Proceedings of the Estonian Academy of Sciences,
2010, **59**, 2, 145–149
doi: 10.3176/proc.2010.2.13
Available online at www.eap.ee/proceedings

MECHANICS

On perturbative solutions for nonlinear waves in inhomogeneous materials

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Received 15 December 2009, accepted 5 March 2010

Abstract. Complex phenomena of wave–prestress and wave–material interactions are studied. Model problems of nonlinear propagation of longitudinal waves in materials (i) with inhomogeneous prestress and (ii) with weakly variable physical properties are posed. In both problems two small parameters of different physical nature appear. The problems are solved using the perturbation technique. The influence of the values of two small parameters on the content of terms of perturbative solutions is analysed. The outcome facilitates utilization of perturbative solutions in composing algorithms for ultrasonic nondestructive testing of materials with variable properties.

Key words: solid mechanics, nonlinear elasticity, longitudinal waves, interaction, perturbative solution, nondestructive testing.

1. INTRODUCTION

Practical application of data about complex phenomena of nonlinear wave–wave, wave–material, and wave–prestress interactions is one of the basic ways of building up the methods for ultrasonic nondestructive characterization of different materials [1–3]. Algorithms for ultrasonic nondestructive characterization of materials are based on analytical solutions to the equations of motion of materials [4,5]. The goal of this paper is to clarify the peculiarities of perturbative solutions for these algorithms, derived in [6,7]. Attention is focused on the role of two essential small parameters in the algorithms and on the influence of these parameters on the amount of information about the material properties in the outcome of the algorithms.

Many techniques of ultrasonic nondestructive characterization of the material and the prestress in the material are based on the results of the analysis of the nonlinear effects that occur in wave interaction processes. Wave–prestress interaction takes place, provided the geometrical or the physical nonlinearity of the problem is taken into account. Wave–material interaction occurs already in the case of the linear model of wave propagation.

Wave–prestress and wave–material interactions are described for nonlinear propagation of longitudinal waves in materials (i) with inhomogeneous prestress and (ii) with weakly variable physical properties. Our goal is to use nonlinear effects of wave propagation in nondestructive evaluation of inhomogeneous properties of the material. It is important that the nonlinear effects of wave propagation contain plenty of information about inhomogeneities. The amount of this information is dependent in case (i) on the ratio of small parameters in the perturbative solution, which characterize strain intensities in the material caused by the propagating wave and predeformation, and in case (ii) on the ratio of the small parameter that characterizes strain intensity in the material caused by the propagating wave to the small parameter that describes the weak variation of material properties. The conclusion is that wave propagation data contain the largest amount of information about the prestressed state and material properties, provided the two different small parameters are of the same order.

2. SUPERPOSITION OR INTERACTION

Ultrasonic nondestructive characterization of the properties and states of materials is based on the data about the recorded parameters of waves. The amount of information about the material is important in these parameters. For example, the recorded wave may not recognize the presence of prestress in the material. Therefore, the criteria for successful nondestructive characterization of the material must be clarified.

The nonlinear theory of elasticity describes the motion of the material by the equation [8]

$$[T^*_{KL}(\delta_{kL}+\delta_{kM}U^*_{M,L})]_{,K}-\rho_0\,\delta_{kM}U^*_{M,tt}=0. \tag{1}$$

Here ρ_0 denotes the density of the material and δ_{kL} and δ_{kM} are Euclidean shifters which connect the Lagrangian Cartesian coordinates X_K and the Eulerian Cartesian coordinates x_k. The indices K, L, and t after a comma indicate differentiation with respect to X_K, X_L, and time t, respectively. The usual summation convention is used and all indices, except time t, run over 1, 2, and 3. Equation (1) describes the motion of the material in the actual state that is characterized by the displacement vector U^*_K and the second Piola–Kirchhoff stress tensor T^*_{KL}.

Theoretically, there are two possibilities of solving the problem of ultrasonic nondestructive characterization of prestress in the material. In both cases the specimen of the material that is in the natural, prestress-free state is considered. At some instant the material is assumed to be in a prestressed state, generated by an appropriate statical loading. The displacement vector and the second Piola–Kirchhoff stress tensor at this state are denoted by $U^0_K(X_J)$ and $T^0_{KL}(X_J)$, respectively. After that, the longitudinal wave process represented by $U_K(X_J,t)$ and $T_{KL}(X_J,t)$ is excited in the prestressed material.

In the first case the displacement at the actual state is expressed as the sum

$$U^*_K(X_J,t)=U^0_K(X_J)+U_K(X_J,t), \tag{2}$$

and in the second case the stress at the actual state is expressed as the sum

$$T^*_{KL}(X_J)=T^0_{KL}(X_J)+T_{KL}(X_J). \tag{3}$$

In both cases the strain is stated by the Green–Lagrange strain tensor [8]

$$2E_{KL}=\frac{\partial U^*_K}{\partial X_L}+\frac{\partial U^*_L}{\partial X_K}+\frac{\partial U^*_M}{\partial X_K}\frac{\partial U^*_M}{\partial X_L}, \tag{4}$$

where the indices K, L, and M range over 1, 2, 3. The constitutive equation for the nonlinear elastic material is formulated by

$$\begin{aligned}T^*_{KL}=&(\lambda I_1+3\nu_1 I_1^2+\nu_2 I_2)\frac{\partial I_1}{\partial E_{KL}}\\&+(\mu+\nu_2 I_1)\frac{\partial I_2}{\partial E_{KL}}+\nu_3\frac{\partial I_3}{\partial E_{KL}}+\cdots.\end{aligned} \tag{5}$$

Here I_1, I_2, and I_3 are invariants of the Green–Lagrange strain tensor E_{KL}. Expression (5) characterizes the five-constant nonlinear theory of elasticity, where λ and μ are Lamé constants or the elastic constants of the second order and ν_1, ν_2, and ν_3 are the elastic constants of the third order.

It is important to pay attention to the discrepancy between the approaches described by Eqs (2) and (3). The closed system of equations of the nonlinear theory of elasticity [8] may be solved for both cases (2) and (3). It is essential that if the displacement U^*_K is given by Eq. (2), the substitution of Eq. (2) into Eq. (4) determines the Green–Lagrange strain tensor E_{KL}. If, instead, the stress T^*_{KL} is given by Eqs (3) and (5) or the strain E_{KL} is given, the determination of the three unknowns U^*_K requires the solution of six partial differential equations (4). Such a system is overdetermined and for the existence of a single-valued continuous displacement field restrictions (compatibility conditions) must be imposed upon E_{KL}. The relevant information is presented, for example, in [8,9]. Subsequently the displacement approach is used, i.e., the displacement U^*_K is given by Eq. (2).

A decisive question is the following: What do we describe by the algorithm of nondestructive testing if we record the displacement U_K evoked by wave motion – is it the superposition of effects caused by different actions or is it the interaction of these effects? Do the recorded data contain information about the prestress in the material?

In ultrasonic nondestructive characterization of prestress the question about superposition or interaction turns into the problem of mutual action of strains caused by wave propagation and prestress. This mutual action is determined by expressions (4) and (5). Analysis of these expressions leads to the conclusion that if the completely linear problem is considered, i.e., the geometrical nonlinearity (the third term in Eq. (4)) and the physical nonlinearity (terms with the third-order elastic constants in Eq. (5)) are neglected, there is no interaction between strains of different origins. The result is that wave propagation is not affected by prestress and does not contain information about prestress.

Interaction of strains caused by wave propagation and prestress can be described by taking the geometrical nonlinearity, physical nonlinearity or both into account. This problem is analysed in [10] where theoretical results are compared with experimental data. The conclusion is that there exist special cases when consideration of only one kind of nonlinearity gives satisfactory results. In most cases it is necessary to take both nonlinearities into account, because often the influence of the geometrical and physical nonlinearity on the final result is of the same order.

3. WAVE–PRESTRESS INTERACTION

Wave–prestress interaction is studied using the perturbative solutions of the governing equations that describe wave motion in a prestressed material. Perturbative solutions form a theoretical basis for the algorithms for ultrasonic nondestructive characterization of prestress [6,11].

Propagation of one-dimensional longitudinal waves in the geometrically and physically nonlinear prestressed material is described by the equation [11]

$$\begin{aligned}(1+k_1U^0_{1,1}+k_2\,U^0_{2,2})\,U_{1,11}&\\+(k_1\,U^0_{1,11}+k_3\,U^0_{1,22}+k_5\,U^0_{2,12})\,U_{1,1}&\\+k_1\,U_{1,11}\,U_{1,1}-c^{-2}\,U_{1,tt}=0,&\qquad(6)\end{aligned}$$

where c denotes the phase velocity and U, U^0 denote displacements caused by the propagating wave and by prestress, respectively. Indices after a comma indicate differentiation with respect to the coordinates X_1 and X_2, and time t. Equation (6) is solved by the assumption that prestress corresponds to plane strain and the displacement U^0 is a solution of the set of two equations

$$\begin{aligned}(1+k_1\,U^0_{I,I}+k_2\,U^0_{J,J})\,U^0_{I,II}+(2\,k_3\,U^0_{I,J}+2\,k_4\,U^0_{J,I})\,U^0_{I,IJ}&\\+(k_7+k_3\,U^0_{I,I}+k_3\,U^0_{J,J})\,U^0_{I,JJ}&\\+(k_4\,U^0_{I,J}+k_3\,U^0_{J,I})\,U^0_{J,II}&\\+(k_3\,U^0_{I,J}+k_4\,U^0_{J,I})\,U^0_{J,JJ}&\\+(k_6+k_5\,U^0_{I,I}+k_5\,U^0_{J,J})\,U^0_{J,JI}&\\=0,&\qquad(7)\end{aligned}$$

where the indices are $I=1$, $J=2$ for the first equation and $I=2$, $J=1$ for the second equation. The constants k_i, $i=1,2,\ldots,7$, in Eqs (6) and (7) are functions of the second- and the third-order elastic constants [6].

Problem (6), (7) is solved by the assumption that the strain evoked in the material by different actions is small but finite. This leads to the idea to solve the problem by making use of the perturbation theory. Solutions of Eqs (6) and (7) are sought by assuming that the displacement due to wave motion can be expressed by the series

$$U_1=\sum_{n=1}^{p}\varepsilon_1^n\,U_1^{(n)}\qquad(8)$$

and the displacement of the prestressed state can be expressed by the series

$$U_K^0=\sum_{m=1}^{p}\varepsilon_2^m\,U_K^{0(m)}.\qquad(9)$$

Here ε_1 and ε_2 are the small parameters ($|\,\varepsilon_1\,|\ll 1$, $|\,\varepsilon_2\,|\ll 1$) that have the physical meaning of small strain evoked in the material by wave motion and prestress, respectively, and p is a positive number.

Below, the perturbation technique is used, i.e., series (8) and (9) are inserted into Eq. (6). Three different cases are considered: case (i) when the strains evoked by wave motion and prestress are of the same order ($\varepsilon_1\approx\varepsilon_2$), case (ii) when a weak wave is excited in the prestressed material ($\varepsilon_1\approx\varepsilon_2^2$), and case (iii) when a strong wave is excited ($\varepsilon_1^2\approx\varepsilon_2$). In all cases the small parameter ε_2 is determined through ε_1 and henceforth the resulting equations contain only one small parameter ε_1. Following the perturbation technique, terms of equal power in ε_1 are equated to zero in all resulting equations. This gives a system of equations to be satisfied by the coefficients of series (8). For each of the three cases a different system is obtained. Consequently, series (8) describes a different behaviour of wave propagation, depending on how the small parameters ε_1 and ε_2 are related to each other. From the point of view of nondestructive testing it is desirable that the displacement U_1 contains the largest amount of information about the prestress described by the static displacement U_K^0.

The problem is studied on the basis of nonlinear propagation of a harmonic wave with the frequency ω. Solution (8) with accuracy of three leading terms may be expressed in the form

$$\begin{aligned}U_1=\sum_{n=2}^{3}\varepsilon_1^n A_0^{(n)}+\sum_{n=1}^{3}\varepsilon_1^n A_1^{(n)}\sin(\omega\zeta^{(n)}+\theta_1^{(n)})&\\+\sum_{n=2}^{3}\varepsilon_1^n A_2^{(n)}\sin(2\omega\zeta^{(n)}+\theta_2^{(n)})&\\+\sum_{n=3}^{3}\varepsilon_1^n A_3^{(n)}\sin(3\omega\zeta^{(n)}+\theta_3^{(n)})\,.&\end{aligned}\qquad(10)$$

Here $A_0^{(n)}$ is the nonperiodic term, $A_J^{(n)}$ and $\theta_J^{(n)}$, $J=1,2,3$, denote the amplitudes and phase shifts of harmonics, and $\zeta^{(n)}=t-X_1/c^{(n)}$. The different phase velocities $c^{(n)}$ are the result of the fact that some constituents in sums of Eq. (10) describe wave motion in the prestress-free material and some of them in the prestressed material.

In all three cases the first term in Eq. (8) describes propagation of the first harmonic in the prestress-free physically linear elastic material. The contribution of the second and the third term in series (8) to the first three terms of Eq. (10) depends on the case. In case (i) the second term in Eq. (8) describes the nonperiodic term in Eq. (10) and the evolution of the second harmonic in the physically nonlinear prestress-free material. In addition, it introduces the influence of prestress on the first harmonic. In case (ii), when the weak wave is excited in the prestressed material, the second term in Eq. (8) introduces the influence of physical nonlinearity and prestress on the propagation of the first harmonic in Eq. (10). It does not describe the nonlinear wave propagation, i.e., the second harmonic in Eq. (10). In case (iii), when the strong wave is excited in the prestressed material, the second term in Eq. (8) does not correct the first harmonic but describes the nonperiodic term in Eq. (10) and the evolution of the second harmonic in the physically nonlinear prestress-free material.

The third term in Eq. (8) has maximum effect on the content of terms in Eq. (10) in case (i). In this case it corrects the first harmonic once again, corrects the nonperiodic term, describes the influence of prestress on the evolution of the second harmonic, and describes the propagation of the third harmonic in the physically nonlinear prestress-free material. By weak wave propagation (case (ii)) the third term corrects the first harmonic in Eq. (10) once again, describes the nonperiodic term, and the evolution of the second harmonic in the prestress-free material. If the strong wave is propagating in the material (case (iii)), the third term introduces the influence of prestress on the first harmonic in Eq. (10) and describes the propagation of the third harmonic in the physically nonlinear prestress-free material.

Consequently, the information about material properties and prestress in periodic terms of solution (10) depends on the values of small parameters ε_1 and ε_2. The best results can be achieved in case (i).

4. WAVE–MATERIAL INTERACTION

Wave–material interaction occurs already in the case of the linear model of wave propagation in materials. The velocity of the one-dimensional longitudinal wave in the homogeneous isotropic material is determined by the formula

$$c = \sqrt{(\lambda + 2\mu)/\rho} \tag{11}$$

that determines the dependence of the wave velocity c on the physical properties of the material. In addition, the stress field evoked in the material by wave propagation acts upon the properties of deformable materials.

Nonlinear propagation of the one-dimensional longitudinal wave in the physically nonlinear inhomogeneous elastic material is described by the equation [7]

$$\begin{aligned} &[1 + k_1(X) U_{,X}(X,t)]\, U_{,XX}(X,t) + k_2(X) U_{,X}(X,t) \\ &\quad + k_3(X)\, [U_{,X}(X,t)]^2 - k_4(X) U_{,tt}(X,t) = 0, \end{aligned} \tag{12}$$

where the coefficients $k_j(X)$, $j = 1,2,...,4$, are functions of the material density $\rho(X)$ and the elastic coefficients $\lambda(X)$, $\mu(X)$, $\nu_1(X)$, $\nu_2(X)$, and $\nu_3(X)$, all of which may depend on the space coordinate X. In the considered one-dimensional case the elastic coefficients are grouped to the linear elastic coefficient $\alpha(X)$ and to the nonlinear elastic coefficient $\beta(X)$ in accordance with the equations

$$\begin{aligned} \alpha(X) &= \lambda(X) + 2\mu(X), \\ \beta(X) &= 2\,[\nu_1(X) + \nu_2(X) + \nu_3(X)]. \end{aligned} \tag{13}$$

Let us assume that the variation of physical properties of the material is weak and can be described by the function

$$\gamma(X) = \gamma^{(1)} + \varepsilon_3\, \gamma^{(2)}(X). \tag{14}$$

Here ε_3 is a dimensionless positive constant that satisfies the condition $\mid \varepsilon_3 \mid \ll 1$. The function $\gamma(X)$ is representative for the material density $\rho(X)$, the Lamé coefficients $\lambda(X)$ and $\mu(X)$, and the third-order elastic coefficients $\nu_1(X)$, $\nu_2(X)$, and $\nu_3(X)$.

The perturbation technique is used and the solution to Eq. (12) is sought in the form of a series with a small parameter ε_4 [7]

$$U = \sum_{n=1}^{p} \varepsilon_4^n U^{(n)}. \tag{15}$$

If the aim is to characterize weakly variable material properties on the basis of wave propagation data, it is necessary also here to pay attention to the values of the dimensionless small parameters ε_3 and ε_4 that have a different physical nature. It turns out that solution (15) contains maximum information about the material properties, provided the small parameters ε_3 and ε_4 are of the same order. The first term in solution (15) describes linear wave propagation in a material with constant properties. Subsequent terms correct the solution by material inhomogeneity and nonlinearity. This correction depends on the values of small parameters similarly to the cases described by wave–prestress interaction.

5. CONCLUSIONS

The specific character of different perturbative solutions derived in [6,7] for algorithms of ultrasonic nondestructive characterization of inhomogeneous prestress and weak material inhomogeneity in the nonlinear elastic material is analysed. It is shown that consideration of the physical and geometrical nonlinearity of the problem in algorithms enhances essentially the possibilities of nondestructive testing.

On the basis of previous analyses the following may be stated. It is important to pay attention to the intensity of the applied wave in ultrasonic nondestructive characterization of prestress or material properties. Using the algorithms that are based on the first terms of the perturbative solution, the best results may be achieved by choosing the values of two essential small parameters in the problem to be of the same order.

ACKNOWLEDGEMENT

The research was supported by the Estonian Science Foundation (grant No. 7728).

REFERENCES

1. McCann, D. M. and Forde, M. C. Review of NDT methods in the assessment of concrete and masonry structures. *NDT&E Int.*, 2001, **34**, 71–84.
2. Achenbach, J. D. Modeling for quantitative non-destructive evaluation. *Ultrasonics*, 2002, **40**, 1–10.
3. Si-Chaib, Med. O., Djelouah, H., and Boutkedjirt, T. Propagation of ultrasonic waves in materials under bending forces. *NDT&E Int.*, 2005, **38**, 283–289.
4. Domanski, W. Propagation and interaction of non-linear elastic plane waves in soft solids. *Int. J. Non-Linear Mech.*, 2009, **44**, 494–498.
5. Gusev, V., Bailliet, H., Lotton, P., and Bruneau, M. Interaction of counterpropagating acoustic waves in media with nonlinear dissipation and in hysteretic media. *Wave Motion*, 1999, **29**, 211–221.
6. Ravasoo, A. Non-linear interaction of waves in prestressed material. *Int. J. Non-Linear Mech.*, 2007, **42**, 1162–1169.
7. Ravasoo, A. Nonlinear waves in characterization of inhomogeneous elastic material. *Mech. Materials*, 1999, **31**, 205–213.
8. Eringen, A. C. *Nonlinear Theory of Continuous Media*. McGraw-Hill, New York, 1962.
9. Reddy, J. N. *An Introduction to Continuum Mechanics with Applications*. Cambridge University Press, 2008.
10. Bobrenko, V. M., Vangeli, M. S., and Kutsenko, A. N. *Acoustic Methods of Control of the Stressed State of Machine Design Materials*. Shtiintsa, Kishinev, 1981 (in Russian).
11. Ravasoo, A. The perturbation technique for wave interaction in prestressed material. In *Applied Wave Mathematics: Selected Topics in Solids, Fluids, and Mathematical Methods* (Quak, E. and Soomere, T., eds). Springer, Berlin, 2009, 31–53.

Häiritusmeetodil baseeruvatest lahenditest mittelineaarse lainelevi kirjeldamisel mittehomogeensetes materjalides

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On uuritud lainelevi ja eelpinge ning lainelevi ja materjali interaktsiooni. On püstitatud mudelülesanded pikilainete mittelineaarsest levist (i) mittehomogeense eelpingega ja (ii) nõrgalt muutuvate füüsikaliste omadustega materjalides. Ülesanded on lahendatud häiritusmeetodit kasutades. On analüüsitud kahe väikese parameetri väärtuste mõju saadud lahendi komponentide sisule. Tulemus lihtsustab häiritusmeetodil baseeruvate lahendite kasutamist muutuvate omadustega materjalide ultraheliga mittepurustavate katsetuste algoritmide koostamisel.