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MATHEMATICS

Equivariant Lie–Rinehart cohomology

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Abstract. In this paper we study Lie–Rinehart cohomology for quotients of singularities by finite groups, and interpret these cohomology groups in terms of integrable connection on modules.

Key words: connections, quotient singularities, integrability, Lie–Rinehart cohomology.

1. INTRODUCTION

In [1], we studied Lie–Rinehart cohomology of singularities, and we gave an interpretation of these cohomology groups in terms of integrable connections on modules of rank one defined on the given singularities. The purpose of this paper is to study equivariant Lie–Rinehart cohomology of the quotient of a singularity by a finite group, and relate these cohomology groups to integrable connections on modules of rank one defined on the quotient.

Let k be an algebraically closed field of characteristic 0, let A be a reduced Noetherian k-algebra, and let (M,∇) be a finitely generated torsion free A-module of rank one with a (not necessarily integrable) connection. There is an obstruction $\mathrm{ic}(M) \in \mathrm{RH}^2(\mathrm{Der}_k(A),\overline{A})$ for the existence of an integrable connection on M, where $\overline{A} = \mathrm{End}_A(M)$, and if $\mathrm{ic}(M) = 0$, then $\mathrm{RH}^1(\mathrm{Der}_k(A),\overline{A})$ is a moduli space for the set of integrable connections on M, up to analytical equivalence; see Theorem 2 in [1].

We recall that an extension $R \subseteq S$ of a normal domain R of dimension two is called Galois if S is the integral closure of R in L, where $K \subseteq L$ is a finite Galois extension of the quotient field K of R, and $R \subseteq S$ is unramified at all prime ideals of height one.

Theorem 1. *Let G be a finite group, let A be a normal domain of dimension two of essential finite type over k with a group action of G, and let M be an A-G module that is finitely generated, maximal Cohen–Macaulay of rank one as an A-module. We assume that the M admits a connection. If $A^G \subseteq A$ is a Galois extension, then the following hold:*

(1) *We have that* $\mathrm{RH}^n(\mathrm{Der}_k(A^G),A^G) \cong \mathrm{RH}^n(\mathrm{Der}_k(A),A)^G$ *for all* $n \geq 0$.
(2) *There is a canonical class* $\mathrm{ic}(M^G) \in \mathrm{RH}^2(\mathrm{Der}_k(A),A)^G$*, called the integrability class, such that* $\mathrm{ic}(M^G) = 0$ *if and only if there exists an integrable connection on* M^G.
(3) *If* $\mathrm{ic}(M^G) = 0$*, then* $\mathrm{RH}^1(\mathrm{Der}_k(A),A)^G$ *is a moduli space for the set of integrable connections on* M^G*, up to analytical equivalence.*

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We use this result to compute $\mathrm{RH}^*(\mathrm{Der}_k(A^G), A^G)$ when $A = k[x_1,x_2,x_3]/(f)$ is a quasi-homogeneous surface singularity and the action of $G = \mathbf{Z}_m$ on A is of type $(m;m_1,m_2,m_3)$ such that $A^G \subseteq A$ is a Galois extension. Explicitly, the action of G on A is given by

$$g{*}x_1 = \xi^{m_1}x_1, \quad g{*}x_2 = \xi^{m_2}x_2, \quad g{*}x_3 = \xi^{m_3}x_3$$

for a cyclic generator g of G and a primitive m'th root of unity $\xi \in k$. In the case when $f = x_1^3+x_2^3+x_3^3$ and $G = \mathbf{Z}_3$ with action of type $(3;1,1,2)$, we compute that $\mathrm{RH}^n(\mathrm{Der}_k(A^G), A^G) = 0$ for $n \geq 1$. In particular, if M is a maximal Cohen–Macaulay A-module of rank one that admits a connection, then the A^G-module M^G admits an integrable connection, unique up to analytic equivalence.

2. BASIC DEFINITIONS

Let k be an algebraically closed field of characteristic 0, let A be a commutative k-algebra, and let M be an A-module. We define a *connection* on M to be an A-linear map $\nabla : \mathrm{Der}_k(A) \to \mathrm{End}_k(M)$ such that

$$\nabla_D(am) = a\nabla_D(m) + D(a)\, m$$

for all $D \in \mathrm{Der}_k(A),\ a \in A,\ m \in M$.

Let ∇ be a connection on M. We define the *curvature* of ∇ to be the A-linear map $R_\nabla : \mathrm{Der}_k(A) \wedge \mathrm{Der}_k(A) \to \mathrm{End}_A(M)$ given by

$$R_\nabla(D \wedge D') = [\nabla_D, \nabla_{D'}] - \nabla_{[D,D']}$$

for all $D, D' \in \mathrm{Der}_k(A)$. Notice that $R_\nabla \in \mathrm{Hom}_A(\wedge_A^2 \mathrm{Der}_k(A), \mathrm{End}_A(M))$. We say that ∇ is an *integrable connection* if $R_\nabla = 0$, i.e. if ∇ is a homomorphism of Lie algebras.

For any integrable connection ∇ on M, we consider the Lie–Rinehart cohomology $\mathrm{RH}^*(\mathrm{Der}_k(A), M, \nabla)$, see [1]. We recall that the Lie–Rinehart cohomology is the cohomology of the Lie–Rinehart complex, given by

$$\mathrm{RC}^n(\mathrm{Der}_k(A), M, \nabla) = \mathrm{Hom}_A(\wedge_A^n \mathrm{Der}_k(A), M)$$

for $n \geq 0$, with differentials $d^n : \mathrm{RC}^n(\mathrm{Der}_k(A), M, \nabla) \to \mathrm{RC}^{n+1}(\mathrm{Der}_k(A), M, \nabla)$ given by

$$d^n(\xi)(D_0 \wedge \cdots \wedge D_n) = \sum_{0 \leq i \leq n} (-1)^i\, \nabla_{D_i}(\xi(D_0 \wedge \cdots \wedge \widehat{D}_i \wedge \cdots \wedge D_n)) + \sum_{0 \leq j < k \leq n} (-1)^{j+k} \xi([D_j, D_k] \wedge D_0 \wedge \cdots \wedge \widehat{D}_j \wedge \cdots \wedge \widehat{D}_k \wedge \cdots \wedge D_n)$$

for all $n \geq 0$ and all $\xi \in \mathrm{RC}^n(\mathrm{Der}_k(A), M, \nabla)$, $D_0, D_1, \dots, D_n \in \mathrm{Der}_k(A)$.

3. GROUP ACTIONS

Let $\sigma : G \to \mathrm{Aut}_k(A)$ be a group action of a group G on the k-algebra A. For simplicity, we shall write $g{*}a = \sigma(g)(a)$ for all $g \in G,\ a \in A$. We remark that σ induces a group action of G on $\mathrm{Der}_k(A)$, given by

$$g{*}D = gDg^{-1} = \{a \mapsto g{*}D(g^{-1}{*}a)\}$$

for $g \in G, D \in \mathrm{Der}_k(A)$. Notice that we have $g{*}(aD) = (g{*}a) \cdot (g{*}D)$ for all $a \in A$, $D \in \mathrm{Der}_k(A)$.

We recall that an A-G module structure on the A-module M is a group action $G \to \mathrm{Aut}_k(M)$ that is compatible with the A-module structure, i.e. a group action such that $g{*}(am) = (g{*}a) \cdot (g{*}m)$ for all $g \in G,\ a \in A,\ m \in M$. Hence $\mathrm{Der}_k(A)$ has a natural A-G module structure induced by σ.

Let M,N be A-G modules, and consider the natural group actions of G on the A-modules $\mathrm{Hom}_A(M,N)$, and $M \wedge_A N$ given by

$$g{*}\phi = g\phi g^{-1} = \{m \mapsto g{*}\phi(g^{-1}{*}m)\} \quad \text{and} \quad g{*}(m \wedge n) = (g{*}m) \wedge (g{*}n)$$

for all $g \in G$, $\phi \in \mathrm{Hom}_A(M,N)$, $m \in M$ and $n \in N$. We remark that this gives $\mathrm{Hom}_A(M,N)$ and $M \wedge_A N$ natural A-G module structures.

Let M be an A-G module. Then there is an induced action of G on the set of connections on M, and $g{*}\nabla$ is given by

$$(g{*}\nabla)_D(m) = g{*}\nabla_{g^{-1}{*}D}(g^{-1}{*}m)$$

for any connection $\nabla : \mathrm{Der}_k(A) \to \mathrm{End}_k(M)$ and for all $g \in G$, $D \in \mathrm{Der}_k(A)$, $m \in M$. A straightforward calculation shows that

$$g{*}R_\nabla = R_{g{*}\nabla}$$

for all $g \in G$. In particular, if ∇ is G-invariant, then the same holds for R_∇.

If G is a finite group, then there is a Reynolds' operator $M \to M^G$ for any A-G module M, given by

$$m \mapsto m' = \frac{1}{|G|} \sum_{g \in G} g{*}m$$

for all $m \in M$. Similarly, if ∇ is a connection on an A-G module M and G is a finite group, then

$$\nabla' = \frac{1}{|G|} \sum_{g \in G} g{*}\nabla$$

is a G-invariant connection on M. However, notice that $R_{\nabla_1+\nabla_2} \neq R_{\nabla_1} + R_{\nabla_2}$ in general. Hence, the Reynolds' type operator $\nabla \mapsto \nabla'$ will not necessarily preserve integrability.

Let M be an A-G module, and let ∇ be an integrable connection on M. Then $\mathrm{RC}^n(\mathrm{Der}_k(A), M, \nabla) = \mathrm{Hom}_A(\wedge^n \mathrm{Der}_k(A), M)$ is an A-G module for all $n \geq 0$. In fact, we have the following result:

Lemma 1. *For any $g \in G$ such that $g{*}\nabla = \nabla$, the diagram*

$$\begin{array}{ccccccc}
M & \xrightarrow{d^0} & \mathrm{Hom}_A(\mathrm{Der}_k(A), M) & \xrightarrow{d^1} & \mathrm{Hom}_A(\wedge_A^2 \mathrm{Der}_k(A), M) & \xrightarrow{d^2} & \cdots \\
\downarrow g{*} & & \downarrow g{*} & & \downarrow g{*} & & \\
M & \xrightarrow[d^0]{} & \mathrm{Hom}_A(\mathrm{Der}_k(A), M) & \xrightarrow[d^1]{} & \mathrm{Hom}_A(\wedge_A^2 \mathrm{Der}_k(A), M) & \xrightarrow[d^2]{} & \cdots
\end{array}$$

commutes. In particular, if ∇ is G-invariant, then G acts on the Lie–Rinehart cohomology $\mathrm{RH}^*(\mathrm{Der}_k(A), M, \nabla)$.

Proof. Let $n \geq 0$ and let $\xi \in \mathrm{RC}^n(\mathrm{Der}_k(A), M, \nabla)$. We must show that if $g{*}\nabla = \nabla$, then $g{*}d^n(\xi) = d^n(g{*}\xi)$, i.e. that $(g{*}d^n(\xi))(D_0 \wedge \cdots \wedge D_n) = (d^n(g{*}\xi))(D_0 \wedge \cdots \wedge D_n)$ for all $D_0, \ldots, D_n \in \mathrm{Der}_k(A)$. To simplify notation, we write

$$\begin{aligned}
m_i &= g{*}\xi(g^{-1}{*}(D_0 \wedge \ldots \widehat{D_i} \wedge \cdots \wedge D_n)), \\
m_{jk} &= g{*}\xi(g^{-1}{*}([D_j, D_k] \wedge D_0 \wedge \cdots \wedge \widehat{D_j} \wedge \cdots \wedge \widehat{D_k} \wedge \cdots \wedge D_n))
\end{aligned}$$

for $0 \leq i \leq n$ and $0 \leq j < k \leq n$. Using this notation, we compute that

$$\begin{aligned}
(g{*}d^n(\xi))(D_0 \wedge \cdots \wedge D_n) &= g{*}d^n(\xi)(g^{-1}{*}(D_0 \wedge \cdots \wedge D_n)) \\
&= \sum_i (-1)^i \, g{*}\nabla_{g^{-1}{*}D_i}(g^{-1}{*}m_i) + \sum_{j<k} (-1)^{j+k} \, m_{jk} \\
&= \sum_i (-1)^i \, (g{*}\nabla)_{D_i}(m_i) + \sum_{j<k} (-1)^{j+k} \, m_{jk}
\end{aligned}$$

and that

$$(d^n(g*\xi))(D_0 \wedge \cdots \wedge D_n) = \sum_i (-1)^i \, \nabla_{D_i}((g*\xi)(D_0 \wedge \ldots \widehat{D_i} \wedge \ldots D_n))$$
$$+ \sum_{j<k} (-1)^{j+k} \, (g*\xi)([D_j, D_k] \wedge D_0 \wedge \cdots \wedge \widehat{D_j} \wedge \cdots \wedge \widehat{D_k} \wedge \cdots \wedge D_n)$$
$$= \sum_i (-1)^i \, \nabla_{D_i}(m_i) + \sum_{j<k} (-1)^{j+k} \, m_{jk}$$

since $g^{-1}[D_j, D_k] = [g^{-1}D_j, g^{-1}D_k]$. But $g*\nabla = \nabla$ by assumption, and it follows that $g*d^n(\xi) = d^n(g*\xi)$. □

4. INTEGRABLE CONNECTIONS AND LIE–RINEHART COHOMOLOGY

In the rest of this paper, we assume that A is a reduced Noetherian k-algebra and that M is a finitely generated torsion free A-module of rank one. Hence there is an isomorphism $f : Q(A) \to Q(A) \otimes_A M$ of $Q(A)$-modules, where $Q(A)$ is the total ring of fractions of A. We identify M with its image in $Q(A) \otimes_A M$, and let $N = f^{-1}(M) \subseteq Q(A)$, so that $f : N \to M$ is an isomorphism of A-modules. Then there is an identification

$$\mathrm{End}_A(M) \cong \overline{A} = \{q \in Q(A) : q \cdot N \subseteq N\}.$$

We consider $\overline{A}$ as a commutative overring with $A \subseteq \overline{A} \subseteq Q(A)$. If A is normal, then $\overline{A} = A$, see [1], Proposition 3.1.

We remark that if ∇ is a connection on M, then there is an induced connection $\overline{\nabla}$ on the A-module $\overline{A} \cong \mathrm{End}_A(M)$. The induced connection $\overline{\nabla}$ is trivial in the sense that

$$\overline{\nabla}_D(q) = \overline{D}(q)$$

for any $D \in \mathrm{Der}_k(A)$ and any $q \in \overline{A}$, where $\overline{D}$ is the natural lifting of D to $\mathrm{Der}_k(Q(A))$. In particular, $\overline{\nabla}$ is an integrable connection on $\overline{A}$.

Since $\overline{\nabla}$ is a canonical integrable connection on $\overline{A}$, it is natural to consider the Lie–Rinehart cohomology $\mathrm{RH}^*(\mathrm{Der}_k(A), \overline{A})$ with values in $(\overline{A}, \overline{\nabla})$. We recall the following result:

Theorem 2 (Eriksen–Gustavsen)**.** *Let A be a reduced Noetherian k-algebra and let M be a finitely generated, torsion free A-module of rank one. We assume that M admits a connection.*
(1) *There is a canonical class* $\mathrm{ic}(M) \in \mathrm{RH}^2(\mathrm{Der}_k(A), \overline{A})$*, called the integrability class, such that* $\mathrm{ic}(M) = 0$ *if and only if there exists an integrable connection on M.*
(2) *If* $\mathrm{ic}(M) = 0$*, then* $\mathrm{RH}^1(\mathrm{Der}_k(A), \overline{A})$ *is a moduli space for the set of integrable connections on M, up to analytical equivalence.*

Proof. See [1], Proposition 3.2 and Theorem 3.4. □

5. THE EQUIVARIANT CASE

Let A be a reduced Noetherian k-algebra with a group action $\sigma : G \to \mathrm{Aut}_k(A)$, and let M be an A-G module that is finitely generated, torsion free of rank one as an A-module. Then there is a natural group action of G on $Q(A)$ induced by σ, given by

$$g*q = g*\left(\frac{a}{s}\right) = \frac{g*a}{g*s}$$

for any $g \in G$ and any $q = a/s \in Q(A)$. Let us consider the (not necessarily equivariant) isomorphism $f : Q(A) \to Q(A) \otimes_A M$ of $Q(A)$-modules, and the submodule $N = f^{-1}(M) \subseteq Q(A)$. Then we have an identification

$$\mathrm{End}_A(M) \cong \overline{A} = \{q \in Q(A) : q \cdot N \subseteq N\}$$

as above, and the group action of G on $\mathrm{End}_A(M)$ induced by the A-G module structure on M coincides with the natural group action on $\overline{A}$ induced by the group action of G on $Q(A)$. In fact, if $\phi \in \mathrm{End}_A(M)$ corresponds to $q \in \overline{A}$, then ϕ is given by $\phi(m) = q \cdot m$ for all $m \in M$, and

$$(g{*}\phi)(m) = g{*}\phi(g^{-1}{*}m) = g{*}(q \cdot (g^{-1}{*}m)) = (g{*}q) \cdot m$$

for all $g \in G$ and $m \in M$. We also remark that since the induced connection $\overline{\nabla}$ on $\overline{A}$ is trivial, it is clear that $\overline{\nabla}$ is G-invariant.

Assume that R is a normal domain of dimension two over k, let $K \subseteq L$ be a finite Galois extension of the quotient field K of R, and let S be the integral closure of R in L. If the extension $R \subseteq S$ is unramified at all prime ideals of height one, we say that it is a *Galois extension*. It is known that if $S = k[x,y]$ is a polynomial ring and $G \subseteq \mathrm{Aut}_k(S)$ is a finite subgroup without pseudo-reflections, then $S^G \subseteq S$ is a Galois extension.

Theorem 3. *Let G be a finite group, let A be a normal domain of dimension two of essential finite type over k with a group action of G, and let M be an A-G module that is finitely generated, maximal Cohen–Macaulay of rank one as an A-module. We assume that the M admits a connection. If $A^G \subseteq A$ is a Galois extension, then the following hold:*
(1) *We have that* $\mathrm{RH}^n(\mathrm{Der}_k(A^G), A^G) \cong \mathrm{RH}^n(\mathrm{Der}_k(A), A)^G$ *for all* $n \geq 0$.
(2) *There is a canonical class* $\mathrm{ic}(M^G) \in \mathrm{RH}^2(\mathrm{Der}_k(A), A)^G$*, called the integrability class, such that* $\mathrm{ic}(M^G) = 0$ *if and only if there exists an integrable connection on* M^G.
(3) *If* $\mathrm{ic}(M^G) = 0$*, then* $\mathrm{RH}^1(\mathrm{Der}_k(A), A)^G$ *is a moduli space for the set of integrable connections on* M^G*, up to analytical equivalence.*

Proof. Since A is normal, we have that $\overline{A} = A$, and therefore $\mathrm{RH}^*(\mathrm{Der}_k(A), A)$ is the Lie–Rinehart cohomology associated with $(\overline{A}, \overline{\nabla})$. The functor $M \mapsto M^G$ is exact since G is a finite group, hence $\mathrm{RH}^*(\mathrm{Der}_k(A), A)^G \cong \mathrm{H}^*(\mathrm{RC}^*(\mathrm{Der}_k(A), A)^G)$. Moreover, it follows as in Proposition 4.4 of [2] that

$$\mathrm{Hom}_A(\wedge^n \mathrm{Der}_k(A), A)^G \cong \mathrm{Hom}_{A^G}(\wedge^n \mathrm{Der}_k(A^G), A^G),$$

and this proves the first part of the theorem. By assumption, A^G is a normal domain, and we notice that M^G is a maximal Cohen–Macaulay A^G-module, see [2], Proposition 4.3. Therefore

$$\nabla' = \frac{1}{|G|} \sum_{g \in G} g{*}\nabla$$

is a connection on M^G, and $\mathrm{RH}^n(\mathrm{Der}_k(A^G), A^G)$ is the Lie–Rinehart cohomology associated with $(\overline{A^G}, \overline{\nabla'})$. The rest of the theorem follows from Theorem 2. □

We remark that this result can be generalized to higher dimensions with suitable conditions on the extension $A^G \subseteq A$. In the case of surface quotient singularities, we have the following corollary:

Corollary 4. *Let $A = k[x_1, x_2]$ be a polynomial algebra, and let $G \subseteq \mathrm{Aut}_k(A)$ be a finite subgroup without pseudo-reflections. Then we have* $\mathrm{RH}^n(\mathrm{Der}_k(A^G), A^G) = 0$ *for* $n \geq 1$. *In particular, any maximal Cohen–Macaulay module over A^G of rank one has an integrable connection, unique up to analytic equivalence.*

6. QUOTIENTS OF QUASI-HOMOGENEOUS SINGULARITIES

Let $A = k[x_1,x_2,x_3]/(f)$ be an integral quasi-homogeneous surface singularity. We write

$$f = \sum_{\alpha \in \mathbf{N_0}^3} c_\alpha \underline{x}^\alpha$$

and define $I(f) = \{\alpha \in \mathbf{N_0}^3 : c_\alpha \neq 0\}$. Then there are integral weights $d = \deg(f)$ and $d_i = \deg(x_i)$ for $i = 1,2,3$ such that $\alpha_1 d_1 + \alpha_2 d_2 + \alpha_3 d_3 = d$ for all $\alpha \in I(f)$.

Let $(m_1,m_2,m_3) \in \mathbf{N_0}^3$ and let $G = \mathbf{Z}_m = \langle g : g^m = 1\rangle$ be the cyclic group of order m. If $\alpha_1 m_1 + \alpha_2 m_2 + \alpha_3 m_3 = m$ for all $\alpha \in I(f)$, then there is a group action of G on A given by

$$g{*}x_1 = \xi^{m_1} \cdot x_1,$$
$$g{*}x_2 = \xi^{m_2} \cdot x_2,$$
$$g{*}x_3 = \xi^{m_3} \cdot x_3,$$

where $\xi \in k$ is a primitive m'th root of unity. We call this a group action of G on A of type $(m;m_1,m_2,m_3)$.

Theorem 5. *Let $A = k[x_1,x_2,x_3]/(f)$ be an integral quasi-homogeneous surface singularity with weights $(d;d_1,d_2,d_3)$, and let G be a cyclic group of order m with a group action on A of type $(m;m_1,m_2,m_3)$. If $A^G \subseteq A$ is a Galois extension, then we have*

(1) $\mathrm{RH}^0(\mathrm{Der}_k(A),A)^G = (A_0)^G = k$,

(2) $\mathrm{RH}^1(\mathrm{Der}_k(A),A)^G = \big(\mathrm{RH}^1(\mathrm{Der}_k(A),A)_0\big)^G = (A_{d-d_1-d_2-d_3} \cdot e_1)^G$,

(3) $\mathrm{RH}^2(\mathrm{Der}_k(A),A)^G = \big(\mathrm{RH}^2(\mathrm{Der}_k(A),A)_0\big)^G = (A_{d-d_1-d_2-d_3} \cdot e_2)^G$

for a generator $e_n \in \mathrm{RH}^n(\mathrm{Der}_k(A),A)_0$ with $g{*}e_n = \xi^{m_1+m_2+m_3-m} \cdot e_n$ for $n = 1,2$.

Proof. In Theorem 6.2 of [1], we considered the case when $A = k[x_1,x_2,x_3]/(f)$ is quasi-homogeneous, and proved that

$$\mathrm{RH}^0(\mathrm{Der}_k(A),A) = A_0 = k,$$
$$\mathrm{RH}^1(\mathrm{Der}_k(A),A) = \mathrm{RH}^1(\mathrm{Der}_k(A),A)_0 = A_{d-d_1-d_2-d_3} \cdot \psi^{(4)},$$
$$\mathrm{RH}^2(\mathrm{Der}_k(A),A) = \mathrm{RH}^2(\mathrm{Der}_k(A),A)_0 = A_{d-d_1-d_2-d_3} \cdot \Delta^*.$$

An explicit description of $e_1 = \psi^{(4)}$ and $e_2 = \Delta^*$ is given in [1], Section 6, and we see that

$$g{*}\psi^{(4)} = \xi^{m_1+m_2+m_3-m} \cdot \psi^{(4)}, \quad g{*}\Delta^* = \xi^{m_1+m_2+m_3-m} \cdot \Delta^*. \qquad \square$$

Example. Let $A = k[x_1,x_2,x_3]/(f)$ with $f = x_1^3 + x_2^3 + x_3^3$, and consider the action of $G = \mathbf{Z}_3$ on A of type $(3;1,1,2)$, given by $g{*}x_i = \xi \cdot x_i$ for $i = 1,2$ and $g{*}x_3 = \xi^2 \cdot x_3$. In this case, it is known that $A^G \subseteq A$ is a Galois extension, see [2]. We have $d - d_1 - d_2 - d_3 = 0$ and $m_1 + m_2 + m_3 - m = 1$, hence it follows from Theorem 5 that G acts non-trivially on $\mathrm{RH}^n(\mathrm{Der}_k(A),A) \cong A_0 = k$ for $n = 1,2$. This implies that $\mathrm{RH}^n(\mathrm{Der}_k(A^G),A^G) \cong \mathrm{RH}^n(\mathrm{Der}_k(A),A)^G = 0$ for $n = 1,2$. In particular, if M is a maximal Cohen–Macaulay A-module of rank one that admits a connection, then M^G admits an integrable connection, unique up to analytic equivalence.

In fact, it is known that A^G is a rational singularity in this case, and it follows from Remark 5.2 in [1] that $\mathrm{RH}^1(\mathrm{Der}_k(A^G),A^G) = 0$. On the other hand, the result that $\mathrm{RH}^2(\mathrm{Der}_k(A^G),A^G) = 0$ is stronger than the results obtained in [1].

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Ekvivariantne Lie-Rineharti kohomoloogia

Eivind Eriksen ja Trond Stølen Gustavsen

On uuritud Lie-Rineharti kohomoloogiat singulaarsuste faktorruumi jaoks lõplike rühmade järgi ja interpreteeritud neid kohomoloogiarühmi integreeruvate seostuste terminites moodulitel.