Group actions, orbit spaces, and noncommutative deformation theory

Arvid Siqveland

Buskerud University College, PoBox 235, 3603 Kongsberg, Norway; arvid.siqveland@hibu.no

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Abstract. Consider the action of a group \( G \) on an ordinary commutative \( k \)-variety \( X = \text{Spec}(A) \). In this note we define the category of \( A \)-\( G \)-modules and their deformation theory. We then prove that this deformation theory is equivalent to the deformation theory of modules over the noncommutative \( k \)-algebra \( A[G] = A \sharp G \). The classification of orbits can then be studied over a commutative ring, and we give an example of this on surface cyclic singularities.

Key words: \( A \)-\( G \) module, noncommutative deformation theory, noncommutative blowup, cyclic surface singularities, orbit closures, swarm of modules, \( r \)-pointed artinian \( k \)-algebras, noncommutative deformation functor, Generalized Matric Massey Products (GMMP), McKay correspondence.

1. INTRODUCTION

Consider the action of a group \( G \) on an ordinary commutative \( k \)-variety \( X = \text{Spec}(A) \). We define the category of \( A \)-\( G \)-modules, Definition 2.1, and their deformation theory. We then prove that this deformation theory is equivalent to the deformation theory of modules over the noncommutative \( k \)-algebra \( A[G] = A \sharp G \). Thus the noncommutative moduli of the one-sided \( A[G] \)-modules can be computed as the noncommutative moduli of \( A \)-modules with \( A \) commutative, invariant under the (dual) action of the group \( G \), which simplify the computations significantly. The orbit closure of \( x \in X \) corresponds to an \( A[G] \)-module \( M_x = A / \mathfrak{a}_x \), so that the classification of closures of orbits can be studied locally by deformation theory of \( M_x \) as an \( A \)-\( G \)-module. Finally, we work through an example of the noncommutative blowup of cyclic surface singularities.

2. MODULES WITH GROUP ACTIONS

Let \( k \) be an algebraically closed field of characteristic 0. Let \( G \) be a finite dimensional reductive algebraic group acting on an affine scheme \( X = \text{Spec}(A) \) a finitely generated (commutative) \( k \)-algebra. Let \( \mathfrak{a}_x \) be the ideal of the closure of the orbit of \( x \) and let \( G \to \text{Aut}_k(A) \) sending \( g \) to \( \nabla_g \) be the induced action of \( G \) on \( A \). Then, as the ideal \( \mathfrak{a}_x \) is invariant under the action of \( G \) on \( A \), we get an induced action on \( A / \mathfrak{a}_x \). The skew group algebra over \( A \) is denoted \( A[G] \). It consists of all formal sums \( \sum_{g \in G} a_g g \) with product defined by

\[
(a_1g_1)(a_2g_2) = a_1 \nabla_{g_1}(a_2)g_1g_2.
\]

For later use notice that this definition extends the definition of the group algebra over \( k \), \( k[G] \). Now, the action of \( A[G] \) on \( M_x \) given by \( (ag)m = a \nabla_g(m) \) defines \( M_x \) as an \( A[G] \)-module because
((a_1g_1)(a_2g_2))m = (a_1\nabla_{g_1}(a_2)g_1g_2)m = a_1\nabla_{g_1}(a_2)\nabla_{g_1g_2}(m) \\
 = a_1\nabla_{g_1}(a_2\nabla_{g_2}(m)) = a_1g_1((a_2g_2)m).

Thus the classification of orbits is the classification of the corresponding $A[G]$-modules $M$. The main issue of this section is the following definition and the lemma proved by the argument above:

**Definition 2.1.** An $A$-$G$-module is an $A$ module with a $G$-action such that the two actions commute, that is

$$\nabla_g(am) = \nabla_g(a)\nabla_g(m).$$

**Lemma 2.1.** The category of $A$-$G$-modules and the category of $A[G]$-modules are equivalent.

### 3. DEFORMATION THEORY

For $A$ a not necessarily commutative $k$-algebra, $V = \{V_i\}_{i=1}^r$ a swarm of right $A$-modules (which means that $\dim_k \operatorname{Ext}_A^i(V_i,V_j) < \infty$ for $1 \leq i, j \leq r$), there exists a well-known deformation theory, see [3]. Let $a_r$ be the category of $r$-pointed artinian $k$-algebras. It consists of the commutative diagrams

$$
\begin{array}{ccc}
k^r & \longrightarrow & R \\
\text{Id} & \downarrow & \rho \\
k^r & \longrightarrow & k^r
\end{array}
$$

such that $\operatorname{rad}(R) = \ker(\rho)$ fulfills $\operatorname{rad}(R)^n = 0$ for some $n$. Generalizing the commutative case, we set $\hat{a}_r$ equal to the category of complete $r$-pointed $k$-algebras $\hat{R}$ such that $\hat{R}/\operatorname{rad}(\hat{R})^n$ is in $a_r$ for all $n$. Letting $R_{ij} = e_i R e_j$, it is easy to see that $\hat{R}$ is isomorphic to the matrix algebra $(R_{ij})$. The noncommutative deformation functor $\operatorname{Def}_V : a_r \rightarrow \text{Sets}$ is given by

$$\operatorname{Def}_V(R) = \{R \otimes_k A^{op}-modules V_R | V_R \cong_R (R_{ij} \otimes_k V_j), k_i \otimes_R V_R \cong V_i\}/ \cong.$$ 

Let $V_R \in \operatorname{Def}_V(R)$. The left $R$-module structure is the trivial one, and the right $A$-module structure is given by the morphisms $\sigma_A^R : V_i \rightarrow R_{ij} \otimes_k V_j$. As in the commutative case, an $(r$-pointed$)$ morphism $\phi : S \rightarrow R$ is small if $\ker(\phi) \cdot \operatorname{rad}(S) = \operatorname{rad}(S) \cdot \ker(\phi) = 0$, and for such morphisms, lifting the $\sigma_A^R$ directly to $S$, the associativity condition gives the obstruction class $o(\phi, V_R) = (\sigma^S_{ab} - \sigma^S_{ac} \sigma^S_{cb}) \in I \otimes_k \operatorname{HH}^2(A, \operatorname{Hom}_k(V_i, V_j))$ where $I = (I_{ij}) = \ker(\phi)$, such that $V_R$ can be lifted to $V_S$ if and only if $o(V_R, \phi) = 0$, see [3] or [1] for details and complete proofs. Obviously, computations are much easier if $A$ is a commutative $k$-algebra. This is possible to achieve when working with $G$-actions and orbit spaces. For a family $V = \{V_i\}_{i=1}^r$ of $A$-$G$-modules, we put

$$\operatorname{Def}_V^G(R) = \{V_R \in \operatorname{Def}_V(R) | \exists A - G$-structure $V : G \rightarrow \operatorname{End}(V_R)\} \subseteq \operatorname{Def}_V(R).$$

In [2,3] Laudal constructs the local formal moduli of $A$-modules. In [5,6] applications in the commutative case are given, and in [7] an easy noncommutative example is worked through. In these cases we start with the $k$-algebra $k[\varepsilon] = k[\varepsilon]/\varepsilon^2$ and use the tangent space

$$\operatorname{Def}_V(k[\varepsilon]) \cong \{\operatorname{HH}^1(A, \operatorname{Hom}_k(V_i, V_j))\} \cong \operatorname{Ext}_A^1(M,M)$$

as dual basis for the local formal moduli $\hat{H}$. The relations among the base elements are given by the obstruction space

$$\operatorname{HH}^2(A, \operatorname{Hom}_k(V_i, V_j)) \cong (\operatorname{Ext}_A^2(V_i, V_j)).$$
4. GENERALIZED MATRIX MASSEY PRODUCTS (GMMP)

Let \( \{V_i\}_{i=1}^r \) be a given swarm of \( A \)-modules. For each \( i \), choose free resolutions \( 0 \leftarrow V_i \xrightarrow{d_i} L_{i,0} \xrightarrow{d_i} L_{i,1} \leftarrow L_{i,2} \leftarrow \cdots \). We write

\[
L_i = \begin{pmatrix}
L_{i,0} & 0 & \cdots & 0 \\
0 & L_{i,1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & L_{i,r}
\end{pmatrix}
\]

and we can prove Lemma 4.1 following the proof in [6] step by step:

**Lemma 4.1.** Let \( V_S \in \text{Def}_V(S) \) and let \( \phi : R \to S \) be a small surjection. Then there exists a resolution \( L^S = (S \otimes_k L_\cdot, d^S) \) lifting the complex \( L_\cdot \), and to give a lifting \( V_R \) of \( V_S \) is equivalent to lift the complex \( L^S \) to \( L^R \).

**Proof.** Generalized from the commutative case, \( M_R \cong_R (R_{ij} \otimes_k M_j) \) is equivalent with \( M_R \) \( R \)-flat. Using this, and tensoring the sequence \( 0 \to I \to R \to S \to 0 \) with \( M_R \) over \( R \), gives the sequence \( 0 \to I \otimes_k M \to M_R \to M_S \to 0 \). Ordinary diagram chasing then proves that the resolution of \( M_S \) can be lifted to an \( R \)-complex \( L^R \) given the resolution \( L^S \) of \( M_S \). Conversely, given a lifting \( L^R \) of the complex \( L^S \) of \( M_S \), the long exact sequence proves that this complex is a resolution, and that \( M_R = H^0(L^R) \) is a lifting of \( M_S \).

If \( M \) is an \( A \to G \) module where \( G \) acts rationally on \( A \) and \( M \) is a rational \( G \)-module, finitely generated as an \( A \)-module, then an \( A \)-free (projective) resolution of \( M \) can be lifted to an \( A \to G \)-free resolution, that is a commutative diagram

\[
\begin{array}{cccccccc}
0 & \leftarrow & V & \leftarrow & A^{n_0} & \leftarrow & A^{n_1} & \leftarrow & A^{n_2} & \leftarrow & \cdots \\
\downarrow \nu_{x} & & \downarrow \nu_{x,0} & & \downarrow \nu_{x,1} & & \downarrow \nu_{x,2} & & & \\
0 & \leftarrow & V & \leftarrow & A^{m_0} & \leftarrow & A^{m_1} & \leftarrow & A^{m_2} & \leftarrow & \cdots \\
\end{array}
\]

This proves that Lemma 4.1 is a particular case of the same lemma with \( \text{Def}_V(S) \) replaced by \( \text{Def}_V^G(S) \). In [7] we give the definition of GMMP. The tangent space of the deformation functor is \( \text{Def}_V^G(E) \cong (\text{Ext}^1_{A \to G}(V_i, V_j)) \), where \( E \) is the noncommutative ring of dual numbers, i.e. \( E = k < t_{ij} > / (t_{ij})^2 \). For computations we note that when \( G \) is reductive and finite dimensional, \( \text{Hom}_{A \to G}(V_i, V_j) \cong \text{Hom}_A(V_i, V_j)^G \) and \( \text{Ext}^1_{A \to G}(V_i, V_j) \cong \text{Ext}^1_A(V_i, V_j)^G \), \( G \) acting by conjugation. Given a small surjection \( \phi : R \to S \), with kernel \( I = (I_{ij}) \), lift \( d^S \) on \( S \otimes_k L \) to \( d^R \) on \( R \otimes_k L \) in the obvious way. Then \( o(\phi, V_S) = \{ d^R_i d^R_{i-1} \}_{i \geq 1} \in (I_{ij} \otimes_k \text{Ext}^2_{A \to G}(V_i, V_j)) \). By the definition of GMMP in [7], these can be read out of the coefficients of a basis in the obstruction space above.

5. THE MCKAY CORRESPONDENCE

Let

\[
G = \mathbb{Z}_2 = \left< \begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix} \right> = < \tau >
\]

act on \( A^2_F \) by \( \tau(a, b) = (-a, -b) \). Our goal is to classify the \( G \)-orbits, and to find a compactification \( M_G \leftarrow P^2_C \) of the orbit space \( M_G \). The existing partial solution is

\[
M_G = \text{Spec}(k[x^2, xy, y^2]) = \text{Spec}(A^G), \ A = k[x, y].
\]
This is an orbit space, but not moduli. Consider the point \( P = (a, b) = (\sqrt{w}, t \sqrt{w}), \ w \neq 0 \). Then
\[
o(P) = \{(\sqrt{w}, t \sqrt{w}), (-\sqrt{w}, -t \sqrt{w})\} = Z(I_t),
\]
where \( I_t = (x^2 - w, y - tx) \). We compute the local formal moduli of the \( A-G \)-module \( M_t = A/I_t \) from the diagram
\[
\begin{array}{cccc}
0 & \rightarrow & A/I_t & \rightarrow & A^{n_1} & \rightarrow & A^{n_2} & \rightarrow & \cdots \\
& & & & \phi & \equiv 0 & & & \\
& & & & & A/I_t & & & \\
\end{array}
\]
where the upper row is a resolution, we see that in general, \( \text{Ext}^1_{A-G}(M_t, I_t^2, A/I_t) \) with the action of \( G \) given by conjugation, that is the composition given in the sequence
\[
I_t \xrightarrow{\varphi} I_t \xrightarrow{\phi} A/I_t \xrightarrow{\varphi^{-1}} A/I_t.
\]
We get
\[
(x^2 - w, y - tx) \xrightarrow{\varphi} (x^2 - w, y - tx) \xrightarrow{\phi} k[x, y]/I_t \xrightarrow{\varphi^{-1}} k[x, y]/I_t
\]
so that \( \phi = (\alpha, \beta x) = \alpha(1, 0) + \beta(0, x) \) is invariant under the action of \( G \). Writing this up in complex form, we get
\[
\begin{array}{cccc}
0 & \rightarrow & M_t & \rightarrow & A^{d_0} & \rightarrow & A^{d_1} & \rightarrow & A \rightarrow 0 \\
& & \xi^1 \rightarrow & \xi^1 \rightarrow & A & \rightarrow & A & \rightarrow & 0 \\
0 & \rightarrow & M_t & \rightarrow & A^{d_0} & \rightarrow & A^{d_1} & \rightarrow & A \rightarrow 0 \\
& & \xi^1 \rightarrow & \xi^1 \rightarrow & A & \rightarrow & A & \rightarrow & 0 \\
\end{array}
\]

\[
d_0 = (x^2 - w y - tx), \ d_1 = \left( \frac{y - tx}{w - x^2} \right), \ \xi^1 = (1, 0), \ \xi^2 = (0, x), \ \xi^2 = \left( \begin{array}{c} 0 \\ -1 \end{array} \right), \ \xi^2 = \left( \begin{array}{c} x \\ 0 \end{array} \right).
\]

We find \( \xi^1 \xi^2 = \xi^1 \xi^2 = \xi^1 \xi^2 + \xi^1 \xi^2 = 0 \), which means that all cup-products are identically zero. Thus \( H_M = k[[t_1, t_2]] \) with algebraization \( H_M = k[t_1, t_2] \). Because the particular point \( 0 = (0, 0) \) corresponds to \( M_0 = k[x,y]/(x,y) \) with \( \text{Ext}^1_{A-G}(M_0, M_0) = 0 \), we understand that \( M_0 \) is a singular point, so that the modulus is \( M_G = (A^2 - \{0\}) \cup \{pt\} \). At least in this case, resolving the singularity is a process of compactifying. Given a family \( V = \{V_i\}_{i=1}^n \) of simple \( A \)-modules, an \( A \)-module \( E \) with composition series \( E = E_0 \supset E_1 \supset \cdots \supset E_i \supset E_{i-1} \supset \cdots \supset E_r \supset 0 \), where \( E_k/E_{k-1} = V_{k_i} \), is called an iterated extension of the family \( V \), and the graph \( \Gamma(E) \) of \( E \) (the representation type) is the graph with nodes in correspondence with \( V \) and arrows \( \rho_{i, j} \) connecting the nodes \( V_i \) and \( V_{i+1} \), identifying arrows if the corresponding extensions are equivalent. In [3] Laudal solves the problem of classifying all indecomposable modules \( E \) with fixed extension graph \( \Gamma \). He proves that for every \( E \) there exists a morphism \( \phi : H(V) \rightarrow k[\Gamma] \) such that \( E \cong M \hat{\otimes}_k k[\Gamma] \), where \( M \) is the versal family, resulting in a noncommutative scheme \( \text{Ind}(\Gamma) \).

In [4], he then proves that the set \( \text{Simp}_n(A) \) of \( n \)-dimensional simple representations of \( A \) with the Jacobson topology has a natural scheme structure. He also proves that when \( \Gamma \) is a representation graph of dimension \( n = \sum_{V \in \Gamma} \dim_k V \), then the set \( \text{Simp}(\Gamma) = \text{Simp}_n(A) \cup \text{Ind}(\Gamma) \) has a natural scheme structure with the Jacobson topology, which is a compactification of \( \text{Simp}_n(A) \). In our present example, we let \( \Gamma \) be the representation type of the regular representation \( k[\Gamma] \). We construct the composition series \( k[\Gamma] \cong k[\tau]/(\tau^2 - 1) \supset (\tau - 1)/(\tau^2 - 1) \supset 0 \). Thus we get \( V_0 = k[\tau]/(\tau - 1) \cong k, V_1 = (\tau - 1)/(\tau^2 - 1) \cong k \) and the action \( \nabla^i_\tau \) of \( \tau \) on \( V_i \) is given by \( \nabla^i_\tau = (-1)^i \). From the sequence \( (x, y) \overset{\varphi}{\longrightarrow} (x, y) \overset{\phi}{\longrightarrow} V_i \overset{\varphi^{-1}}{\longrightarrow} V_i \)
we immediately see that \( \text{Ext}^1_{A - G}(V_i, V_j) = \alpha(1,0) + \beta(0,1) \) when \( i \neq j \), 0 if \( i = j \). Writing up the corresponding diagram and multiplying as in the previous example, we get

\[
H(V_1, V_2) = \begin{pmatrix}
\begin{pmatrix}
< t_{21}(1), t_{21}(2) > & < t_{12}(1), t_{12}(2) > \\
\end{pmatrix} \\
\begin{pmatrix}
t_{12}(1)t_{21}(2) - t_{12}(2)t_{21}(1) \\
t_{21}(1)t_{12}(2) - t_{21}(2)t_{12}(1)
\end{pmatrix}
\end{pmatrix}.
\]

The versal family is given as the cokernel of the morphism

\[
\Psi: \left( \begin{array}{cc}
A^2 & 0 \\
0 & A^2
\end{array} \right) \rightarrow \left( \begin{array}{cc}
H_{11} \otimes A & H_{12} \otimes A \\
H_{21} \otimes A & H_{22} \otimes A
\end{array} \right),
\]

\[
\Psi = \begin{pmatrix}
1 \otimes (x,y) & t_{12}(1) \otimes (1,0) + t_{12}(2) \otimes (0,1) \\
t_{21}(1) \otimes (1,0) + t_{21}(2) \otimes (0,1) & 1 \otimes (x,y)
\end{pmatrix}.
\]

Now, as \( k[\Gamma] = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix} \), \( \phi : H \rightarrow k[\Gamma] \) sends both \( t_{21}(1) \) and \( t_{21}(2) \) to 0. The isomorphism classes of indecomposable \( A[G] \)-modules with representation type \( \Gamma \) are thus given by

\[
V_i = \left( \begin{array}{ccc}
x & y & 0 & 0 \\
-1 & -t & x & y
\end{array} \right),
\]

\[
V_\infty = \left( \begin{array}{ccc}
x & y & 0 & 0 \\
0 & -1 & x & y
\end{array} \right).
\]

The inherited group action is \( \nabla : \left( \begin{array}{c}
1 \\
0 \\
-1
\end{array} \right) \) on \( k^2 \). To find \( \text{Simp}(\Gamma) \), we start by computing the local formal moduli of the (worst) module \( V_i \), following the algorithm in [2]. We find

\[
\text{Ext}^1_{A - G}(V_i, V_i) = \text{Der}_k(A, \text{End}_k(V_i))/\text{Triv} = \left\{ \delta \mid \delta(x) = \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix}, \delta(y) = \begin{pmatrix} 0 & w(t + v) \\ v & 0 \end{pmatrix} \right\}
\]

by using (in particular) the fact that \( xy = yx \) in \( A \). Then \( H(V_i)^\text{com} = k[v, w] \) with versal family \( \begin{pmatrix} x & y & -w & -w(t + v) \\ 1 & -(t + v) & x & y \end{pmatrix} \), computed by again using the fact that \( xy = yx \) in \( A \). While \( w = 0 \) gives the indecomposable module \( V_{x+t} \), \( w \neq 0 \) gives a simple two-dimensional \( A-G \)-module given by \( x^2 = w, y^2 = (t + v)^2w \). This gives an embedding \( A^2 = k[s_0, s_1, s_2]/(s_0s_1 - s_2^2) = k[x^2, xy, y^2] \hookrightarrow k[v, w] \) inducing the morphism \( \text{Simp}_\Gamma \rightarrow \text{Spec}(A_G) \) which is the ordinary blowup of the singular point. The exceptional fibre is \( \left( \begin{array}{ccc}
x & y & 0 & 0 \\
-1 & -t & x & y
\end{array} \right) \cup V_\infty \cong \mathbb{P}^1 \).

REFERENCES

Rühmatoimed, orbiitruumid ja mittekommutatiivne deformatsiooniteooria

Arvid Siqveland