Generalized Sasakian space forms with semi-symmetric non-metric connections

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Abstract. We introduce generalized Sasakian space forms with semi-symmetric non-metric connections. We show the existence of a generalized Sasakian space form with a semi-symmetric non-metric connection and give some examples by warped products endowed with semi-symmetric non-metric connections.

Key words: generalized Sasakian space form, warped product, semi-symmetric non-metric connection.

1. INTRODUCTION

A semi-symmetric linear connection in a differentiable manifold was introduced by Friedmann and Schouten in [5]. Hayden [6] introduced the idea of a metric connection with torsion in a Riemannian manifold. In [15], Yano studied a semi-symmetric metric connection in a Riemannian manifold. In [1], Agashe and Chafle introduced the notion of a semi-symmetric non-metric connection and studied some of its properties.

Furthermore, in [2], Alegre, Blair, and Carriazo introduced the notion of a generalized Sasakian space form and gave many examples of these manifolds by using some different geometric techniques.

In [11], the present authors studied a warped product manifold endowed with a semi-symmetric metric connection and found relations between curvature tensors, Ricci tensors, and scalar curvatures of the warped product manifold with this connection. Moreover, in [12], we considered generalized Sasakian space forms with semi-symmetric metric connections.

Motivated by the above studies, in the present paper, we consider generalized Sasakian space forms admitting semi-symmetric non-metric connections. We obtain the existence theorem of a generalized Sasakian space form with a semi-symmetric non-metric connection and give some examples by the use of warped products.

The paper is organized as follows: In Section 2, we give a brief introduction to the semi-symmetric non-metric connection. In Section 3, the definition of a generalized Sasakian space form is given and we introduce generalized Sasakian space forms endowed with semi-symmetric non-metric connections. In the last section, the existence theorem of a generalized Sasakian space form with a semi-symmetric non-metric connection is given by warped product $\mathbb{R} \times fN$, where $N$ is a generalized complex space form. In that section we obtain some examples of generalized Sasakian space forms with non-constant functions with respect to semi-symmetric non-metric connections.

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2. SEMI-SYMMETRIC NON-METRIC CONNECTION

Let \( M \) be an \( n \)-dimensional Riemannian manifold with Riemannian metric \( g \). If \( \nabla \) is the Levi-Civita connection of a Riemannian manifold \( M \), a linear connection \( \overset{\circ}{\nabla} \) is given by

\[
\overset{\circ}{\nabla}_X Y = \nabla_X Y + \eta(Y)X,
\]

where \( \eta \) is a 1-form associated with the vector field \( \xi \) on \( M \) defined by

\[
\eta(X) = g(X, \xi),
\]

(see [1]). By the use of (1), the torsion tensor \( T \) of the connection \( \overset{\circ}{\nabla} \)

\[
T(X,Y) = \overset{\circ}{\nabla}_X Y - \overset{\circ}{\nabla}_Y X - [X,Y]
\]

satisfies

\[
T(X,Y) = \eta(Y)X - \eta(X)Y.
\]

A linear connection \( \overset{\circ}{\nabla} \) satisfying (4) is called a semi-symmetric connection. \( \overset{\circ}{\nabla} \) is called a metric connection if

\[
\overset{\circ}{\nabla} g = 0.
\]

If \( \overset{\circ}{\nabla} g \neq 0 \), then \( \overset{\circ}{\nabla} \) is said to be a non-metric connection. In view of (1), it is easy to see that

\[
(\overset{\circ}{\nabla}_X g)(Y,Z) = -\eta(Y)g(X,Z) - \eta(Z)g(X,Y)
\]

(5)

for all vector fields \( X, Y, Z \) on \( M \).

Therefore, in view of (4) and (5), \( \overset{\circ}{\nabla} \) is a semi-symmetric non-metric connection.

Let \( R \) and \( \overset{\circ}{R} \) be curvature tensors of \( \nabla \) and \( \overset{\circ}{\nabla} \) of a Riemannian manifold \( M \), respectively. Then \( R \) and \( \overset{\circ}{R} \) are related by

\[
\overset{\circ}{R}(X,Y)Z = R(X,Y)Z - \alpha(Y,Z)X + \alpha(X,Z)Y
\]

for all vector fields \( X, Y, Z \) on \( M \), where \( \alpha \) is a \((0,2)\)-tensor field denoted by

\[
\alpha(X,Y) = (\nabla_X \eta)Y - \eta(X)\eta(Y),
\]

(see [15]).

3. GENERALIZED SASAKIAN SPACE FORMS

Let \( M \) be an \( n \)-dimensional almost contact metric manifold with an almost contact metric structure \((\phi, \xi, \eta, g)\) consisting of a \((1, 1)\) tensor field \( \phi \), a vector field \( \xi \), a 1-form \( \eta \), and a Riemannian metric \( g \) on \( M \) satisfying

\[
\phi^2X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)
\]

for all vector fields \( X, Y \) on \( M \) [4].

An almost contact metric structure of \( M \) is said to be normal if

\[
[\phi, \phi](X, Y) = -2d\eta(X,Y)\xi,
\]

for any vector fields \( X, Y \) on \( M \), where \( [\phi, \phi] \) denotes the Nijenhuis torsion of \( \phi \), given by

\[
[\phi, \phi](X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y].
\]

A normal contact metric manifold is called a Sasakian manifold [4].
It is well known that an almost contact metric manifold is Sasakian if and only if \((\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X\). Moreover, the curvature tensor \(R\) of a Sasakian manifold satisfies \(R(X, Y)\xi = \eta(Y)X - \eta(X)Y\). An almost contact metric manifold \(M\) is a **trans-Sasakian manifold** [9] if there exist two functions \(\alpha\) and \(\beta\) on \(M\) such that

\[
(\nabla_X \phi)Y = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X]
\]

for any vector fields \(X, Y\) on \(M\). From (7) it follows that

\[
\nabla_X \xi = -\alpha \phi X + \beta [X - \eta(X)\xi].
\]

If \(\beta = 0\) (resp. \(\alpha = 0\)), then \(M\) is said to be an \(\alpha\)-Sasakian manifold (resp. \(\beta\)-Kenmotsu manifold). Sasakian manifolds (resp. Kenmotsu manifolds [7]) appear as examples of \(\alpha\)-Sasakian manifolds (\(\beta\)-Kenmotsu manifolds), with \(\alpha = 1\) (resp. \(\beta = 1\)).

Another kind of trans-Sasakian manifolds is that of **cosymplectic manifolds** [3], obtained for \(\alpha = \beta = 0\). From (8), for a cosymplectic manifold it follows that

\[
\nabla_X \xi = 0.
\]

For an almost contact metric manifold \(M\), a **\(\phi\)-section** of \(M\) at \(p \in M\) is a section \(\pi \subseteq T_pM\) spanned by a unit vector \(X_p\) orthogonal to \(\xi_p\) and \(\phi X_p\). The **\(\phi\)-sectional curvature** of \(\pi\) is defined by \(K(X \wedge \phi X) = R(X, \phi X, \phi X, X)\). A Sasakian manifold with constant \(\phi\)-sectional curvature \(c\) is called a **Sasakian space form**. Similarly, a Kenmotsu manifold with constant \(\phi\)-sectional curvature \(c\) is called a **Kenmotsu space form**. A cosymplectic manifold with constant \(\phi\)-sectional curvature \(c\) is called a **cosymplectic space form**.

Given an almost contact metric manifold \(M\) with an almost contact metric structure \((\phi, \xi, \eta, g)\), \(M\) is called a **generalized Sasakian space form** if there exist three functions \(f_1, f_2, f_3\) on \(M\) such that

\[
R(X, Y)Z = f_1\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}
\]

for any vector fields \(X, Y, Z\) on \(M\), where \(R\) denotes the curvature tensor of \(M\). If \(f_1 = \frac{c+3}{4}, f_2 = f_3 = \frac{c-1}{4}\), then \(M\) is a Sasakian space form; if \(f_1 = \frac{c+3}{4}, f_2 = f_3 = \frac{c+1}{4}\), then \(M\) is a Kenmotsu space form; if \(f_1 = f_2 = f_3 = \frac{c}{4}\), then \(M\) is a cosymplectic space form.

Let \(\nabla\) be semi-symmetric non-metric connection on an almost contact metric manifold \(M\). We define \(M\) as a **generalized Sasakian space form with semi-symmetric non-metric connection** if there exist four functions \(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4\) on \(M\) such that

\[
\tilde{R}(X, Y)Z = \tilde{f}_1\{g(Y, Z)X - g(X, Z)Y\} + \tilde{f}_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} + \tilde{f}_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} + \tilde{f}_4\{g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}
\]

for any vector fields \(X, Y, Z\) on \(M\), where \(\tilde{R}\) denotes the curvature tensor of \(M\) with respect to semi-symmetric non-metric connection \(\tilde{\nabla}\).

**Example 3.1.** A cosymplectic space form with a semi-symmetric non-metric connection is a generalized Sasakian space form with a semi-symmetric non-metric connection such that \(f_1 = f_2 = f_3 = \frac{c}{4}\) and \(f_3 = \frac{c+1}{4}\).

**Example 3.2.** A Kenmotsu space form with a semi-symmetric non-metric connection is a generalized Sasakian space form with a semi-symmetric non-metric connection such that \(f_1 = f_3 = \frac{c+1}{4}\) and \(f_2 = f_4 = \frac{c+1}{4}\).
**Remark 3.3.** A Sasakian space form with a semi-symmetric non-metric connection is not a generalized Sasakian space form with a semi-symmetric non-metric connection.

If \((M,J,g)\) is a Kaehlerian manifold (i.e., a smooth manifold with a \((1,1)\)-tensor field \(J\) and a Riemannian metric \(g\) such that \(J^2 = -I\), \(g(JX,JY) = g(X,Y)\), \(\nabla J = 0\) for arbitrary vector fields \(X,Y\) on \(M\), where \(I\) is identity tensor field and \(\nabla\) the Riemannian connection of \(g\)) with constant holomorphic sectional curvature \(K(X\wedge JX) = c\), then it is said to be a complex space form if its curvature tensor is given by

\[
R(X,Y)Z = \frac{c}{4} \{g(Y,Z)X - g(X,Z)Y + g(X,JZ)JY - g(Y,JZ)JY + 2g(X,JY)JZ\}.
\]

Models for these spaces are \(\mathbb{C}^n\), \(\mathbb{CP}^n\), and \(\mathbb{CH}^n\), depending on \(c = 0\), \(c > 0\), or \(c < 0\).

More generally, if the curvature tensor of an almost Hermitian manifold \(M\) satisfies

\[
R(X,Y)Z = F_1\{g(Y,Z)X - g(X,Z)Y\} + F_2\{g(X,JZ)JY - g(Y,JZ)JY + 2g(X,JY)JZ\},
\]

where \(F_1\) and \(F_2\) are differentiable functions on \(M\), then \(M\) is said to be a generalized complex space form (see [13] and [14]).

**4. EXISTENCE OF A GENERALIZED SASAKIAN SPACE FORM WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION**

Let \((M_1,g_{M_1})\) and \((M_2,g_{M_2})\) be two Riemannian manifolds and \(f\) a positive differentiable function on \(M_1\). Consider the product manifold \(M_1 \times M_2\) with its projections \(\pi: M_1 \times M_2 \rightarrow M_1\) and \(\sigma: M_1 \times M_2 \rightarrow M_2\). The warped product \(M_1 \times_f M_2\) is the manifold \(M_1 \times M_2\) with the Riemannian structure such that

\[
||X||^2 = ||\pi'(X)||^2 + f^2(\pi(p)) ||\sigma'(X)||^2
\]

for any tangent vector \(X \in TM\). Thus we have that

\[
g = g_{M_1} + f^2 g_{M_2}
\]

holds on \(M\). The function \(f\) is called the warping function of the warped product [8].

We need the following lemma from [10] for later use:

**Lemma 4.1.** Let \(M = M_1 \times_f M_2\) be a warped product and \(R\) and \(\tilde{R}\) denote the Riemannian curvature tensors of \(M\) with respect to the Levi-Civita connection and the semi-symmetric non-metric connection, respectively. If \(X,Y,Z \in \chi(M_1), U,V,W \in \chi(M_2)\) and \(\xi \in \chi(M_1)\), then

(i) \(\tilde{R}(X,Y)Z \in \chi(M_1)\) is the lift of \(\hat{R}(X,Y)Z\) on \(M_1\),

(ii) \(\tilde{R}(V,X)Y = [-H'(X,Y)/f - g(Y,\nabla_X \xi) + \eta(X)\eta(Y)]V\),

(iii) \(\tilde{R}(X,Y)V = 0\),

(iv) \(\tilde{R}(V,W)X = 0\),

(v) \(\tilde{R}(X,V)W = -g(V,W)[(\nabla_X \text{grad } f)/f + (\xi f/f)X]\),

(vi) \(\tilde{R}(U,V)W = \{\text{grad } f\}^2/f^2 + (\xi f/f)\} [g(V,W)U - g(U,W)V]\).

Now, let us begin with the existence theorem of a generalized Sasakian space form with a semi-symmetric non-metric connection:
Theorem 4.2. Let \( N(F_1, F_2) \) be a generalized complex space form. Then the warped product \( M = \mathbb{R} \times_f N \) endowed with the almost contact metric structure \((\varphi, \xi, \eta, g)\) with a semi-symmetric non-metric connection is a generalized Sasakian space form with a semi-symmetric non-metric connection such that

\[
\tilde{f}_1 = \frac{(F_1 \circ \pi)}{f^2} - \left( \frac{f'}{f} \right)^2, \quad \tilde{f}_2 = \frac{(F_2 \circ \pi)}{f^2},
\]

\[
\tilde{f}_3 = \frac{(F_1 \circ \pi)}{f^2} - \left( \frac{f'}{f} \right)^2 + \left( \frac{f'' - f}{f} \right), \quad \tilde{f}_4 = \frac{(F_2 \circ \pi)}{f^2} - \left( \frac{f'}{f} \right)^2 - \left( \frac{f''}{f} \right).
\]

**Proof.** For any vector fields \( X, Y, Z \) on \( M \), we can write

\[
X = \eta(X) \xi + U, \quad Y = \eta(Y) \xi + V,
\]

and

\[
Z = \eta(Z) \xi + W,
\]

where \( U, V, W \) are vector fields on a generalized complex space form \( N \). Since the structure vector field \( \xi \) is on \( \mathbb{R} \), then in view of Lemma 4.1 we have

\[
\tilde{R}(X, Y)Z = \eta(X) \eta(Z) \left[ \frac{H_f(\xi, \xi)}{f} \right] - 1 \right] V - \eta(X) g(V, W) \left[ \nabla_\xi \operatorname{grad} f + \frac{f}{f} (\xi, f) \right] \eta(Z)
\]

\[
- \eta(Y) \eta(Z) \left[ \frac{H_f(\xi, \xi)}{f} \right] - 1 \right] U + \eta(Y) g(U, W) \left[ \nabla_\xi \operatorname{grad} f + \frac{f}{f} (\xi, f) \right] \eta(Z)
\]

\[
+ N R(U, V)W - \{ \| \operatorname{grad} f \|^2 / f^2 + (\xi, f / f) \} \eta(V, W) - g(U, W) V.
\] (11)

Since \( f = f(t) \), \( \operatorname{grad} f = f' \xi \), we get

\[
\nabla_\xi \operatorname{grad} f = f'' \xi + f' \nabla_\xi \xi.
\]

By virtue of Proposition 35 on page 206 in [8], since \( \nabla_\xi \xi = 0 \), the above equation reduces to

\[
\nabla_\xi \operatorname{grad} f = f'' \xi.
\] (12)

Moreover, we have

\[
H_f(\xi, \xi) = g(\nabla_\xi \operatorname{grad} f, \xi) = f'',
\] (13)

\[
\| \operatorname{grad} f \|^2 = (f')^2, \quad \xi f = g(\operatorname{grad} f, \xi) = f'.
\] (14)

By virtue of equations (10), (12), (13), and (14) in (11) and by using the fact that \( N \) is a generalized complex space form, we have

\[
\tilde{R}(X, Y)Z = \left( \frac{f'' - f}{f} \right) \{ \eta(X) \eta(Z) V - \eta(Y) \eta(Z) U \}
\]

\[
+ \left( \frac{f'' + f'}{f} \right) \{ f^2 g_{M_1}(U, W) \eta(Y) \xi - f^2 g_{M_2}(V, W) \eta(X) \xi \}
\]

\[
+ (F_1 \circ \pi) \{ g_{M_2}(V, W) U - g_{M_2}(U, W) V \}
\]

\[
+ (F_2 \circ \pi) \{ g_{M_2}(U, JV) JU - g_{M_2}(U, JW) JV + 2 g_{M_2}(U, JW) JW \}
\]

\[
+ \left( \frac{f'}{f} \right)^2 \{ f^2 g_{M_2}(U, W) V - f^2 g_{M_2}(V, W) U \}.
\]
In view of Equation (10) and by the use of the relations between the vector fields $X, Y, Z$ and $U, V, W$, the above equation reduces to

$$\nabla^\circ R(X, Y)Z = \left( \frac{(F_1 \circ \pi)}{f^2} - \left[ \frac{(f')^2}{f} + \frac{f'}{f} \right] \right) \{g(Y, Z)X - g(X, Z)Y\}$$

$$+ \left( \frac{(F_2 \circ \pi)}{f^2} \right) \{g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X + 2g(X, \varphi Y)\varphi Z\}$$

$$+ \left( \frac{(F_1 \circ \pi)}{f^2} \right) \left[ \frac{(f')^2}{f} + \frac{f'}{f} \right] \{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\}$$

$$+ \left( \frac{(F_1 \circ \pi)}{f^2} \right) \left[ \frac{(f')^2}{f} - \frac{f''}{f} \right] \{g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi\}.$$ 

Therefore, we complete the proof of the theorem. 

So we can state the following corollaries:

**Corollary 4.3.** If $N(a, b)$ is a generalized complex space form with constant functions, then we have a generalized Sasakian space form with a semi-symmetric non-metric connection with non-constant functions such that

$$\tilde{f}_1 = \frac{a}{\pi} - \left[ \frac{(f')^2}{f} + \frac{f'}{f} \right], \quad \tilde{f}_2 = \frac{b}{\pi},$$

$$\tilde{f}_3 = \frac{a}{\pi} - \left[ \frac{(f')^2}{f} + \frac{f''}{f} \right], \quad \tilde{f}_4 = \frac{a}{\pi} - \left[ \frac{(f')^2}{f} - \frac{f''}{f} \right].$$

**Corollary 4.4.** If $N(c)$ is a complex space form, we have

$$\tilde{f}_1 = \frac{c}{\pi^2} - \left[ \frac{(f')^2}{f} + \frac{f'}{f} \right], \quad \tilde{f}_2 = \frac{c}{\pi^2},$$

$$\tilde{f}_3 = \frac{c}{\pi^2} - \left[ \frac{(f')^2}{f} + \frac{f''}{f} \right], \quad \tilde{f}_4 = \frac{c}{\pi^2} - \left[ \frac{(f')^2}{f} - \frac{f''}{f} \right].$$

Hence, the warped product $M = \mathbb{R} \times \tilde{f} N(c)$ is a generalized Sasakian space form with a semi-symmetric non-metric connection $\nabla^\circ$.

Thus, for example, the warped product $\mathbb{R} \times \tilde{f} \mathbb{C}^n$ with non-constant functions

$$\tilde{f}_1 = - \left[ \frac{(f')^2}{f} + \frac{f'}{f} \right], \quad \tilde{f}_2 = 0,$$

$$\tilde{f}_3 = - \left[ \frac{(f')^2}{f} + \frac{f''}{f} \right], \quad \tilde{f}_4 = - \left[ \frac{(f')^2}{f} - \frac{f''}{f} \right],$$

the warped product $\mathbb{R} \times \tilde{f} \mathbb{CP}^n(4)$ with non-constant functions

$$\tilde{f}_1 = \frac{1}{\pi} - \left[ \frac{(f')^2}{f} + \frac{f'}{f} \right], \quad \tilde{f}_2 = \frac{1}{\pi},$$
\[ \tilde{f}_3 = \frac{1}{f^2} - \left( \frac{f'}{f} \right)^2 + \frac{f''}{f}, \quad \tilde{f}_4 = \frac{1}{f^2} - \left( \frac{f'}{f} \right)^2 - \frac{f''}{f}, \]

and the warped product \( \mathbb{R} \times \mathcal{C}(\mathbb{H}^{n}(-4)) \) with non-constant functions

\[ \tilde{f}_1 = -\frac{1}{f^2} - \left( \frac{f'}{f} \right)^2, \quad \tilde{f}_2 = -\frac{1}{f^2}, \]

\[ \tilde{f}_3 = -\frac{1}{f^2} - \left( \frac{f'}{f} \right)^2 + \frac{f''}{f}, \quad \tilde{f}_4 = -\frac{1}{f^2} - \left( \frac{f'}{f} \right)^2 - \frac{f''}{f} \]

are generalized Sasakian space forms with semi-symmetric non-metric connections, respectively.

Hence, this method gives us some examples of generalized Sasakian space forms with semi-symmetric non-metric connections with arbitrary dimensions and non-constant functions.

5. CONCLUSION

Generalized Sasakian space forms with semi-symmetric non-metric connections are introduced. It is shown that if \( N(F_1, F_2) \) is a generalized complex space form, then the warped product \( M = \mathbb{R} \times f N \) endowed with the almost contact metric structure \( (\phi, \xi, \eta, g) \) with a semi-symmetric non-metric connection is a generalized Sasakian space form with a semi-symmetric non-metric connection. Using this method, we obtain some examples of generalized Sasakian space forms with semi-symmetric non-metric connections with arbitrary dimensions and non-constant functions.

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