Necessary conditions for inclusion relations for double absolute summability

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Abstract. We establish necessary conditions for a general inclusion theorem involving a pair of doubly triangular matrices. As corollaries we obtain inclusion results for some special classes of doubly triangular matrices.

Key words: absolute summability factors, doubly triangular summability.

1. INTRODUCTION

A doubly infinite matrix \(A = (a_{mni j})\) is said to be doubly triangular if \(a_{mni j} = 0\) for \(i > m\) and \(j > n\). The \(mn\)-th term of the \(A\)-transform of a double sequence \(\{s_{mn}\}\) is defined by

\[
T_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{mni j} s_{ij}.
\]

A series \(\sum \sum c_{mn}\), with partial sums \(s_{mn}\) is said to be absolutely \(A\)-summable, of order \(k \geq 1\), if

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{k-1} |\Delta_{11} T_{m-1,n-1}|^k < \infty,
\]

where, for any double sequence \(\{u_{mn}\}\), and for any fourfold sequence \(\{a_{mni j}\}\), we define

\[
\Delta_{11} u_{mn} = u_{mn} - u_{m+1,n} - u_{m,n+1} + u_{m+1,n+1},
\]

\[
\Delta_{11} a_{mni j} = a_{mni j} - a_{m+1,n,i,j} - a_{m,n+1,i,j} + a_{m+1,n+1,i,j},
\]

\[
\Delta_{ij} a_{mni j} = a_{mni j} - a_{m,n+1,i,j} - a_{m,n,i+1,j} + a_{m,n,i,j+1},
\]

\[
\Delta_{i0} a_{mni j} = a_{mni j} - a_{m,n,i+1,j},
\]

\[
\Delta_{0j} a_{mni j} = a_{mni j} - a_{m,n,i,j+1}.
\]

The one-dimensional version of (1) appears in [1].

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Associated with $A$ are two matrices $\tilde{A}$ and $\hat{A}$ defined by

$$\tilde{a}_{mni} = \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu}$$

and

$$\hat{a}_{mni} = \Delta_{1} \tilde{a}_{m-1,n-1,i,j}, \ 0 \leq i \leq m, \ 0 \leq j \leq n, \ m,n = 0,1,\ldots,$$

and

$$\hat{a}_{mni} = \Delta_{1} \tilde{a}_{m-1,n-1,i,j}, \ 0 \leq i \leq m, \ 0 \leq j \leq n, \ m,n = 1,2,\ldots.$$  

It is easily verified that $\hat{a}_{0000} = \hat{a}_{0000} = a_{0000}$. In [3] it is shown that

$$\hat{a}_{mni} = \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \Delta_{1} a_{m-1,n-1,\mu,\nu}.$$  

Thus $\hat{a}_{m00} = \hat{a}_{m00} = 0$.

Let $x_{mn}$ denote the $mth$ term of the $A$-transform of the sequence of partial sums $\{s_{mn}\}$ of the series $\sum \sum c_{mn}$. Then

$$x_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{mni} s_{ij} = \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{\mu=0}^{i} \sum_{\nu=0}^{j} a_{\mu\nu} c_{\mu\nu}$$

and

$$x_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{n} \tilde{a}_{mni} c_{ij},$$

and a direct calculation verifies that

$$X_{mn} := \Delta_{1} x_{m-1,n-1} = \sum_{i=0}^{m} \sum_{j=0}^{n} \hat{a}_{mni} c_{ij},$$

since

$$\tilde{a}_{m-1,n-1,m,j} = a_{m-1,n-1,i,n} = \hat{a}_{m,n-1,i,n} = \hat{a}_{m-1,n,m,n} = 0.$$

2. MAIN RESULT

We have the following theorem

**Theorem 1.** Let $1 < k \leq s < \infty$, $A$ and $B$ be doubly triangular matrices with $A$ satisfying

$$\sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{k-1} |\Delta_{uv} \hat{a}_{mnuv}|^{k} = O(M^{k}(\hat{a}_{uvuv})), \quad (3)$$

where

$$M(\hat{a}_{uvuv}) := \max \{ |\hat{a}_{uvuv}|, |\Delta_{0} \hat{a}_{uv+1,v,u,v}|, |\Delta_{0} \hat{a}_{u+1,u,v,v}| \}.$$  

Then necessary conditions for $\sum \sum c_{mn}$ summable $|A|_{k}$ to imply that $\sum \sum c_{mn}$ is summable $|B|_{s}$ are:

(i) $|\bar{b}_{uvuv}| = O((uv)^{1/s-1/k}M(\hat{a}_{uvuv}))$,

(ii) $|\Delta_{0} \bar{b}_{uv+1,v,u,v}| = O((uv)^{1/s-1/k}M(\hat{a}_{uvuv}))$,

(iii) $|\Delta_{0} \bar{b}_{u+1,u,v,v}| = O((uv)^{1/s-1/k}M(\hat{a}_{uvuv}))$,

(iv) $\sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{s-1} |\Delta_{uv} \bar{b}_{mnuv}|^{s} = O((uv)^{s-1/k}M^{s}(\hat{a}_{uvuv}))$,

(v) $\sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{s-1} |\bar{b}_{m,n,u+1,v+1}|^{s} = O\left( \sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{k-1} |\hat{a}_{m,n,u+1,v+1}|^{k} \right)^{s/k}$.
Proof. We are given that
\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{x-1} |Y_{mn}|^s < \infty, \tag{4}
\]
whenever
\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{k-1} |X_{mn}|^k < \infty, \tag{5}
\]
where
\[
Y_{mn} = \Delta_{11}y_{m-1,n-1},
\]
\[
y_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{n} \tilde{p}_{mn}c_{ij}.
\]
The space of sequences satisfying (5) is a Banach space if normed by
\[
\|X\| = \left( |X_{00}|^k + |X_{01}|^k + |X_{10}|^k + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{k-1} |X_{mn}|^k \right)^{1/k}. \tag{6}
\]
We shall also consider the space of sequences \( \{Y_{mn}\} \) that satisfy (4). This space is also a BK-space with respect to the norm
\[
\|Y\| = \left( |Y_{00}|^s + |Y_{01}|^s + |Y_{10}|^s + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{s-1} |Y_{mn}|^s \right)^{1/s}. \tag{7}
\]
The transformation \( y_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{n} \tilde{p}_{mn}c_{ij} \) maps sequences satisfying (5) into sequence spaces satisfying (4). By the Banach–Steinhaus Theorem there exists a constant \( K > 0 \) such that
\[
\|Y\| \leq K\|X\|. \tag{8}
\]
For fixed \( u,v \), the sequence \( \{c_{ij}\} \) is defined by \( c_{uv} = c_{u+1,v+1} = 1, c_{u+1,v} = c_{u,v+1} = -1, c_{ij} = 0 \), otherwise, gives
\[
X_{mn} = \begin{cases} 
0, & m \leq u, \ n < v, \\
0, & m < u, \ n \leq v, \\
\hat{a}_{mn}a_{uv}, & m = u, \ n = v, \\
\Delta_{a0}\hat{a}_{mn}, & m = u + 1, \ n = v, \\
\Delta_{a0}\hat{a}_{mn}, & m = u, \ n = v + 1, \\
\Delta_{av}\hat{a}_{mn}, & m > u, \ n > v 
\end{cases}
\]
and
\[
Y_{mn} = \begin{cases} 
0, & m \leq u, \ n < v, \\
0, & m < u, \ n \leq v, \\
\hat{b}_{mn}b_{uv}, & m = u, \ n = v, \\
\Delta_{b0}\hat{b}_{mn}, & m = u + 1, \ n = v, \\
\Delta_{b0}\hat{b}_{mn}, & m = u, \ n = v + 1, \\
\Delta_{bv}\hat{b}_{mn}, & m > u, \ n > v 
\end{cases}
\]
From (6) and (7) it follows that
\[
\|X\| = \left( (uv)^{k-1}|\hat{a}_{uv}|^k + ((u+1)v)^{k-1}|\Delta_{a0}a_{u+1,v,u,v}|^k \\
+ (u(v+1))^{k-1}|\Delta_{a0}a_{u,v+1,u,v}|^k + \sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{k-1}|\Delta_{av}\hat{a}_{mn}a_{uv}|^k \right)^{1/k}. \tag{9}
\]
and
\[ ||Y|| = \left\{ (uv)^{s-1} |\hat{b}_{uvv}|^s + ((u+1)v)^{s-1} |\Delta_0 \hat{b}_{u+1,vu,v}|^s \\
+ (u(v+1))^{s-1} |\Delta_0 \hat{b}_{u,v+1,u,v}|^s + \sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{s-1} |\Delta_{mn} \hat{a}_{mnuv}|^s \right\}^{1/s}. \] (10)

Substituting (9) and (10) into (8), along with (3), gives
\[
(\text{uv})^{s-1} |\hat{b}_{uvv}|^s + ((u+1)v)^{s-1} |\Delta_0 \hat{b}_{u+1,vu,v}|^s + (u(v+1))^{s-1} |\Delta_0 \hat{b}_{u,v+1,u,v}|^s \\
+ \sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{s-1} |\Delta_{mn} \hat{a}_{mnuv}|^s \leq K^s \left\{ (uv)^{k-1} |\hat{a}_{uvuv}|^k \\
+ ((u+1)v)^{k-1} |\Delta_0 \hat{a}_{u+1,vu,v}|^k + (u(v+1))^{k-1} |\Delta_0 \hat{a}_{u,v+1,u,v}|^k \\
+ \sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{k-1} |\Delta_{mn} \hat{a}_{mnuv}|^k \right\}^{s/k} \\
= K^s \left\{ O(1)(uv)^{k-1} M^k (\hat{a}_{uvuv})^{s/k} \right\}^{s/k}.
\]

The above inequality implies that each term of the left-hand side is \( O((uv)^{k-1} M^k (\hat{a}_{uvuv})^{s/k}) \).

Using the first term, one obtains
\[
(\text{uv})^{s-1} |\hat{b}_{uvv}|^s = O((uv)^{s-k-1} M^k (\hat{a}_{uvuv})^{s/k}),
\]
or
\[
|\hat{b}_{uvv}|^s = O((uv)^{s-k-s+1} M^k (\hat{a}_{uvuv})).
\]

Thus
\[
|\hat{b}_{uvv}| = O((uv)^{1/s-1/k} M (\hat{a}_{uvuv})),
\]
which is condition (i).

In a similar manner one obtains conditions (ii)–(iv). Using the sequence defined by \( c_{u+1,v+1} = 1 \), and \( a_{ij} = 0 \) otherwise yields
\[
X_{mn} = \begin{cases} 
0, & m \leq u + 1, \ n \leq v, \\
0, & m \leq u, \ n \leq v + 1, \\
\hat{a}_{m,n,u+1,v+1}, & m \geq u + 1, \ n \geq v + 1
\end{cases}
\]
and
\[
Y_{mn} = \begin{cases} 
0, & m \leq u + 1, \ n \leq v, \\
0, & m \leq u, \ n \leq v + 1, \\
\hat{b}_{m,n,u+1,v+1}, & m \geq u + 1, \ n \geq v + 1.
\end{cases}
\]

The corresponding norms are
\[
||X|| = \left\{ \sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{k-1} |\hat{a}_{m,n,u+1,v+1}|^k \right\}^{1/k}
\]
and
\[
||Y|| = \left\{ \sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{s-1} |\hat{b}_{m,n,u+1,v+1}|^s \right\}^{1/s}.
\]

Applying (8), one obtains
\[
\sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{s-1} |\hat{b}_{m,n,u+1,v+1}|^s \leq K^s \left\{ \sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{k-1} |\hat{a}_{m,n,u+1,v+1}|^k \right\}^{s/k},
\]
which is equivalent to (v). \( \square \)
Corollary 1. Let $1 \leq k < \infty$, $A$ and $B$ be two doubly triangular matrices, $A$ satisfying (3). Then necessary conditions for $\sum c_{mn}$ summable $|A|_k$ to imply that $\sum c_{mn}$ is summable $|B|_k$ are

(i) $|\hat{b}_{uvv}| = O(M(\hat{a}_{uvv}))$,
(ii) $|\Delta_{00}\hat{a}_{u+1,v,v}| = O(M(\hat{a}_{uvv}))$,
(iii) $|\Delta_{00}\hat{a}_{u,v+1,u,v}| = O(M(\hat{a}_{uvv}))$,
(iv) $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\Delta_{uv}\hat{a}_{m,n,v+1}| = O(M(\hat{a}_{uvv}))$,
(v) $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\Delta_{uv}\hat{a}_{m,n,v+1}| = O(M(\hat{a}_{uvv}))$.

Proof. To prove Corollary 1, simply set $s = k$ in Theorem 1.

We shall call a doubly infinite matrix a product matrix if it can be written as the termwise product of two singly infinite matrices $F$ and $G$; i.e., $a_{mij} = f_{mi}g_{nj}$ for each $i,j,m,n$.

A doubly infinite weighted mean matrix $P$ has nonzero entries $p_{ij}/P_{mn}$, where $p_{00}$ is positive and all of the other $p_{ij}$ are nonnegative, and $P_{mn} := \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} p_{ij}$. If $P$ is a product matrix, then the nonzero entries are $p_i q_j / P_m Q_n$, where $p_0 > 0, p_i > 0$ for $i > 0, q_0 > 0, q_j > 0$ for $j > 0$ and $P_m := \sum_{i=0}^{\infty} p_i, Q_n := \sum_{j=0}^{\infty} q_j$.

Corollary 2. Let $1 \leq k < \infty$, $P$ be a product weighted mean matrix, $B$ be a doubly triangular matrix with $P$ satisfying

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (mn)^{k-1} |\Delta_{uvv} P_m q_n P_{u-1} Q_{v-1} P_u Q_v| = O\left(\frac{p_u q_v}{P_u Q_v}\right).$$

Then necessary conditions for $\sum c_{mn}$ summable $|P|_k$ to imply that $\sum c_{mn}$ is summable $|B|_s$ are:

(i) $|\hat{b}_{uvv}| = O\left((uv)^{1-s/k} \frac{p_u q_v}{P_u Q_v}\right)$,
(ii) $|\Delta_{00}\hat{a}_{u+1,v,v}| = O\left((uv)^{1-s/k} \frac{p_u q_v}{P_u Q_v}\right)$,
(iii) $|\Delta_{00}\hat{a}_{u,v+1,u,v}| = O\left((uv)^{1-s/k} \frac{p_u q_v}{P_u Q_v}\right)$,
(iv) $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (mn)^{s-1} |\Delta_{uv} B_{mnv}| = O\left((uv)^{s-1/k} \left(\frac{p_u q_v}{P_u Q_v}\right)^s\right)$, and
(v) $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (mn)^{s-1} |\hat{b}_{m,n,u+1,v+1}| = O(1)$.

Proof. From [3]

$$\hat{b}_{uvv} = \sum_{i=0}^{u-1} \sum_{j=0}^{v-1} \Delta_{11} p_{u-1,v-1,i,j}. \quad (12)$$

Note that

$$\Delta_{11} p_{u-1,v-1,i,j} = p_{u-1,v-1,i,j} - p_{u,v-1,i,j} - p_{u-1,v,i,j} + p_{u,i,j}$$

$$= \frac{p_{i,j} q_j}{P_{u-1} Q_{v-1}} - \frac{p_{i,j} q_j}{P_{u-1} Q_v} - \frac{p_{i,j} q_j}{P_{u} Q_{v-1}} + \frac{p_{i,j} q_j}{P_{u-1} Q_v}$$

$$= \frac{p_{i,j} q_j}{P_{u-1} P_u Q_{v-1} Q_v}. \quad (13)$$
Therefore

\[
\hat{\beta}_{uv} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{P_q j P_u q_v}{P_{u-1} Q_{v-1} Q_v} = \frac{P_u q_v}{P_u Q_v}.
\] (14)

From (12) and (13),

\[
\hat{\beta}_{u+1,v,u,v} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Delta_{11} P_{u,v-1,i,j} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{P_q j P_{u+1} q_v}{P_{u} P_{v+1} Q_{v-1} Q_v} = \frac{P_{u+1} q_v}{P_{u+1} Q_v}.
\]

Using (2) and (14),

\[
\Delta_{00} \hat{\beta}_{u+1,v,u,v} = \hat{\beta}_{u+1,v,u,v} - \hat{\beta}_{u+1,v,u+1,v} = \frac{P_{u+1} q_v}{P_{u+1} Q_v} - \frac{P_{u+1} q_v}{P_{u+1} Q_v} = 0.
\]

Similarly, \(\Delta_{00} p_{u+1,v,u,v} = 0\). Thus

\[
M(\hat{\beta}_{uv}) = \frac{P_u q_v}{P_u Q_v},
\]

and conditions (i)–(v) take the form represented.

**Corollary 3.** Let \(B\) be a doubly triangular matrix, \(P\) a product weighted mean matrix satisfying (11). Then necessary conditions for \(\sum \sum c_{mn}\) summable \(|P|_k\) to imply that \(\sum \sum c_{mn}\) is summable \(|B|_k\) are

(i) \(|\hat{\beta}_{uv}| = O\left(\frac{P_u q_v}{P_u Q_v}\right)\),

(ii) \(|\Delta_{00} \hat{\beta}_{u+1,v,u,v}| = O\left(\frac{P_u q_v}{P_u Q_v}\right)\),

(iii) \(|\Delta_{00} \hat{\beta}_{u,v+1,u,v}| = O\left(\frac{P_u q_v}{P_u Q_v}\right)\),

(iv) \(\sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{k-1} |\Delta_{uv} \hat{\beta}_{uv}|^k = O\left(\left(\frac{P_u q_v}{P_u Q_v}\right)^k\right)\), and

(v) \(\sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{k-1} |\hat{\beta}_{m,n,u+1,v+1}|^k = O(1)\).

**Proof.** In Corollary 2 set \(s = k\).

The results of this paper for single summability are available in [2].

### 3. CONCLUSION

Let \(\sum a_n\) denote a series with partial sums \(s_n\). For an infinite matrix \(A\), the \(n\)th term of the \(A\)-transform of \(\{s_n\}\) is denoted by

\[
t_n = \sum_{v=0}^{\infty} t_m s_v.
\]

Recently, Savas [2] established a general absolute inclusion theorem involving a pair of triangles. But the necessary conditions for a general inclusion theorem involving a pair of doubly triangular matrices has not been studied so far. The present paper has filled in a gap in the existing literature.
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