogous results are considered by Karaev for the Banach space $C_A^{(n)}(\mathbb{D})$.

4. ON THE NORM OF INNER DERIVATION OPERATOR

Let A be a fixed linear bounded operator acting on the Banach space $C_A^{(n)}(\mathbb{D})$, which is the space of all n-times continuously differentiable functions on $\overline{\mathbb{D}}$ that are holomorphic on the unit disc $\mathbb{D}$. In [7], Karaev proved that $C_A^{(n)}(\mathbb{D})$ is an algebra with multiplication of the Duhamel product

$$(f \circledast g)(z) = \frac{d}{dz}\int_0^z f(z-t)\,g(t)\,dt. \tag{3}$$

Thus, the Duhamel operator $\mathscr{D}_f$ defined on $C_A^{(n)}(\mathbb{D})$ by $\mathscr{D}_f g := f \circledast g$ is bounded and $\|\mathscr{D}_f\| \leq \|f\|$. On the other hand, it is clear from (3) that $f = f \circledast 1$, and therefore $\|\mathscr{D}_f\| = \|f\|$. In this section, by using this formula we will estimate the norm of the inner derivation operator Δ_A defined on the Banach algebra $\mathscr{B}\left(C_A^{(n)}(\mathbb{D})\right)$ by the formula

$$\Delta_A(X) := AX - XA.$$

Obviously

$$\|\Delta_A\| \leq 2\|A\|. \tag{4}$$

The following theorem gives some lower estimate for $\|\Delta_A\|$ in terms of A.

Theorem 3. *Let $A \in \mathscr{B}\left(C_A^{(n)}(\mathbb{D})\right)$ be a fixed operator. Suppose that for every $X \in \mathscr{B}\left(C_A^{(n)}(\mathbb{D})\right)$ there exists a nonzero function $f := f_X \in C_A^{(n)}(\mathbb{D})$ such that*

$$((AX - XA)f)(0) \neq 0.$$

Then there exists a constant $C_A > 0$ such that

$$C_A \leq \|\Delta_A\| \leq 2\|A\|.$$

Proof. According to (4), there remains only to prove the left inequality. Indeed, let us denote

$$(AX - XA)f(z) := g(z). \tag{5}$$

Clearly, $g = g_{A,X}$. Since $g(0) \neq 0$, by the result of paper [8, Theorem 1], the Duhamel operator $\mathscr{D}_g$ is invertible in $C_A^{(n)}(\mathbb{D})$. Therefore, there exists a unique $G \in C_A^{(n)}(\mathbb{D})$ such that $G \circledast g = g \circledast G = 1$. Hence, $f \circledast G \circledast g = f$. Thus, it follows from (5) that

$$\mathscr{D}_F\left(AX-XA\right)f=f, \tag{6}$$

where $F:=f\circledast G$. Clearly, $F=F_{A,X}$. The equality (6) shows that $1\in\sigma_p\left(\mathscr{D}_F\left(AX-XA\right)\right)$, that is, 1 is the eigenvalue of the operator $\mathscr{D}_F\Delta_A\left(X\right)$. Therefore,

$$\begin{aligned}1 &\leq r\left(\mathscr{D}_F\left(AX-XA\right)\right)\leq\left\|\mathscr{D}_F\left(AX-XA\right)\right\|\\ &\leq\left\|\mathscr{D}_F\right\|\left\|AX-XA\right\|=\left\|F\right\|_{C_A^{(n)}(\mathbb{D})}\left\|\Delta_A\left(X\right)\right\|_{\mathscr{B}\left(C_A^{(n)}(\mathbb{D})\right)};\end{aligned}$$

here $r\left(.\right)$ denotes the spectral radius of the operator. Hence

$$\frac{1}{\left\|F\right\|_{C_A^{(n)}(\mathbb{D})}}\leq\left\|\Delta_A\left(X\right)\right\|.$$

By taking supremum over the operators X with $\|X\|\leq 1$, we have from this inequality that

$$\sup_{\|X\|\leq 1}\frac{1}{\left\|F_{A,X}\right\|_{C_A^{(n)}(\mathbb{D})}}\leq\sup_{\|X\|\leq 1}\left\|\Delta_A\left(X\right)\right\|=\left\|\Delta_A\right\|,$$

that is

$$\frac{1}{\inf_{\|X\|\leq 1}\left\|F_{A,X}\right\|_{C_A^{(n)}(\mathbb{D})}}\leq\left\|\Delta_A\right\|.$$

By denoting $C_A:=\frac{1}{\inf_{\|X\|\leq 1}\left\|F_{A,X}\right\|_{C_A^{(n)}(\mathbb{D})}}>0$, we have the desired result. The theorem is proved. □

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Mõnedest operaatorvõrranditest analüütiliste funktsioonide ruumis

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On uuritud teatavate konvolutsioonioperaatorite laiendatud omaväärtusi, laiendatud omavektoreid ja tsüklivektoreid ning nendega seonduvaid küsimusi.