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MECHANICS

Hierarchical structures in complex solids with microscales

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Abstract. In nonclassical mechanics it is natural to deal with the problem of the propagation of nonlinear waves in solids with different internal structural scales (Engelbrecht, J., Pastrone, F., Braun, M., and Berezovski, A. Hierarchy of waves in nonclassical materials. In *Universality of Nonclassical Nonlinearity* (Delsanto, P. P., ed.). Springer, 2007, 29–48). The choice of suitable microstrain functions of the microdisplacements, of their time derivatives as strain velocities, allows us to obtain the field equations via a variational principle (see Pastrone, F., Cermelli, P., and Porubov, A. V. Nonlinear waves in 1-D solids with microsctructure. *Mater. Phys. Mech.*, 2004, **7**, 9–16; Casasso, A. and Pastrone, F. Wave propagation in solids with vectorial microstructures. *Wave Motion*, doi: 10.1016/j.wavemoti.2009.12.006; Porubov, A. V., Pastrone, F., and Maugin, G. A. Selection of two-dimensional nonlinear strain waves in micro-structured media. *C. R. Acad. Sci. Paris*, 2004, Ser. I 337, 513–518) in three different cases: one-dimensional solids with two different microscales, two-dimensional solids with microstructures, and plane granular media. In all cases the hierarchical structure of equations due to the scales in materials is evident.

Key words: microstructures, nonlinear elasticity, hierarchical structures, granular media.

1. INTRODUCTION

Recently, the problem of complexity revealed its importance in continuum mechanics and, in particular, in nonlinearly elastic structures. The problem of scale-depending phenomena has become more and more relevant and has been given different names according to the different microscales used. ‘Mesomechanics’, ‘nanomechanics’, and ‘microstructure theory’ are the terms widely employed, sometimes with a kind of overlapping of models. The material continuum which exhibits such a structure is called a complex system. The influence of micro(or meso-, or nano-)structures on the behaviour of the macrostructures is of great importance in applications in very different fields: fluids with bubbles, microcrack distribution in solids, crystal fluids, dislocations and disclinations, granular solids, porous media, and so on. In many cases the mathematical theory does not provide suggestions on how to perform experiments in order to exhibit the existence, consistency, and influence of the microstructure over the macrobody. In this sense we can talk of nonclassical, nonlinear elasticity, in the sense that some of the pillars of the classical exact theory of elasticity are relaxed. For instance, the Cauchy stress tensor is no more symmetric because of the presence of microstructures described by additional internal degrees of freedom, which implies the presence of applied couples in bulk and at the surface.

As extensively described in [5], the cornerstones for describing dynamic processes of microstructured materials at intensive and high-speed deformations are the following:

(i) nonclassical theory of continua able to account for internal scales;
(ii) hierarchical structure of waves due to the scales in materials;
(iii) nonlinearities caused by large deformation and character of stress–strain relations.

The second point mentioned above is the hierarchy of waves. The concept of hierarchy of waves was introduced by Whitham [13]. High intensities of external forces and high deformation rates (high speeds of deformation) dictate the need to consider nonlinearities in governing equations. One should distinguish between geometrical (large deformation) and physical (stress–strain relation) nonlinearities (see [3]). In terms of wave characteristics, there are many physical effects due to the microstructure and its possible structural changes in the wave field. In addition, the influence of nonlinearities causes nonadditivity of other physical effects. Leaving aside more complicated effects like phase transition, kinetic localization of damage, shear bands, etc., even the basic dissipative and dispersive effects are strongly influenced by nonlinearities. There are many studies concerning the dissipative effects combined with nonlinearities.

We examine three examples where the hierarchical structure of field equations is briefly outlined. In Section 2 we deal with one-dimensional elastic bodies with different scales of the microstructure, in Section 3 with the more general problem of two-dimensional microstructured media, and in Section 4 with nondissipative plane granular media.

2. ONE-DIMENSIONAL SOLIDS WITH TWO MICROSCALES

In this section we consider a one-dimensional microstructured model with two different scale levels of the microstructure. So, instead of the two-scale elastic system containing the macro- and microstructure, we introduce a material, which is supposed to be composed by a macrostructure, a first-level microstructure and a second-level microstructure at a much smaller scale. The last may be interpreted as a nanostructure, to the extent (see [2,5]).

In this model we deal with three different scalar functions: one for the macrostructure and two for the microstructure, one for each scale level. In the model under examination the body is a one-dimensional manifold, so we consider the material coordinates x and t, and the functions $v = v(x,t)$ for the macrostructure, $\varphi = \varphi(x,t)$ and $\psi = \psi(x,t)$, respectively, for the first and the second scale level.

The macrobody is supposed to be elastic. The first- and second-level microstructures satisfy the same generalized elasticity hypothesis as well, such that we can assume the existence of an internal strain energy.

In general, the strain energy function in elastic solids with microstructures is assumed to be a function of the vector fields and their gradients [6]. Because of objectivity, we can write this function as a function of the scalar components only, namely,

$$W = W(v, v_x, \varphi, \varphi_x, \psi, \psi_x, x).$$

The kinetic energy is a quadratic form in v_t, φ_t, ψ_t:

$$K = \frac{1}{2}(\rho v_t^2 + I_1 \varphi_t^2 + I_2 \psi_t^2),$$

where ρ is the one-dimensional mass density and, since we deal with a Lagrangian formulation and the body is supposed to be homogeneous, ρ is constant, being the density in the reference configuration; I_1 and I_2 are the inertia terms connected with the two different scale levels of the microstructure. If we consider model without dissipation, the field equations take the following form, as proved in [2]:

$$\begin{cases} \rho v_{tt} = \left(\dfrac{\partial W}{\partial v_x}\right)_x - \dfrac{\partial W}{\partial v}, \\ I_1 \varphi_{tt} = \left(\dfrac{\partial W}{\partial \varphi_x}\right)_x - \dfrac{\partial W}{\partial \varphi}, \\ I_2 \psi_{tt} = \left(\dfrac{\partial W}{\partial \psi_x}\right)_x - \dfrac{\partial W}{\partial \psi}, \end{cases} \tag{1}$$

where v is the displacement field, and the subscripts mean derivatives with respect to time t or to the spatial coordinate x.

The particular choice of the strain energy function W gives rise to different nonlinear models; in this paper we consider the following form:

$$W = \frac{1}{2}\alpha v_x^2 + \frac{1}{3}\beta v_x^3 - A_1\varphi v_x + \frac{1}{2}B_1\varphi^2 + \frac{1}{2}C_1\varphi_x^2 - A_2\varphi_x\psi + \frac{1}{2}B_2\psi^2 + \frac{1}{2}C_2\psi_x^2.$$

This choice is the generalization of the strain energy function for nonlinear elastic solids with one microstructure level to our case, where the introduction of the cubic term v_x^3 means that the behaviour of the matrix is nonlinear.

To obtain the governing equation in dimensionless form, it is necessary to introduce some parameters and constants. So we pose

$$C_1 = C_1^* l_1^2, \quad I_1 = \rho l_1^2 I_1^*, \quad A_1 = l_1 A_1^* \tag{2}$$

for the first level of the microstructure and

$$C_2 = C_2^* l_2^2, \qquad I_2 = \rho l_2^2 I_2^*, \qquad A_2 = l_2 A_2^* \tag{3}$$

for the second scale level. Values l_1 and l_2 represent the size of the microstructure elements. Then we introduce two different parameters δ_i, $i = 1,2$, characterizing the ratio between the microstructure and the wavelength, and ε accounting for elastic strain; in detail we have

$$\delta_1 = \left(\frac{l_1}{L}\right)^2, \qquad \delta_2 = \left(\frac{l_2}{L}\right)^2, \qquad \varepsilon = \frac{v_0}{L}, \tag{4}$$

where L is the wavelength and v_0 the intensity of the initial excitation. Field equations can be written as

$$\begin{cases} \rho v_{tt} = \alpha v_{xx} + \left(\beta v_x^2\right)_x - A_1\varphi_x, \\ I_1\varphi_{tt} = C_1\varphi_{xx} + A_1 v_x - B_1\varphi - A_2\psi_x, \\ I_2\psi_{tt} = C_2\psi_{xx} + A_2\varphi_x - B_2\psi, \end{cases} \tag{5}$$

where α, β and A_i, B_i, C_i $(i = 1,2)$ denote material constants.

Introducing the macrostrain $\nu = v_x$, the dimensionless variables

$$u = \frac{\nu}{v_0}, \quad X = \frac{x}{L}, \quad T = \frac{c_0}{L}t,$$

and substituting the parameters (2), (3), and (4) into the previous system, we get the dimensionless equations

$$\begin{cases} u_{TT} = \dfrac{\alpha}{\rho c_0^2}u_{XX} + \dfrac{\beta v_0}{\rho c_0^2}\left(u^2\right)_{XX} - \dfrac{A_1}{v_0\rho c_0^2}\varphi_{XX}, \\ \varphi = \dfrac{A_1 v_0}{B_1}u - \dfrac{A_2^*\sqrt{\delta_2}}{B_1}\psi_X + \dfrac{\delta_1}{B_1}\left[C_1^*\varphi_{XX} - \rho I_1^* c_0^2\varphi_{TT}\right], \\ \psi = \dfrac{A_2^*\sqrt{\delta_2}}{B_2}\varphi_X + \dfrac{\delta_2}{B_2}\left[C_2^*\psi_{XX} - \rho I_2^* c_0^2\psi_{TT}\right]. \end{cases} \tag{6}$$

The slaving principle [3] can now be used. This procedure allows us to write one function in terms of the other; in this way we can obtain the governing equation that depends only on the function $u(x,t)$. To this end, we determine the variable ψ in terms of φ and its derivatives from $(6)_3$. Then Eq. $(6)_2$ is used to express φ in terms of derivatives of u. This expression is finally inserted into Eq. $(6)_1$, obtaining a single differential equation for u.

First, from $(6)_3$ we obtain

$$\psi = \frac{A_2^*\sqrt{\delta_2}}{B_2}\varphi_x + \frac{A_2^*\sqrt{\delta_2}\delta_2}{B_2^2}\left[C_2^*\varphi_{XXX} - \rho I_2^* c_0^2 \varphi_{XTT}\right]$$

and inserting this expression in $(6)_2$, we have

$$\varphi = \frac{A_1 v_0}{B_1}u + \frac{\delta_1 A_1 v_0}{B_1^2}\left[C_1^* u_{XX} - \rho I_1^* c_0^2 u_{TT}\right] - \frac{\delta_2^2 A_1 (A_2^*)^2 v_0}{B_1^2 B_2^2}\left[C_2^* u_{XXXX} - \rho I_2^* c_0^2 u_{XXTT}\right].$$

Finally, substituting in $(6)_1$, we obtain the partial differential equation

$$u_{TT} = \left(\frac{\alpha B_1 - A_1^2}{\rho c_0^2 B_1}\right) u_{XX} + \frac{\beta v_0}{\rho c_0^2}(u^2)_{XX} + \frac{\delta_1 A_1^2}{\rho c_0^2 B_1^2}\left[\rho I_1^* c_0^2 u_{TT} - C_1^* u_{XX}\right]_{XX}$$
$$- \frac{\delta_2^2 A_1^2 (A_2^*)^2}{\rho c_0^2 B_1^2 B_2^2}\left[\rho I_2^* c_0^2 u_{TT} - C_2^* u_{XX}\right]_{XXXX}. \quad (7)$$

Equation (7) can be rewritten as

$$u_{TT} + \alpha_1 u_{XX} + \alpha_2 (u^2)_{XX} + (\alpha_3 u_{XX} + \alpha_4 u_{TT})_{XX} + (\alpha_5 u_{XX} + \alpha_6 u_{TT})_{XXXX} = 0, \quad (8)$$

where we have defined

$$\alpha_1 = -\frac{\alpha B_1 - A_1^2}{\rho c_0^2 B_1}, \quad \alpha_2 = -\frac{\beta v_0}{\rho c_0^2}, \quad \alpha_3 = \frac{\delta_1 A_1^2 C_1^*}{\rho c_0^2 B_1^2},$$
$$\alpha_4 = -\frac{\delta_1 A_1^2 I_1^*}{B_1^2}, \quad \alpha_5 = -\frac{\delta_2^2 A_1^2 (A_2^*)^2 C_2^*}{\rho c_0^2 B_1^2 B_2^2}, \quad \alpha_6 = \frac{\delta_2^2 A_1^2 (A_2^*)^2 I_2^*}{B_1^2 B_2^2}.$$

Equation (8) is the hierarchical equation in terms of u, where the two different levels of the microstructure are reflected in special wave operators.

We have found a sixth-order partial differential equation that is hardly to be solved explicitly. However, in a forthcoming paper, some exact solutions will be given to an ODE corresponding to the PDE mentioned, when Eq. (7) is reformulated in terms of the phase variable $z = x \pm Vt$, where V is the velocity of propagation

$$(V^2 + \alpha_1)u^{(\mathrm{II})} + \alpha_2 (u^2)^{(\mathrm{II})} + \left(\alpha_3 + V^2\alpha_4\right) u^{(\mathrm{IV})} + \left(\alpha_5 + V^2 \alpha_6\right) u^{(\mathrm{VI})} = 0.$$

If the nonlinearity is neglected, (7), or (8), are equivalent to eq. (3.57) of [5]. On the other hand, we can assume nonlinearity also in the microstructures, hence add in the strain energy function W two terms, $B_3\varphi_x^3$ and $C_3\psi_x^3$. The governing equations (1), (5) will contain now terms $B_3\varphi_x\varphi_{xx}$ and $C_3\psi_x\psi_{xx}$.

3. TWO-DIMENSIONAL MICROSTRUCTURED MEDIA

In this section we refer mainly to [12], where the interest was in weak transverse variation, but we obtain a result that we can define 'complementary'.

One of the most important problems in a microstructured medium is to define the values of the parameters of a microstructure. One possibility is to evaluate the parameters of strain waves propagating in such a medium. Indeed, the amplitude and velocity of the wave depend upon the parameters of the microstructure. Usually wave propagation and the shape of the waves depend on suitable balance conditions among nonlinearity, dispersion, dissipation, and energy input. The hierarchical structure of the governing equations can help in detecting the relevance of the various parameters, splitting the contribution of the different levels of the microstructures. The balances define the shape of the wave.

The governing equations are obtained using the model developed in [10]. The fundamental strains are given by the Cauchy–Green macrostrain tensor, the distortion tensor, and the microdisplacement gradient tensor. The macromotion is supposed to be small but finite, and the Murnaghan model is used to describe the so-called physical nonlinearity in the expansion of free or potential energy. The microstructure is assumed sufficiently weak to be considered in the linear approximation. A dissipation and an energy input are introduced through the additive linear terms in all three tensors, the simplest extension of the Hooke law to viscoelastic media.

Let us denote displacements along the x- and y-axes by $\mathscr{U}(x,y,t)$, $\mathscr{V}(x,y,t)$, respectively. Then a scale $\mathscr{W}$ is introduced as for longitudinal strains $v = \mathscr{U}_x$, and $\mathscr{W} \ll 1$ that is natural for the Murnaghan materials. The scale for another strain $w = \mathscr{V}_y$ is chosen equal to $\kappa\mathscr{W}$. Also L/c_0 is used as a scale for time t, $c_0^2 = (\lambda + 2\mu)/\rho$ is a characteristic velocity, λ, μ are the Lame coefficients, ρ is the macrodensity. We also introduce a typical size p of a microstructure element and the dissipation parameter d having the dimension of a length. Three positive dimensionless parameters will be used in the following: $\varepsilon = \mathscr{W} \ll 1$, accounting for elastic strains; $\delta = p^2/L^2 \ll 1$, characterizing the ratio between the microstructure size and the wavelength; $\gamma = d/L$, characterizing the influence of dissipation.

Now we are interested in longitudinal waves. We can assume $w \approx 0$, hence in eqs (1) and (2) in [12] the terms in w disappear and, if we assume that nonlinearity, dispersion, and dissipation are taken into account, we have

$$v_{tt} - v_{xx} + \delta\alpha(v_{xx} - v_{tt})_{xx} + \gamma\delta\beta(v_{tt} - v_{xx})_{xxt} + \varepsilon\alpha_1(v^2)_{xx} = 0,$$

where the nonlinear term coefficient α_1 depends upon the Murnaghan moduli.

We can use an asymptotic approximation

$$v = v_0(x,t) + \gamma v_1(x,t) + \dots$$

valid for a small dissipation, and obtain two equations

$$\begin{aligned} &v_{0,tt} - v_{0,xx} + \delta\alpha(v_{0,xx} - v_{0,tt})_{xx} + \varepsilon\alpha_1(v_0^2)_{xx} = 0, \\ &v_{1,tt} - v_{1,xx} + \delta\alpha(v_{1,xx} - v_{1,tt})_{xx} + 2\varepsilon\alpha_1(v_0 v_1)_{xx} + \gamma\delta\beta(v_{0,tt} - v_{0,xx})_{txx} = 0. \end{aligned} \tag{9}$$

Introducing the variable $\theta = x - t$, the slow variable $T = \varepsilon\gamma t$, and the phase variable ζ, such that $\zeta_x = 1$, $\zeta_t = c(T)$, we obtain

$$cv_{0,\zeta\zeta} + \alpha_1(v_0^2)_{\zeta\zeta} + \delta\alpha v_{0,\zeta\zeta\zeta\zeta} = 0$$

that admits a solitary wave solution with slowly varying parameters (see [10]).

Equation (9) reads

$$cv_{1,\zeta\zeta} + \delta\alpha\beta v_{0,\zeta\zeta\zeta\zeta\zeta} = 0,$$

namely, a fifth-order linear differential equation for v_1, much simpler than eq. (7) in [10]. It seems feasible to obtain similar results as in [10], but we stop here our analysis.

4. NONDISSIPATIVE PLANE GRANULAR MEDIA

For simplicity sake, as explained in [1], we shall use the notation $X^1 = x$, $X^2 = y$, $x^1 = u$, $x^2 = v$, hence we consider the vector $r = r(x,y,t) = u(x,y,t)e_1 + v(x,y,t)e_2$ for the macrostructure and, for the microstructure, the function $\theta = \theta(x,y,t)$ that represents the angle of rotation of the particle with respect to the fixed basis. In the following the lower indices x,y,t will denote differentiation. The kinetic energy density reads

$$T = \frac{1}{2}\left[\rho\left(u_t^2+v_t^2\right)+I\theta_t^2\right].$$

The strain energy density is chosen in the form

$$\begin{aligned} W = &\frac{1}{2}\alpha\left(u_x^2+v_x^2\right)+\frac{1}{2}\beta\left(u_y^2+v_y^2\right)+\frac{1}{6}\gamma(u_x^3+u_y^3+v_x^3+v_y^3)\\ &+\frac{1}{2}\gamma(u_x^2v_x+u_y^2v_y+v_x^2u_x+v_y^2u_y)-A\theta\left(u_x+u_y+v_x+v_y\right)\\ &+\frac{1}{2}B\theta^2+\frac{1}{2}C\left(\theta_x^2+\theta_y^2\right)+\frac{1}{3}D\left(\theta_x^3+\theta_y^3\right). \end{aligned}$$

We assume that the dissipation is negligible and calculate the Lagrange equations

$$\begin{cases} \rho u_{tt} = \alpha u_{xx}+\beta u_{yy}-A(\theta_x+\theta_y)+\frac{1}{2}\gamma\left[\left[(u_x+v_x)^2\right]_x+\left[(u_y+v_y)^2\right]_y\right],\\ \rho v_{tt} = \alpha v_{xx}+\beta v_{yy}-A(\theta_x+\theta_y)+\frac{1}{2}\gamma\left[\left[(u_x+v_x)^2\right]_x+\left[(u_y+v_y)^2\right]_y\right],\\ I\theta_{tt} = C(\theta_{xx}+\theta_{yy})+D\left[(\theta_x^2)_x+(\theta_y^2)_y\right]+A(u_x+u_y+v_x+v_y)-B\theta. \end{cases}$$

We introduce a new variable $w = u+v$. Therefore, we add the first two equations of the previous system and get

$$\begin{cases} \rho w_{tt} = \alpha w_{xx}+\beta w_{yy}-2A(\theta_x+\theta_y)+\gamma\left[(w_x^2)_x+(w_y^2)_y\right],\\ I\theta_{tt} = C(\theta_{xx}+\theta_{yy})+D\left[(\theta_x^2)_x+(\theta_y^2)_y\right]+A(w_x+w_y)-B\theta. \end{cases} \tag{10}$$

For further analysis the dimensionless variables are introduced

$$\mathscr{W} = \frac{w}{\mathscr{W}_0},\quad X = \frac{x}{L},\quad Y = \frac{y}{L},\quad T = \frac{c_0}{L}t,$$

where c_0, $\mathscr{W}_0$, L are physically meaningful constants (velocity, intensity, and wavelength of the initial excitation). We also need a scale for the microstructure l and then two dimensionless parameters can be introduced

$$\begin{cases} \delta \sim \left(\dfrac{l}{L}\right)^2 & \text{characterizing the ratio between the} \\ & \text{microstructure and the wavelength,}\\ \varepsilon \sim \left(\dfrac{\mathscr{W}_0}{L}\right) & \text{accounting for elastic strain,} \end{cases}$$

where δ has the relevant meaning of a characteristic length.

Following [10], we suppose $I = \rho l^2 I^*$, $C = l^2 C^*$, $D = l^2 D^*$, where I^* is dimensionless and C^* and D^* have the dimension of the stress. Then (10) yields

$$\begin{cases} \mathscr{W}_{TT} = \dfrac{1}{\rho c_0^2}(\alpha\mathscr{W}_{XX}+\beta\mathscr{W}_{YY})-\dfrac{2A}{\rho c_0^2\varepsilon}(\theta_X+\theta_Y)+\dfrac{\varepsilon^2\gamma}{L}\left[(\mathscr{W}_X^2)_X+(\mathscr{W}_Y^2)_Y\right],\\ \theta_{TT} = \dfrac{1}{\rho c_0^2 I^*}\left[\dfrac{\varepsilon A}{B}(\mathscr{W}_X+\mathscr{W}_Y)+\dfrac{\delta}{B}\left[C^*(\theta_{XX}+\theta_{YY})+\dfrac{D^*}{L}\left[(\theta_X^2)_X+(\theta_Y^2)_Y\right]\right]-\theta\right]. \end{cases} \tag{11}$$

If we consider the expansion in terms of the characteristic length δ: $\theta = \theta_0 + \delta\theta_1 + \ldots$ and equalize the coefficients of the power of δ, we obtain the system

$$\theta_0 = \varepsilon\,\tilde{\theta}_0 = \frac{\varepsilon A}{B}(\mathscr{W}_X + \mathscr{W}_Y),$$

$$\theta_1 = \varepsilon\,\tilde{\theta}_1 + \varepsilon^2\,\tilde{\theta}_2 = \frac{\varepsilon A}{B^2}\left[C^*(\mathscr{W}_{XXX} + \mathscr{W}_{XXY} + \mathscr{W}_{XYY} + \mathscr{W}_{YYY}) - \rho c_0^2 I^*(\mathscr{W}_{XTT} + \mathscr{W}_{YTT})\right]$$
$$+\frac{\varepsilon^2 A^2 D^*}{B^3 L}\left\{\left[(\mathscr{W}_{XX} + \mathscr{W}_{XY})^2\right]_X + \left[(\mathscr{W}_{XY} + \mathscr{W}_{YY})^2\right]_Y\right\}.$$

Let us consider the following approximation for θ:

$$\theta \simeq \frac{\varepsilon A}{B}(\mathscr{W}_X + \mathscr{W}_Y) + \frac{\delta\varepsilon}{B^2}\left[C^*(\mathscr{W}_{XXX} + \mathscr{W}_{XXY} + \mathscr{W}_{XYY} + \mathscr{W}_{YYY}) - \rho c_0^2 I^*(\mathscr{W}_{XTT} + \mathscr{W}_{YTT})\right],$$

namely $\theta \cong \varepsilon(\tilde{\theta}_0 + \tilde{\theta}_1)$. Equation $(11)_1$ becomes

$$\mathscr{W}_{TT} = \frac{1}{\rho c_0^2}\left(\alpha - \frac{2A^2}{B}\right)\mathscr{W}_{XX} + \frac{1}{\rho c_0^2}\left(\beta - \frac{2A^2}{B}\right)\mathscr{W}_{YY}$$
$$- \frac{4A^2}{\rho c_0^2 B}\mathscr{W}_{XY} + \frac{2\delta A^2 I^*}{B^2}(\mathscr{W}_{XX} + 2\mathscr{W}_{XY} + \mathscr{W}_{YY})_{TT}$$
$$- \frac{2\delta A^2 C^*}{\rho c_0^2 B^2}[(\mathscr{W}_{XX} + 2\mathscr{W}_{XY} + \mathscr{W}_{YY})_{XX} + (\mathscr{W}_{XX} + 2\mathscr{W}_{XY} + \mathscr{W}_{YY})_{YY}].$$

This equation describes longitudinal wave propagation only if the movement is provided along the x-axis. Otherwise it accounts for both the longitudinal and the shear horizontal waves. As shown in [11], we can see that in the 1D case the wave evolution is described by the 'Double Dispersion Equation':

$$\mathscr{W}_{TT} = \alpha_1 \mathscr{W}_{XX} + \alpha_2 (\mathscr{W}_X^2)_X + \alpha_3 \mathscr{W}_{XXXX} + \alpha_4 \mathscr{W}_{XXTT},$$

where $\alpha_1 = \frac{1}{\rho c_0^2}\left(\alpha - \frac{2A^2}{B}\right)$, $\alpha_2 = \frac{\varepsilon^2\gamma}{L}$, $\alpha_3 = -\frac{2\delta A^2 C^*}{\rho c_0^2 B^2}$, $\alpha_4 = \frac{2\delta A^2 I^*}{B^2}$.

5. CONCLUSIONS

The field equations obtained in the previous sections can be used to study the propagation of nonlinear waves, as done in the papers [1,4,5,9–11]. In all cases the hierarchical structures of field equations are reflected in a possible hierarchy of waves in the sense of Whitham [13], where the influence of the different levels of scale parameters on wave propagation is clearly seen. Even the possibility of propagation of bell-shaped or kink-shaped solitons depends on the balance of suitable parameters related to the different scales of macro- and microstructures. The use of the 'slaving principle', asymptotic approximations, and, at least in one case, the possibility of reducing the main field equation obtained via the previous methods to an ODE (see [10]) which can be integrated allow us to obtain further information about the wave profiles and other properties depending on the different scales. We can guess that in many other cases of physical interest this approach can be fruitfully used and our purpose is to develop further this kind of analysis, where the fundamental model of solids with vectorial microstructures as developed in [2,7,8] plays an important role.

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Hierarhilised struktuurid mikroskalaarsetes keerukates tahkistes

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Mitteklassikalises mehaanikas on tavapärane käsitleda mittelineaarsete lainete levi erinevate sisemiste struktuursete skaaladega tahkistes [5]. Mikrodeformatsiooni funktsioonide, mikrosiirde ja nende aja järgi tuletiste kui deformatsioonikiiruste sobiv valik võimaldab variatsiooniprintsiibi abil saada välju kirjeldavad võrrandid (vt [2,9,12]) kolmel erineval juhul: kahe erineva mikroskaalaga ühemõõtmeliste tahkiste, mikrostruktuuridega kahemõõtmeliste tahkiste ning tasapinnalise granuleeritud keskkonna puhul. Kõikidel juhtudel on materjalide struktuursete skaalade mõju võrrandite hierarhilisele struktuurile ilmne.