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MATHEMATICS

On an (ε, δ)-trans-Sasakian structure

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Abstract. In this paper we investigate (ε, δ)-trans-Sasakian manifolds which generalize the notion of (ε)-Sasakian and (ε)-Kenmotsu manifolds. We prove the existence of such a structure by an example and we consider ϕ-recurrent, pseudo-projectively flat and pseudo-projective semi-symmetric (ε, δ)-trans-Sasakian manifolds.

Key words: trans-Sasakian manifold, Einstein manifold, ϕ-recurrent, (α, δ)-trans-Sasakian, pseudo-projective Ricci tensor.

1. INTRODUCTION

The study of manifolds with indefinite metrics is of interest from the standpoint of physics and relativity. Manifolds with indefinite metrics have been studied by several authors. In 1993, Bejancu and Duggal [1] introduced the concept of (ε)-Sasakian manifolds and Xufeng and Xiaoli [6] established that these manifolds are real hypersurfaces of indefinite Kahlerian manifolds. Kumar et al. [4] studied the curvature conditions of these manifolds. Tripathi et al. [5] introduced and studied (ε)-almost para contact manifolds. Recently De and Sarkar [3] introduced (ε)-Kenmotsu manifolds and studied conformally flat, Weyl semisymmetric, ϕ-recurrent (ε)-Kenmotsu manifolds. The existence of a new structure on indefinite metrics influences the curvature. Motivated by the above studies, in this paper we introduce the concept of (ε, δ)-trans-Sasakian manifolds which generalizes the notion of (ε)-Sasakian as well as (ε)-Kenmotsu manifolds.

The paper is organized as follows. Section 2 covers some preliminary facts on (ε, δ)-trans-Sasakian structures. In Section 3, we give an example of such a structure and present some basic results. Also an explicit formula for the curvature tensor and Ricci tensors are obtained. Section 4 is devoted to ϕ-recurrent (ε, δ)-trans-Sasakian manifolds. In Section 5, we consider pseudo-projectively flat (ε, δ)-trans-Sasakian manifolds and prove that these manifolds are Einstein. In the last section, we consider (ε, δ)-trans-Sasakian manifolds with the condition $R.\overline{P} = 0$ and prove that these manifolds are pseudo-projectively flat.

2. PRELIMINARIES

Let M be an almost contact metric manifold of dimension n equipped with an almost contact metric structure (φ, ξ, η, g) consisting of a (1, 1) tensor field φ, a vector field ξ, a 1-form η, and a Riemannian metric g satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \tag{2.1}$$

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$$\eta(\xi)=1, \tag{2.2}$$

$$\varphi\xi=0,\ \eta\circ\varphi=0. \tag{2.3}$$

An almost contact metric manifold M is called an (ε)-almost contact metric manifold if

$$g(\xi,\xi)=\varepsilon, \tag{2.4}$$

$$\eta(X)=\varepsilon g(X,\xi), \tag{2.5}$$

$$g(\phi X,\phi Y)=g(X,Y)-\varepsilon\eta(X)\eta(Y),\ \ \forall\ X,Y\in TM, \tag{2.6}$$

where $\varepsilon=g(\xi,\xi)=\pm 1$.

An (ε)-almost contact metric manifold is called an $(\varepsilon,\ \delta)$-trans-Sasakian manifold if

$$(\nabla_X\phi)Y=\alpha(g(X,Y)\xi-\varepsilon\eta(Y)X)+\beta(g(\phi X,Y)\xi-\delta\eta(Y)\phi X) \tag{2.7}$$

holds for some smooth functions α and β on M and $\varepsilon=\pm 1$, $\delta=\pm\ 1$. For $\beta=0$, $\alpha=1$, an $(\varepsilon,\ \delta)$-trans-Sasakian manifold reduces to an (ε)-Sasakian and for $\alpha=0$, $\beta=1$ it reduces to a (δ)-Kenmotsu manifold.

3. (ε,δ)-TRANS-SASAKIAN MANIFOLDS

Let M be an n-dimensional (ε,δ)-trans-Sasakian manifold. Taking $Y=\xi$ in (2.7) and making use of (2.3), (2.4), and (2.5), we obtain

$$\phi(\nabla_X\xi)=\varepsilon\alpha(X-\eta(X)\xi)+\delta\beta\phi X.$$

Applying ϕ on both sides, we have from (2.1) and (2.3)

$$\nabla_X\xi=-\varepsilon\alpha\phi X-\beta\delta\phi^2X. \tag{3.1}$$

Conversely, suppose that (3.1) holds. Since $\nabla_X(\eta\wedge\phi)=0$, we have

$$\begin{aligned}(\nabla_X\eta)(Y)\Phi(Z,W)&+\eta(Y)(\nabla_X\Phi)(Z,W)\\&+(\nabla_X\eta)(Z)\Phi(W,Y)+\eta(Z)(\nabla_X\Phi)(W,Y)\\&+(\nabla_X\eta)(W)\Phi(Y,Z)+\eta(W)(\nabla_X\Phi)(Y,Z)=0,\end{aligned} \tag{3.2}$$

where $\Phi(X,Y)=g(\phi X,Y)$. Taking $W=\xi$ in (3.2), we obtain

$$(\nabla_X\phi)(Y)=\varepsilon g(\phi(\nabla_X\xi),Y)\xi-\eta(Y)\phi(\nabla_X\xi). \tag{3.3}$$

Using (3.1) in (3.3), we have (2.7), provided $\varepsilon\delta=1$. Thus we have

Lemma 3.1. *An ε-almost contact metric manifold M is an (ε,δ)-trans-Sasakian manifold if and only if*

$$\nabla_X\xi=-\varepsilon\alpha\phi X-\beta\delta\phi^2X \tag{3.4}$$

holds in M.

From (3.4) it follows that

$$(\nabla_X\eta)Y=\delta\beta(\varepsilon g(X,Y)-\eta(X)\eta(Y))-\alpha g(\phi X,Y). \tag{3.5}$$

Example. Let (x,y,z) be Cartesian coordinates in $\mathbb{R}^3$ and let

$$e_1 = \frac{x}{z}\frac{\partial}{\partial x},\ e_2 = \frac{y}{z}\frac{\partial}{\partial y},\ \ e_3 = -(\varepsilon+\delta)\frac{\partial}{\partial z}.$$

Then e_1, e_2, e_3 are linearly independent at each point of M. We define

$$\xi = e_3,\ \eta = \frac{-\delta}{2}dz,\ \ \varphi = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } g = (dx)^2 + (dy)^2 + \varepsilon(dz)^2.$$

Then the (φ,ξ,η,g) structure is an (ε)-almost contact metric structure in $\mathbb{R}^3$.

Further from Koszul's formula

$$2g(\nabla_X Y,Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y])$$

we have

$$\nabla_{e_1}e_3 = -\frac{(\in+\delta)}{z}e_1,\ \nabla_{e_1}e_3 = -\frac{(\in+\delta)}{z}\ e_1,\ \nabla_{e_1}e_3 = 0.$$

Using the above relations, for any vector field X on M we have $\nabla_X\xi = -\varepsilon\alpha\phi X - \beta\delta\phi^2 X$, where $\alpha = 1/z$ and $\beta = -1/z$. Hence the (φ,ξ,η,g) structure defines the (ε,δ)-trans-Sasakian structure in R^3.

Lemma 3.2. *In an $(\varepsilon,\ \delta)$-trans-Sasakian manifold M, the curvature R satisfies*

$$\begin{aligned} R(X,Y)\xi =& \varepsilon((Y\alpha)\phi X - (X\alpha)\phi Y) + (\beta^2-\alpha^2)(\eta(X)Y - \eta(Y)X) \\ &- \delta((X\beta)\phi^2 Y - (Y\beta)\phi^2 X) + 2\delta\varepsilon\alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y) \\ &+ 2\alpha\beta(\delta-\varepsilon)g(\phi X,Y)\xi. \end{aligned} \tag{3.6}$$

Consequently

$$\begin{aligned} R(\xi,X)Y =& \varepsilon[(grad\alpha)g(\phi Y,X) + (Y\alpha)\phi X] + (\alpha^2-\beta^2)[g(Y,X)\xi - \eta(Y)X] \\ &+ \delta[(grad\beta)g(\phi^2 Y,X) - (Y\beta)\phi^2 X] + 2\varepsilon\delta\alpha\beta[g(\phi Y,X)\xi + \eta(Y)\phi X] \\ &+ 2(\delta-\varepsilon)\alpha\beta g(X,\xi)\phi Y \end{aligned} \tag{3.7}$$

$\forall$ vector fields X and Y on M.

Proof. By definition

$$R(X,Y)\xi = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi.$$

Using (3.1), the above equation becomes

$$\begin{aligned} R(X,Y)\xi =& \nabla_X(-\varepsilon\alpha\phi Y - \delta\beta\phi^2 Y) - \nabla_Y(-\varepsilon\alpha\phi X - \delta\beta\phi^2 X) \\ &- (-\varepsilon\alpha\phi[X,Y] - \delta\beta\phi^2[X,Y]). \end{aligned}$$

Using (2.7), the above relation yields (3.6).

From (3.6) and $g(R(\xi,X)Y,Z) = g(R(Y,Z)\xi,X)$, we obtain (3.7). □

We note that for constants α and β,

$$R(X,Y)\xi = (\beta^2-\alpha^2)(\eta(X)Y - \eta(Y)X). \tag{3.8}$$

Lemma 3.3. *In an $(\varepsilon,\ \delta)$-trans-Sasakian manifold, we have*

$$
\begin{aligned}
R(X,Y)\varphi Z =& \phi R(X,Y)Z + (X\alpha)(g(Y,Z)\xi - \varepsilon\eta(Z)Y) - (Y\alpha)(g(X,Z)\xi - \varepsilon\eta(Z)X) \\
& - \varepsilon(\alpha^2-\beta^2)(g(Y,Z)\phi X - g(X,Z)\phi Y) + 2\alpha\beta\delta(g(Y,Z)X - g(X,Z)Y) \\
& + \alpha\beta(\delta-\varepsilon)(g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi) \\
& + (X\beta)(g(\phi Y,Z)\xi - \delta\eta(Z)\phi Y) + (Y\beta)(g(\phi X,Z)\xi - \delta\eta(Z)\phi(X)) \\
& + 2\beta^2(\varepsilon-\delta)g(\phi X,Y)\eta(Z)\xi + \alpha\beta(\varepsilon+\delta)(g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X) \\
& + (\varepsilon\alpha^2 - \delta\beta^2)(g(\phi X,Z)Y - g(\phi Y,Z)X).
\end{aligned} \tag{3.9}
$$

Consequently,

$$
\begin{aligned}
\eta(R(X,Y)Z) =& 2\alpha\beta(\delta-\varepsilon)g(\phi X,Y)\eta(Z) - (X\alpha)g(Y,\phi Z) \\
& + (Y\alpha)g(X,\phi Z) - \alpha\beta(\delta+\varepsilon)(g(Y,\phi Z)\eta(X) \\
& - g(X,\phi Z)\eta(Y)) - (\varepsilon\alpha^2-\delta\beta^2)(g(X,Z)\eta(Y) - g(Y,Z)\eta(X)) \\
& - (X\beta)(g(Y,Z) - \varepsilon\eta(Y)\eta(Z)) + (Y\beta)(g(X,Z) - \varepsilon\eta(X)\eta(Z)).
\end{aligned} \tag{3.10}
$$

Proof. By definition,

$$R(X,Y)\phi Z = \nabla_X\nabla_Y\phi Z - \nabla_Y\nabla_X\phi Z - \nabla_{[X,Y]}\phi Z.$$

Using (2.7) in the above equation, we have

$$
\begin{aligned}
R(X,Y)\phi Z =& \nabla_X(\phi(\nabla_Y Z) + \alpha(g(Y,Z)\xi - \varepsilon\eta(Z)Y) + \beta(g(\phi Y,Z)\xi - \delta\eta(Z)\phi Y)) \\
& + \nabla_Y(\phi(\nabla_X Z) + \alpha(g(X,Z)\xi - \varepsilon\eta(Z)X) + \beta(g(\phi X,Z)\xi - \delta\eta(Z)\phi X)) \\
& - (\phi(\nabla_{[X,Y]}Z) + \alpha(g([X,Y],Z)\xi - \varepsilon\eta(Z)[X,Y]) + \beta(g(\phi[X,Y],Z)\xi - \delta\eta(Z)\phi[X,Y])).
\end{aligned}
$$

Again using (2.7) and in view of

$$\nabla_X g(Y,Z) = g(\nabla_X Y,Z) - g(Y,\nabla_X Z),$$

we obtain (3.9).

Replacing Z by ϕZ in (3.9) and making use of (2.1), we obtain

$$
\begin{aligned}
-R(X,Y)Z + \eta(Z)R(X,Y)\xi =& \phi R(X,Y)\phi Z + (X\alpha)g(Y,\phi Z)\xi - (Y\alpha)g(X,\phi Z)\xi \\
& + \varepsilon(\alpha^2-\beta^2)(g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X) \\
& + (\varepsilon\alpha^2-\delta\beta^2)(g(\phi X,\phi Z)Y - g(\phi Y,\phi Z)X) \\
& + \alpha\beta(\delta-\varepsilon)(g(X,\phi Z)\eta(Y) - g(Y,\phi Z)\eta(X))\xi \\
& + \alpha\beta(\varepsilon+\delta)(g(\phi X,\phi Z)\phi Y - g(\phi Y,\phi Z)\phi X) \\
& + 2\alpha\beta\delta(g(Y,\phi Z)X - g(X,\phi Z)Y) \\
& + (X\beta)g(\phi Y,\phi Z)\xi - (Y\beta)g(\phi X,\phi Z)\xi.
\end{aligned}
$$

Contracting the above with ξ and using (2.4) and (2.5), we obtain (3.10). □

Lemma 3.4. *In an (ε,δ)-trans-Sasakian manifold the following relations hold.*

$$S(X,\xi) = -(\phi X)\alpha + ((n-1)(\varepsilon\alpha^2 - \beta^2\delta) - (\xi\beta))\eta(X) - (n-2)(X\beta) \tag{3.11}$$

and

$$\xi\alpha + 2\delta\alpha\beta = 0. \tag{3.12}$$

Proof. Taking $Y = Z = e_i$ in (3.10) and using (2.3), we obtain (3.11).

Taking $X = \xi$ in (3.6) and using (2.3), we have

$$\begin{aligned} R(\xi,X)\xi &= -\varepsilon(\xi\alpha)\phi X + (\beta^2-\alpha^2)(X-\eta(X)\xi) - \delta(\xi\beta)\phi^2 X - 2\varepsilon\delta\alpha\beta\phi X \\ &= -\varepsilon(\xi\alpha + 2\delta\alpha\beta)\phi X - ((\beta^2-\alpha^2) + \delta(\xi\beta))\phi^2 X. \end{aligned} \tag{3.13}$$

Taking $Y = \xi$ in (3.7), we obtain

$$\begin{aligned} R(\xi,X)\xi &= \varepsilon(\xi\alpha)\phi X + (\beta^2-\alpha^2)(X-\eta(X)\xi) - \delta(\xi\beta)\phi^2 X + 2\varepsilon\delta\alpha\beta\phi X \\ &= \varepsilon(\xi\alpha + 2\delta\alpha\beta)\phi X + ((\alpha^2-\beta^2) - \delta(\xi\beta))\phi^2 X. \end{aligned} \tag{3.14}$$

Comparing (3.13) and (3.14), we obtain (3.12). □

4. ϕ-RECURRENT (ε,δ)-TRANS-SASAKIAN MANIFOLDS

Let M be an (ε,δ)-trans-Sasakian manifold. Then M is said to be a ϕ-recurrent manifold if there exists a nonzero 1-form A such that

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z \tag{4.1}$$

for arbitrary vector fields X,Y,Z,W on M. If the 1-form vanishes identically, then the manifold will be called a ϕ-symmetric manifold.

Suppose that the (ε,δ)-trans-Sasakian manifold under consideration is ϕ-recurrent. Then from (4.1) and (2.1), we obtain

$$(\nabla_W R)(X,Y)Z = \eta((\nabla_W R)(X,Y)Z)\xi - A(W)R(X,Y)Z. \tag{4.2}$$

By the above relation, Bianchi's identity yields

$$A(W)\eta(R(X,Y)Z) + A(X)\eta(R(Y,W)Z) + A(Y)\eta(R(W,X)Z) = 0. \tag{4.3}$$

Taking $Y = Z = e_i$ in the above equation, where (e_i) is an orthogonal basis of the tangent space at each point of the manifold and using (3.10) in (4.3), we obtain

$$A(W)\eta(X) = A(X)\eta(W). \tag{4.4}$$

For $X = \xi$, this equation yields

$$A(W) = A(\xi)\eta(W). \tag{4.5}$$

Now

$$(\nabla_W R)(X,Y)\xi = \nabla_W R(X,Y)\xi - R(\nabla_W X,Y)\xi - R(X,\nabla_W Y)\xi - R(X,Y)\nabla_W\xi. \tag{4.6}$$

If α and β are constants, then from (2.6) and (3.8) in (4.6), we obtain

$$\begin{aligned} (\nabla_W R)(X,Y)\xi =& (\beta^2-\alpha^2)((\nabla_W\eta)X)Y - ((\nabla_W\eta)Y)X \\ & - (-\varepsilon\alpha R(X,Y)\phi W + \beta\delta R(X,Y)W - \beta\delta\eta(W)R(X,Y)\xi). \end{aligned} \tag{4.7}$$

If we consider X,Y orthogonal to ξ, then in view of (3.8), we have $\eta((\nabla_W R)(X,Y)\xi)=0$.
Hence

$$\eta((\nabla_{\phi W}R)(X,Y)\xi)=0. \tag{4.8}$$

From (4.7), we obtain

$$\begin{aligned}(\nabla_{\phi W}R)(X,Y)\xi =&(\beta^2-\alpha^2)((\nabla_{\phi W}\eta)X.Y-(\nabla_{\phi W}\eta)Y.X)\\ &-(-\varepsilon\alpha R(X,Y)\phi^2 W+\beta\delta R(X,Y)\phi W-\beta\delta\eta(\phi W)R(X,Y)\xi).\end{aligned} \tag{4.9}$$

Suppose that the manifold is ϕ-recurrent. Then using (4.2), we obtain from the above equation

$$\begin{aligned}&-\eta((\nabla_{\phi W}R)(X,Y)\xi)+A(\phi W)R(X,Y)\xi\\ &=(\beta^2-\alpha^2)(\delta\beta\varepsilon g(\phi W,X)+\alpha g(W,X))Y-(\beta^2-\alpha^2)(\delta\beta\varepsilon g(\phi W,Y)+\alpha g(W,Y))X\end{aligned} \tag{4.10}$$

or

$$0=(\beta^2-\alpha^2)\{\delta\beta\varepsilon[g(\phi W,X)Y-g(\phi W,Y)X]+\alpha[g(W,X)Y-g(W,Y)X]\}, \tag{4.11}$$

i.e.

$$\begin{aligned}&\varepsilon\alpha R(X,Y)W+\alpha(\beta^2-\alpha^2)(g(Y,W)X-g(W,X)Y)\\ &=\beta\delta[-R(X,Y)\phi W+\varepsilon(\beta^2-\alpha^2)(g(\phi W,X)Y-g(\phi W,Y)X)].\end{aligned} \tag{4.12}$$

If $\alpha=0$ and $\beta\neq 0$, then we have

$$R(X,Y)\phi W=\varepsilon\beta^2(g(\phi W,X)Y-g(\phi W,Y)X). \tag{4.13}$$

Change W to ϕ W to get

$$R(X,Y)W=-\varepsilon\beta^2(g(Y,W)X-g(X,W)Y), \tag{4.14}$$

i.e. the manifold M reduces to a β-Kenmotsu manifold of constant curvature. If $\beta=0$ and $\alpha\neq 0$, then we obtain

$$R(X,Y)W=\varepsilon\alpha^2(g(Y,W)X-g(W,X)Y), \tag{4.15}$$

i.e. the manifold M reduces to an α-Sasakian manifold of constant curvature. Thus we can state that

Theorem 4.1. *For constants α and β, an (ε,δ)-trans-Sasakian ϕ-recurrent manifold reduces to a β-Kenmotsu manifold of constant curvature for $\alpha=0$ (respectively an α-Sasakian manifold of constant curvature for $\beta=0$).*

5. PSEUDO-PROJECTIVELY FLAT (ε,δ)-TRANS-SASAKIAN MANIFOLDS

Let M be an (ε,δ)-trans-Sasakian manifold. The pseudo-projective curvature tensor in M is given by [2]

$$\begin{aligned}\overline{P}(X,Y)Z =&aR(X,Y)Z+b[S(Y,Z)X-S(X,Z)Y]\\ &-\frac{r}{n}\left(\frac{a}{n-1}+b\right)[g(Y,Z)X-g(X,Z)Y],\end{aligned} \tag{5.1}$$

where R and S are respectively Riemann and Ricci curvature tensors. Suppose $\overline{P}=0$, then from (5.1) we have

$$R(X,Y)Z=\frac{b}{a}(S(Y,Z)X-S(X,Z)Y)-\frac{r}{an}\left(\frac{a}{n-1}+b\right)(g(Y,Z)X-g(X,Z)Y). \tag{5.2}$$

From (5.2) we have

$$
\begin{aligned}
'R(X,Y,Z,W) =& \frac{b}{a}(S(Y,Z)g(X,W) - S(X,Z)g(Y,W)) \\
& - \frac{r}{an}\left(\frac{a}{n-1} + b\right)(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)),
\end{aligned} \tag{5.3}
$$

where $'R(X,Y,Z,W) = g(R(X,Y)Z,W)$.

Putting $W = \xi$ in (5.3), we obtain

$$
\begin{aligned}
\eta(R(X,Y)Z) =& \frac{b}{a}\varepsilon(S(Y,Z)\eta(X) - S(X,Z)\eta(Y)) \\
& - \frac{r}{an}\left(\frac{a}{n-1} + b\right)\varepsilon(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)).
\end{aligned} \tag{5.4}
$$

Again taking $X = \xi$ in (5.4) and using (3.6) and (3.11), we obtain

$$
\begin{aligned}
S(Y,Z) =& \frac{a}{b}\left[\varepsilon\left(\alpha^2 - \beta^2\right) + \frac{r}{an}\left(\frac{a}{n-1} + b\right)\right] g(Y,Z) \\
& - \frac{a}{b}\left[1 + \frac{b}{a}\left(\alpha^2 - \beta^2\right)(n-1) + \varepsilon\frac{r}{an}\left(\frac{a}{n-1} + b\right)\right]\eta(Z)\eta(Y).
\end{aligned}
$$

Therefore the manifold M is η-Einstein.

6. (ε,δ)-TRANS-SASAKIAN MANIFOLDS SATISFYING $R(X,Y).\overline{P} = 0$

Let M be an n-dimensional (ε,δ)-trans-Sasakian manifold. Using (2.1), (2.2), (2.4), (2.5), and (2.6) in (3.4), we obtain

$$
\begin{aligned}
\eta(\overline{P}(X,Y)Z) =& a(\alpha^2 - \beta^2)[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)] + b[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)] \\
& - \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)].
\end{aligned} \tag{6.1}
$$

Putting $Z = \xi$ in (6.1), we obtain

$$
\eta(\overline{P}(X,Y)\xi) = 0. \tag{6.2}
$$

Again taking $X = \xi$ in (6.1) and using (2.5), (2.6), and (3.11), we have

$$
\eta(\overline{P}(\xi,Y)Z) = \left[a\varepsilon(\alpha^2 - \beta^2) + \frac{r}{n}\left(\frac{a}{n-1} + b\right)\right] g(\phi Y,\phi Z) + b[S(Y,Z) - 2n(\alpha^2 - \beta^2)\eta(Y)\eta(Z)]. \tag{6.3}
$$

Suppose $R(X,Y).P = 0$. Then we have

$$
R(X,Y)\overline{P}(U,V)Z - \overline{P}(R(X,Y)U,V)Z - \overline{P}(U,R(X,Y)V)Z - \overline{P}(U,V)R(X,Y)Z = 0. \tag{6.4}
$$

Contracting (6.4) with respect to ξ and taking $X = \xi$, we obtain

$$
\begin{aligned}
& g(R(\xi,Y)\overline{P}(U,V)Z,\xi) - g(\overline{P}(R(\xi,Y)U,V)Z,\xi) \\
& - g(\overline{P}(U,R(\xi,Y)V)Z,\xi) - g(\overline{P}(U,V)R(\xi,Y)Z,\xi) = 0.
\end{aligned} \tag{6.5}
$$

From this it follows that

$$\begin{aligned}(\alpha^2-\beta^2)'\overline{P}(U,V,Z,Y)&-\varepsilon(\alpha^2-\beta^2)\eta(Y)\eta(\overline{P}(U,V)Z)\\&-(\alpha^2-\beta^2)g(U,Y)\eta(\overline{P}(\xi,V)Z)+\varepsilon(\alpha^2-\beta^2)\eta(U)\eta(\overline{P}(Y,V)Z)\\&-(\alpha^2-\beta^2)g(V,Y)\eta(\overline{P}(U,\xi)Z)+\varepsilon(\alpha^2-\beta^2)\eta(V)\eta(\overline{P}(U,Y)Z)\\&-(\alpha^2-\beta^2)g(Z,Y)\eta(\overline{P}(U,V)\xi)+\varepsilon(\alpha^2-\beta^2)\eta(Z)\eta(\overline{P}(U,V)Y)=0,\end{aligned}\tag{6.6}$$

where $'\overline{P}(U,V,Z,Y)=g(\overline{P}(U,V)Z,Y)$.

Putting $Y=U$ in (6.6), we obtain

$$\begin{aligned}'\overline{P}(U,V,Z,U)&-g(U,U)\eta(\overline{P}(\xi,V)Z)-g(V,U)\eta(\overline{P}(U,\xi)Z)\\&+\varepsilon\eta(V)\eta(\overline{P}(U,U)Z)-g(Z,Y)\eta(\overline{P}(U,V)\xi)+\varepsilon\eta(Z)\eta(\overline{P}(U,V)U)=0.\end{aligned}\tag{6.7}$$

Let $\{e_i\}$, $i=1,2,\dots,n$ be an orthonormal basis of the tangent space at any point. Then the sum for $1\le i\le n$ of the relation (6.7) for $U=e_i$ yields

$$\begin{aligned}(n-1)\eta(\overline{P}(\xi,V)Z)=&(a+(n-1)b)S(V,Z)-\frac{r}{n}(a+(n-1)b)g(V,Z)\\&+\varepsilon(\varepsilon(n-1)(\alpha^2-\beta^2)(b-a)-\frac{r}{n}(b-a))\eta(V)\eta(Z).\end{aligned}\tag{6.8}$$

From (6.3) and (6.8), we have

$$S(V,Z)=\varepsilon(n-1)(\alpha^2-\beta^2)g(V,Z)-((n-1)(\alpha^2-\beta^2)n-\varepsilon r)\frac{b}{a}\eta(V)\eta(Z).\tag{6.9}$$

Taking $Z=\xi$ in (6.9) and using (3.11), we obtain

$$r=\varepsilon n(n-1)(\alpha^2-\beta^2).\tag{6.10}$$

Now using (6.1), (6.2), (6.9), and (6.10) in (6.6), we obtain

$$'\overline{P}(U,V,Z,Y)=0.\tag{6.11}$$

From (6.11), it follows that

$$\overline{P}(U,V)Z=0.$$

Thus we have

Theorem 6.2. *An (ε,δ)-trans-Sasakian manifold with $R.\overline{P}=0$ is projectively flat.*

From the pseudo-projective curvature tensor as given in (5.1), a symmetric tensor of type $(0,2)$ can be defined as follows:

$$Ric\overline{P}(X,Y)='\overline{P}(X,\ e_i,e_i,Y),\tag{6.12}$$

where $'\overline{P}(X,Y,Z,U)=g('\overline{P}(X,Y)Z,U)$ and (e_i) is an orthonormal basis of the tangent space at any point and summing over $1\le i\le n$ in (6.12). From (5.1) and (6.12), we have

$$Ric\overline{P}(X,Y)=(a-b)S(X,Y)+\frac{b-a}{n}rg(X,Y).\tag{6.13}$$

If $R(X,Y).Ric\overline{P}=0$, then we have

$$Ric\overline{P}(R(X,Y)U,V)+Ric\overline{P}(U,R(X,Y)V)=0.\tag{6.14}$$

For constants α and β, taking $X=\xi$ in (6.14) and using (3.7), we have

$$(\alpha^2-\beta^2)g(U,Y)Ric\overline{P}(\xi,V)-\eta(U)Ric\overline{P}(Y,V)+(\alpha^2-\beta^2)g(V,Y)Ric\overline{P}(U,\xi)-\eta(V)Ric\overline{P}(U,Y)=0.$$

Then either $\alpha=\pm\beta$ or

$$g(U,Y)Ric\overline{P}(\xi,V)-\eta(U)Ric\overline{P}(Y,V)+g(V,Y)Ric\overline{P}(U,\xi)-\eta(V)Ric\overline{P}(U,Y)=0. \tag{6.15}$$

By using (6.13) and (3.11) in (6.15), we obtain

$$(a-b)\left[\left(\varepsilon(n-1)(\varepsilon\alpha^2-\delta\beta^2)+\frac{r}{n}\right)(g(Y,V)\eta(U)+g(Y,U)\eta(V))\right] - (a-b)(S(Y,V)\eta(U)+S(Y,U)\eta(V))=0. \tag{6.16}$$

Taking $V=\xi$ in (6.16) and using (2.5) and (3.11), we obtain

$$S(Y,U)=(n-1)\varepsilon(\varepsilon\alpha^2-\delta\beta^2)g(Y,U), \tag{6.17}$$

i.e. M is an Einstein manifold.

Taking $Y=U=e_i$, and summing over $i=1,...,n$, we obtain

$$r=n(n-1)\varepsilon(\varepsilon\alpha^2-\delta\beta^2).$$

Thus we have

Theorem 6.3. *Let M be an (ε,δ)-trans-Sasakian manifold and let $R(X,Y).Ric\overline{P}=0$ hold in M. Then either $\alpha=\pm\beta$ or M is an Einstein manifold in which case the curvature is given by $r=n(n-1)\varepsilon(\varepsilon\alpha^2-\delta\beta^2)$.*

7. CONCLUSIONS

The (ε,δ)-trans-Sasakian manifolds generalize the (ε)-Sasakian and the (ε)-Kenmotsu manifolds. The indefinite metrics which arose during the study of physics and relativity from the geometric point of view influences the curvature. In this paper we proved under certain conditions that the (ε,δ)-trans-Sasakian manifolds reduce to the manifolds of constant curvature. Further we showed that (ε,δ)-trans-Sasakian manifolds with a pseudo-projective curvature tensor are Einstein. The study of these structures with semi-symmetry conditions would give interesting results.

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(ε,δ)-trans-Sasaki muutkondadest

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On uuritud (ε,δ)-trans-Sasaki muutkondi, mis üldistavad (ε)-Sasaki ja (δ)-Kenmotsu muutkondade mõisteid. Näite varal on tõestatud taolise struktuuri olemasolu ja uuritud ϕ-kordse, pseudoprojektiivselt lamedaid ning pseudoprojektiivselt poolsümmeetrilisi (ε,δ)-trans-Sasaki muutkondi.