Proceedings of the Estonian Academy of Sciences,
2010, **59**, 2, 93–98
doi: 10.3176/proc.2010.2.06
Available online at www.eap.ee/proceedings

MECHANICS

# The generalized dressing method with applications to the integration of variable-coefficient Toda equations

*Dedicated to Jüri Engelbrecht on the occasion of his 70th birthday*

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Received 15 December 2009, accepted 3 February 2010

**Abstract.** Integrable variable-coefficient 2D Toda lattice equations are proposed by utilizing a generalized version of the dressing method. Compatibility conditions are given, which ensures that these equations are integrable. Further, soliton solutions for the new type of equations are shown in explicit forms.

**Key words:** variable-coefficient Toda equation, generalized dressing method, integrability.

## 1. INTRODUCTION

The dressing method based on the triangular factorization of Volterra integrable operators was first introduced by Zakharov and Shabat [1,2] for generating integrable nonlinear evolution equations and constructing their multi-soliton solutions. A number of authors have used this method to study various integrable equations. Chowdhury and Basak [3] applied it to obtain the soliton solution of the Hirota–Satsuma coupled system of the KdV equations. Dye and Parker [4] examined regularized long-wave (RLW) equation and its explicit solutions by using this method. Further, with the aid of this method, Parker [5] studied the Sawada–Kotera equation and gave a reformulation of the dressing method via Hirota's formulation. In [1,2] authors only transformed constant-coefficient operators into dressed constant-coefficient ones. Dai and Jeffrey [6] and Jeffrey and Dai [7] extended the dressing method to a variable-coefficient and generalized version and constructed the inverse scattering transformations for certain types of variable-coefficient KdV equation. The generalization provided a procedure for construction of integrable variable-coefficient equations and gave their explicit solutions. In the present work we develop the generalization to the discrete version for generating an integrable variable-coefficient Toda equation. Also, we shall represent the one-soliton and two-soliton solutions in explicit forms.

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## 2. A GENERALIZED VERSION OF THE DRESSING METHOD

In this section we extend the generalized version of the dressing method for discrete systems.

First we consider three linear difference operators

$$\begin{aligned}\mathbf{F}(n,m,t,y)\psi_n &= \sum_{-\infty}^{\infty} F(n,m,t,y)\psi_m,\\ \mathbf{K}_+(n,m,t,y)\psi_n &= \sum_{n}^{\infty} K_+(n,m,t,y)\psi_m,\\ \mathbf{K}_-(n,m,t,y)\psi_n &= \sum_{-\infty}^{n} K_-(n,m,t,y)\psi_m.\end{aligned} \tag{1}$$

Similar to the generalized dressing method for continuous systems, we introduce the triangular factorization about the operator '**F**'

$$\mathbf{I}+\mathbf{F} = (\mathbf{I}+\mathbf{K}_+)^{-1}(\mathbf{I}+\mathbf{K}_-), \tag{2}$$

where **I** is the identity operator, $K_+(n,m,t,y) = 0$ for $m < n$ and $K_-(n,m,t,y) = 0$ for $m > n$. It is assumed that

$$\sup\sum_{n_0}^{\infty} |K_\pm(n,m,t,y)|\psi_m < \infty, \quad \sup\sum_{n_0}^{\infty} |F(n,m,t,y)|\psi_m < \infty,$$

for all $n_0 > -\infty$. For convenience, we denote $F(n,m) = F(n,m,t,y)$, $K_\pm(n,m) = K_\pm(n,m,t,y)$.

The discrete Gel′fand–Levitan–Marchenko equation can be obtained from (2), which reads (cf. [1])

$$F(n,m) + K_+(n,m) + \sum_{s=n}^{\infty} K_+(n,s)F(s,m) = 0. \tag{3}$$

We introduce two differential-difference operators $\mathbf{M_1}$ and $\mathbf{M_2}$ defined by

$$\mathbf{M_1} = \alpha_1\partial_y + \beta_1\partial_t + a_1\mathbf{E} + a_{-1}\mathbf{E^{-1}},\ \mathbf{M_2} = \alpha_2\partial_y + \beta_2\partial_t + b_{-1}\mathbf{E^{-1}}, \tag{4}$$

where **E** is the shift operator of the discrete variable $n$, defined by $\mathbf{E}^k f(n) = f(n+k)$, $k \in \mathbb{Z}$; $t$ and $y$ are continuous variables, $a_1, a_{-1}, b_{-1}, \alpha_1, \alpha_2, \beta_1$, and $\beta_2$ are functions of $t$ and $y$.

Suppose that the operator **F** commutes with $\mathbf{M_1}$ and $\mathbf{M_2}$, i.e.,

$$[\mathbf{M_1},\mathbf{F}] = \mathbf{M_1F} - \mathbf{FM_1} = 0, \quad [\mathbf{M_2},\mathbf{F}] = \mathbf{M_2F} - \mathbf{FM_2} = 0. \tag{5}$$

From (4) and (5) we can obtain two equations for $F$:

$$\alpha_1 F_y(n,m) + \beta_1 F_t(n,m) + a_1 F(n+1,m) + a_{-1}F(n-1,m) - F(n,m-1)a_1 - F(n,m+1)a_{-1} = 0, \tag{6}$$

$$\alpha_2 F_y(n,m) + \beta_2 F_t(n,m) + b_{-1}F(n-1,m) - F(n,m+1)b_{-1} = 0. \tag{7}$$

The dressing operators $\mathbf{N_1}$ and $\mathbf{N_2}$ are introduced from the relations

$$\mathbf{N_1}(\mathbf{I}+\mathbf{K}_+(n,m)) - (\mathbf{I}+\mathbf{K}_+(n,m))\mathbf{M_1} = 0, \tag{8}$$

$$\mathbf{N_2}(\mathbf{I}+\mathbf{K}_+(n,m)) - (\mathbf{I}+\mathbf{K}_+(n,m))\mathbf{M_2} = 0. \tag{9}$$

Similar to a theorem in [1] for continuous systems, it can be proved that $\mathbf{N_1}$ and $\mathbf{N_2}$ are differential-difference operators. For the sake of simplicity, we denote $K(n,m) = K_+(n,m)$ and $\hat{K} = K(n,m)|_{m=n}$.

We write

$$\mathbf{N_1} = \mathbf{M_1} + \mathbf{D_1}, \quad \mathbf{N_2} = \mathbf{M_2} + \mathbf{D_2}. \tag{10}$$

Then, from (8) and (9), after some calculations, we find that

$$\mathbf{D_1} = c_{-1}\mathbf{E^{-1}} + c_0, \quad \mathbf{D_2} = d_{-1}\mathbf{E^{-1}}, \tag{11}$$

and

$$\alpha_1 \widehat{K}_y + \beta_1 \widehat{K}_t + a_{-1}(K(n-1,n) - K(n,n+1)) + c_0(1+\widehat{K}) + c_{-1}K(n-1,n) = 0, \tag{12}$$

$$c_{-1}(1 + K(n-1,n-1)) - a_{-1}(K(n,n) - K(n-1,n-1)) = 0, \tag{13}$$

$$d_{-1}(1 + K(n-1,n-1)) - b_{-1}(K(n,n) - K(n-1,n-1)) = 0. \tag{14}$$

The following theorem in [7] is an extension of the original dressing method, which can yield a wide range of integrable variable-coefficient nonlinear evolution equations.

**Theorem.** *If the operators* $\mathbf{M_1}$ *and* $\mathbf{M_2}$ *satisfy*

$$[\mathbf{M_1}, \mathbf{M_2}] = \rho_1 \mathbf{M_1} + \rho_2 \mathbf{M_2}, \tag{15}$$

*where* $\rho_1$ *and* $\rho_2$ *are arbitrary functions of* $t$ *and* $y$, *then their corresponding dressing operators satisfy*

$$[\mathbf{N_1}, \mathbf{N_2}] = \rho_1 \mathbf{N_1} + \rho_2 \mathbf{N_2}. \tag{16}$$

Actually, variable-coefficient nonlinear evolution equations are obtained from (16).

In fact, from (15) we find that $a_1, a_{-1}, b_{-1}, \alpha_1, \alpha_2, \beta_1, \beta_2, \rho_1$, and $\rho_2$ satisfy

$$\begin{aligned} \alpha_1\alpha_{2y} + \beta_1\alpha_{2t} - \alpha_2\alpha_{1y} - \beta_2\alpha_{1t} &= \rho_1\alpha_1 + \rho_2\alpha_2, \\ \alpha_1\beta_{2y} + \beta_1\beta_{2t} - \alpha_2\beta_{1y} - \beta_2\beta_{1t} &= \rho_1\beta_1 + \rho_2\beta_2, \\ \alpha_1 b_{-1,y} + \beta_1 b_{-1,t} - \alpha_2 a_{-1,y} - \beta_2 a_{-1,t} &= \rho_1 a_{-1} + \rho_2 b_{-1}, \\ -\alpha_2 a_{1y} - \beta_2 a_{1t} &= \rho_1 a_1. \end{aligned} \tag{17}$$

These are the compatibility conditions for (16) to be integrable.

Using (16), we obtain the nonlinear evolution equations

$$a_1 \Delta d_{-1} - \alpha_2 c_{0y} - \beta_2 c_{0t} - \rho_1 c_0 = 0, \tag{18}$$

$$\alpha_1 d_{-1,y} + \beta_1 d_{-1,t} - \alpha_2 c_{-1,y} - \beta_2 c_{-1,t} + (b_{-1} + d_{-1})(c_0 - E^{-1}c_0) - \rho_1 c_{-1} - \rho_2 d_{-1} = 0, \tag{19}$$

where $\Delta$ is a difference operator, which is defined as $\Delta\psi(n) = \psi(n+1) - \psi(n)$ for any function $\psi$.

Let

$$u_n = \frac{1 + K(n,n)}{1 + K(n-1,n-1)}, \quad c_0 = v_n. \tag{20}$$

Utilizing (13), (14), and (20), equations (18) and (19) can be changed to

$$a_1 b_{-1} \Delta u_n - \alpha_2 v_{n,y} - \beta_2 v_{n,t} - \rho_1 v_n = 0, \tag{21}$$

$$(\alpha_1 b_{-1} - \alpha_2 a_{-1}) u_{n,y} + (\beta_1 b_{-1} - \beta_2 a_{-1}) u_{n,t} + b_{-1} u_n (v_n - v_{n-1}) = 0. \tag{22}$$

We further let

$$u_n = e^{x_{n-1}-x_n}, \quad v_n = \left(\alpha_1 - \alpha_2\frac{a_{-1}}{b_{-1}}\right)x_{n,y} + \left(\beta_1 - \beta_2\frac{a_{-1}}{b_{-1}}\right)x_{n,t}. \tag{23}$$

Substitution of (23) into (21) and (22) yields the integrable 2D variable-coefficient Toda lattice equation

$$\begin{aligned} a_1 b_{-1}\Delta e^{x_{n-1}-x_n} &- \left[\alpha_2\left(\alpha_1 - \alpha_2\frac{a_{-1}}{b_{-1}}\right)_y + \beta_2\left(\alpha_1 - \alpha_2\frac{a_{-1}}{b_{-1}}\right)_t + \rho_1\left(\alpha_1 - \alpha_2\frac{a_{-1}}{b_{-1}}\right)\right]x_{n,y} \\ &- \left[\alpha_2\left(\beta_1 - \beta_2\frac{a_{-1}}{b_{-1}}\right)_y + \beta_2\left(\beta_1 - \beta_2\frac{a_{-1}}{b_{-1}}\right)_t + \rho_1\left(\beta_1 - \beta_2\frac{a_{-1}}{b_{-1}}\right)\right]x_{n,t} \\ &- \left[\alpha_2\left(\beta_1 - \beta_2\frac{a_{-1}}{b_{-1}}\right) + \beta_2\left(\alpha_1 - \alpha_2\frac{a_{-1}}{b_{-1}}\right)\right]x_{n,ty} - \alpha_2\left(\alpha_1 - \alpha_2\frac{a_{-1}}{b_{-1}}\right)x_{n,yy} \\ &- \beta_2\left(\beta_1 - \beta_2\frac{a_{-1}}{b_{-1}}\right)x_{n,tt} = 0. \end{aligned} \tag{24}$$

For the $N$-soliton solution of the integrable equation (24), we let $F$ be

$$F(n,m) = \sum_{j=1}^{N} f_j(t,y,n)g_j(t,y,m), \tag{25}$$

where $f_j(t,y,n)$ and $g_j(t,y,m)$ are some $l \times l$ matrices, whose expressions can be obtained from (6) and (7). Moveover, we suppose that

$$K(n,m) = \sum_{j=1}^{N} k_j(t,y,n)g_j(t,y,m). \tag{26}$$

Substituting (25) and (26) into the discrete GLM equation (3) gives

$$K(n,n) = \sum_{j=1}^{N} k_j(t,y,n)g_j(t,y,n) = -(f_1, f_2, \cdots, f_N)L^{-1}(g_1, g_2, \cdots, g_N)^T, \tag{27}$$

where $L$ is defined by

$$L_{jl} = \delta_{jl} + \sum_{s=n}^{\infty} g_j(t,y,s)f_l(t,y,s), \quad 1 \le j,l \le N.$$

The $N$-soliton solution for (24) can be obtained from (20) and (27).

In the following section we consider some special forms of $\mathbf{M_1}$ and $\mathbf{M_2}$.

## 3. AN INTEGRABLE 2D VARIABLE-COEFFICIENT TODA LATTICE EQUATION

Let differential-difference operators be

$$\mathbf{M_1} = \alpha_1\partial_y + a_1\mathbf{E} + a_{-1}\mathbf{E^{-1}}, \quad \mathbf{M_2} = \beta_2\partial_t + b_{-1}\mathbf{E^{-1}}. \tag{28}$$

From (17) we have

$$\begin{gathered} -\beta_2\alpha_{1t} = \rho_1\alpha_1, \quad \alpha_1\beta_{2y} = \rho_2\beta_2, \\ -\beta_2 a_{1t} = \rho_1 a_1, \quad \alpha_1 b_{-1,y} - \beta_2 a_{-1,t} = \rho_1 a_{-1} + \rho_2 b_{-1}. \end{gathered} \tag{29}$$

Then, from (24), after some calculations, we obtain the integrable 2D variable-coefficient Toda lattice equation

$$e^{x_n-x_{n+1}}-e^{x_{n-1}-x_n}-h_1(t)h_2(y)x_{n,yt}+h_1^2(t)h_2(y)h_3(y)x_{n,tt}+\frac{1}{2}(h_1^2(t))_t h_2(y)h_3(y)x_{n,t}=0, \tag{30}$$

where $h_1(t)=\frac{\beta_2}{b_{-1}}, h_2(y)=\frac{\alpha_1}{a_1}$, and $h_3(y)=\frac{a_{-1}}{\alpha_1}$ are arbitrary functions. The above equation becomes the well-known 2D Toda lattice for $\alpha_1=\beta_2=a_1=a_{-1}=b_{-1}=1$ , $\xi=y-t$.

**Case 1. One-soliton solution**

We take $N=1$ in (25). From (6) and (7) we have the special solution

$$F(n,m)=e^{w(t)+q(y)+p_1m+p_2n}, \tag{31}$$

with $q(y)=(e^{-p_1}-e^{p_2})\int h_2^{-1}(y)dy+(e^{p_1}-e^{-p_2})\int h_3(y)dy, \quad w(t)=(e^{p_1}-e^{-p_2})\int h_1^{-1}(t)dt$.

We obtain the one-soliton solution of equation (30)

$$u_n=\frac{(1-e^{p_1+p_2}+e^{w(t)+q(y)+(p_1+p_2)(n-1)})(1-e^{p_1+p_2}+e^{w(t)+q(y)+(p_1+p_2)(n+1)})}{(1-e^{p_1+p_2}+e^{w(t)+q(y)+(p_1+p_2)n})^2}. \tag{32}$$

**Case 2. Two-soliton solution**

We take $N=2$ in (25). From (6) and (7) we have the following special solution:

$$F(n,m)=\sum_{j=1}^{2}f_j(t,y,n)g_j(m,t,y)=e^{w_1(t)+q_1(y)+p_1^{(2)}n}e^{p_1^{(1)}m}+e^{w_2(t)+q_2(y)+p_2^{(2)}n}e^{p_2^{(1)}m}, \tag{33}$$

with

$$q_j=(e^{-p_j^{(1)}}-e^{p_j^{(2)}})\int h_2^{-1}(y)dy+(e^{p_j^{(1)}}-e^{-p_j^{(2)}})\int h_3(y)dy, \quad w_j=(e^{p_j^{(1)}}-e^{-p_j^{(2)}})\int h_1^{-1}(t)dt, j=1,2,$$

$p_j^{(1)}$ and $p_j^{(2)}$ are arbitrary negative constants.

Then, from (27) we have

$$\begin{aligned}K(n,n)=-\frac{1}{|L|}\Bigg[&e^{w_1(t)+q_1(y)+p_1^{(2)}n}+e^{w_2(t)+q_2(y)+p_2^{(2)}n}\\&+\frac{(e^{p_2^{(1)}+p_2^{(2)}}-e^{p_2^{(1)}+p_1^{(2)}})e^{w_1(t)+w_2(t)+q_1(y)+q_2(y)+(p_1^{(2)}+p_2^{(1)}+p_2^{(2)})n}}{(1-e^{p_2^{(1)}+p_2^{(2)}})(1-e^{p_2^{(1)}+p_1^{(2)}})}\\&+\frac{(e^{p_1^{(1)}+p_2^{(2)}}-e^{p_1^{(1)}+p_1^{(2)}})e^{w_1(t)+w_2(t)+q_1(y)+q_2(y)+(p_1^{(2)}+p_1^{(1)}+p_2^{(2)})n}}{(1-e^{p_1^{(1)}+p_1^{(2)}})(1-e^{p_1^{(1)}+p_2^{(2)}})}\Bigg],\end{aligned} \tag{34}$$

with

$$\begin{aligned}|L|=1&+\frac{e^{w_2(t)+q_2(y)+(p_2^{(1)}+p_2^{(2)})n}}{1-e^{p_2^{(1)}+p_2^{(2)}}}+\frac{e^{w_1(t)+q_1(y)+(p_1^{(1)}+p_1^{(2)})n}}{1-e^{p_1^{(1)}+p_1^{(2)}}}\\&+\frac{e^{w_1(t)+w_2(t)+q_1(y)+q_2(y)+(p_1^{(1)}+p_1^{(2)}+p_2^{(1)}+p_2^{(2)})n}}{(1-e^{p_1^{(1)}+p_1^{(2)}})(1-e^{p_2^{(1)}+p_2^{(2)}})}-\frac{e^{w_1(t)+w_2(t)+q_1(y)+q_2(y)+(p_1^{(1)}+p_1^{(2)}+p_2^{(1)}+p_2^{(2)})n}}{(1-e^{p_1^{(1)}+p_2^{(2)}})(1-e^{p_2^{(1)}+p_1^{(2)}})}.\end{aligned}$$

Using (20), we can have the two-soliton solution of equation (30).

## ACKNOWLEDGEMENT

The work described in the paper is supported by a grant from City University of Hong Kong (Project No. 7002366).

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## Üldistatud rüütamismeetod muutuvate kordajatega Toda võrrandi integreerimiseks

Hui-Hui Dai ja Ting Su

Zakharov ja Shabat [1,2] konstrueerisid nn rüütamismeetodi (ingl *dressing method*) integreeruvate evolutsioonivõrrandite solitoni-tüüpi lahendite leidmiseks. Meetodi idee seisneb mittelineaarsete võrrandite teisendamises (rüütamises) lihtsamalt lahenduvaks lineaarsete integraalvõrrandite süsteemiks. Dai ja Jeffrey [6,7] on seda meetodit üldistanud muutuvate kordajatega evolutsioonivõrrandite analüüsiks. Selle üldistuse baasil on käesolevas artiklis lahendatud muutuvate kordajatega Toda võrrand ja esitatud vastavad pidevustingimused. Näitena on konstrueeritud ühe- ja kahesolitonilised lahendid.