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MECHANICS

On periodic waves governed by the extended Korteweg–de Vries equation

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Abstract. The evolution equation describing the propagation of one-dimensional waves in a microstructured material has the form of an extended Korteweg–de Vries equation, where the additional term reflects the influence of micrononlinearity. As shown by Janno and Engelbrecht (*J. Phys. A: Math. Gen.*, 2005, **38**, 5159–5172), solitary waves in a microstructured material become asymmetric if nonlinearities are taken into account in both macro- and microscale. The present paper generalizes previous results to periodic waves which, in the KdV case, have the form of cnoidal waves. It is shown that, due to the nonlinearity in microscale, these waves become inclined in the same manner as solitary waves, while the relations between the period, amplitude, and velocity are not affected.

Key words: materials with microstructure, cnoidal waves, solitary waves, KdV equation.

1. INTRODUCTION

A linear theory of microstructured solids was proposed by Mindlin [1] in 1964. Engelbrecht and Pastrone [2] specialized this theory to one dimension and, at the same time, generalized it by including nonlinear terms at both macro- and microlevel. To describe the motion of the one-dimensional microstructured solid, they complemented the macroscopic displacement by the microstrain, both of which are considered as functions of the space coordinate and time. The governing equations appear as a system of coupled partial differential equations for the two field variables. Using the so-called slaving principle, Engelbrecht and Pastrone [2] distilled from it a single partial differential equation, which governs mainly the macrodisplacement while retaining, in a first approximation, the influence of the microstructure.

On the basis of this equation the propagation of solitary waves was studied by Janno and Engelbrecht [3]. They showed that the wave profile becomes asymmetric. The evolution equation of these waves assumes the form of an extended Korteweg–de Vries (KdV) equation, where the additional, higher-order term reflects the influence of micrononlinearity. An approximate solution of this equation in analytical form has been provided by Randrüüt and Braun [4].

Besides solitary waves, the KdV equation admits a whole family of periodic solutions, the so-called cnoidal waves, of which the solitary wave is just the limit if the period tends to infinity. The aim of the present paper is to study how these periodic waves are affected if micrononlinearity is taken into account. As can be expected, the waves stay periodic but become inclined in the same manner as solitary waves.

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2. EXTENDED KdV EQUATION

As mentioned before, the basic governing equations are not treated directly. Rather, by using the slaving principle, a single partial differential equation is extracted. This, in turn, is analysed via the reductive perturbation method which finally provides an evolution equation describing the perturbation of the wave profile. For the sake of brevity, this whole procedure is not duplicated here. The application of the slaving principle is explained in detail in [2–4], and the evolution equation is derived in [4,5]. It has the form of an extended KdV equation. By scaling the variables appropriately, the evolution equation can be reduced to the standardized form

$$y_t + 3\left(y^2\right)_x + y_{xxx} + 3\varepsilon\left(y_x^2\right)_{xx} = 0. \tag{1}$$

The variable y represents a scaled macrostrain. The independent variables are a dimensionless moving space coordinate x and the dimensionless slow time t. The evolution equation describes the slow variation of the wave profile as observed in a frame which is travelling along with the wave at its basic propagation speed. The evolution equation (1) is the starting point of our analysis.

We look for solutions of the form

$$y(x,t) = q(\theta), \quad \theta = x - ct, \tag{2}$$

representing undistorted waves propagating at the velocity c within the moving reference frame. The function $q = q(\theta)$ will then satisfy an ordinary differential equation, which can be integrated twice to result in a first-order differential equation of the form

$$q'^2 + 4\varepsilon q'^3 = f(q), \quad f(q) = 2B + 2Aq + cq^2 - 2q^3. \tag{3}$$

The cubic polynomial $f(q)$ on the right-hand side contains three parameters: the velocity c of the wave profile relative to the moving frame and two constants of integration, A and B. Instead of these parameters one can also introduce the three roots of the cubic polynomial and write the polynomial in the form

$$f(q) = 2(q-q_1)(q-q_2)(q_3-q). \tag{4}$$

We assume the roots $q_1 \leq q_2 \leq q_3$ to be real. It can be easily shown that, if two roots become conjugate complex, there will be no finite solutions of the differential equation (3).

In principle, equation (3) has to be solved for q' and then integrated. However, it is unlikely that this integration can be performed in closed form. Therefore we confine ourselves to an *approximate* solution, assuming the parameter ε to be small. Expanding the roots of the cubic equation (3) in powers of ε, one obtains

$$q' = \pm\sqrt{f(q)}\left\{1 \mp 2\varepsilon\sqrt{f(q)} + 10\varepsilon^2 f(q) \mp 64\varepsilon^3\,[f(q)]^{3/2}\right\} + O(\varepsilon^4), \tag{5}$$

where $f(q)$ is the cubic polynomial defined by $(3)_2$. Although, at first glance, this differential equation for $q(\theta)$ seems even more complicated than the original one, it can be integrated in closed form.

3. PHASE PORTRAIT

Before going on with the integration the behaviour of the phase curves $q'(q)$ will be analysed in detail. The polynomial $f(q)$ involves three parameters. In order to get a one-parameter family of curves, two constants should be fixed. Let us suppose that the minimum and the maximum of the polynomial are located at $q = 0$ and $q = a$, respectively, where a is an arbitrary but fixed value. Then the cubic polynomial admits the representation

$$f(q) = b^2 - (2q+a)(q-a)^2, \tag{6}$$

where b is considered as the only free parameter of the function. The phase portrait depicts the family of phase curves $q'(q)$ for different values of the parameter b, while a and ε are kept fixed.

As has been shown in [4], solitary waves are possible solutions of the extended KdV equation only if

$$\varepsilon \le \varepsilon_{\max} = \frac{1}{2}(3a)^{-3/2}. \tag{7}$$

Figure 1 shows the phase portrait for $\varepsilon = 0.8\,\varepsilon_{\max}$. There is a pronounced asymmetry which increases with growing values of ε, while for $\varepsilon = 0$ the symmetric phase portrait of the standard KdV equation would be retained. In principle, the whole (q,q')-plane is filled by phase curves. Only those, however, which do not extend to infinity correspond to finite solutions $q = q(\theta)$ of the evolution equation. As can be seen from the figure, it is only the shaded part of the phase plane which contains closed phase curves representing finite waves. Those curves which intersect the q-axis twice at right angles correspond to periodic waves. The limiting curve forms a homoclinic orbit starting and ending at the origin, which means that $q = q' = 0$ is attained asymptotically for $\theta \to \pm\infty$. This curve corresponds to the limiting solitary wave.

The final integration uses the series expansion (5) rather than the exact phase curves $q'(q)$. In Fig. 2 the exact solution of the cubic equation (3) is contrasted with the approximations (5) allowing for different powers of ε. The $O(1)$ approximation neglects the influence of micrononlinearity and gives the symmetric phase curves of the KdV case. Taking into account the corrections (5) with increasing powers of ε leads to the asymmetric phase curves which are characteristic of the extended KdV equation. The approximations converge to the exact solution. In the upper half-plane the convergence is alternating, in the lower the curves approach the limit from above. Even for $\varepsilon = \varepsilon_{\max}$ the approximation is acceptable for periodic waves. It is still poor at the kink of the phase curve representing the solitary wave. This, however, is the worst case.

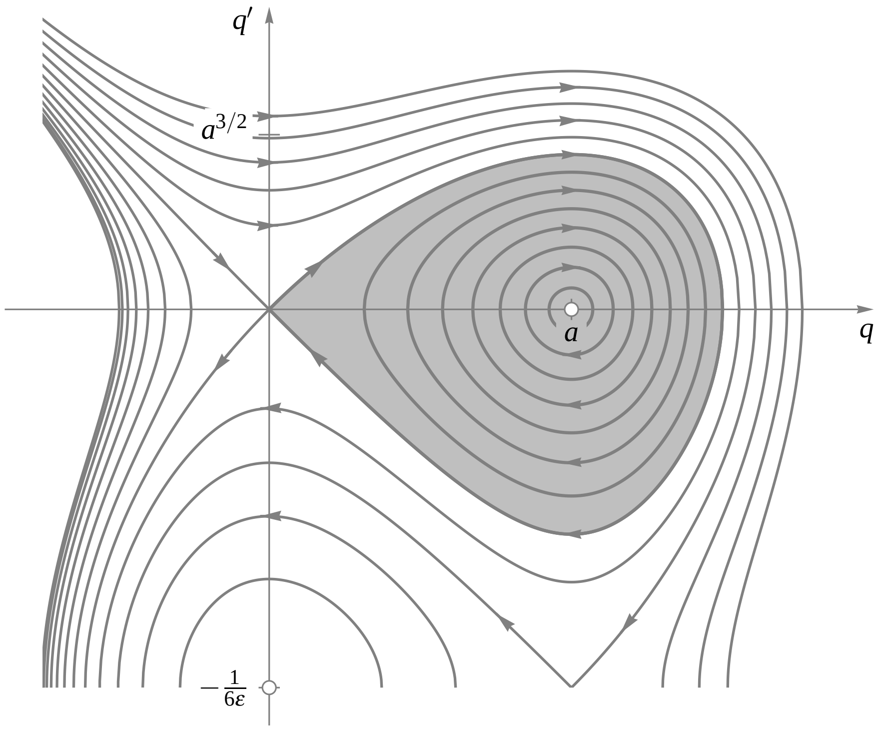

Fig. 1. Phase portrait of the extended KdV equation for $\varepsilon = 0.8\,\varepsilon_{\max}$.

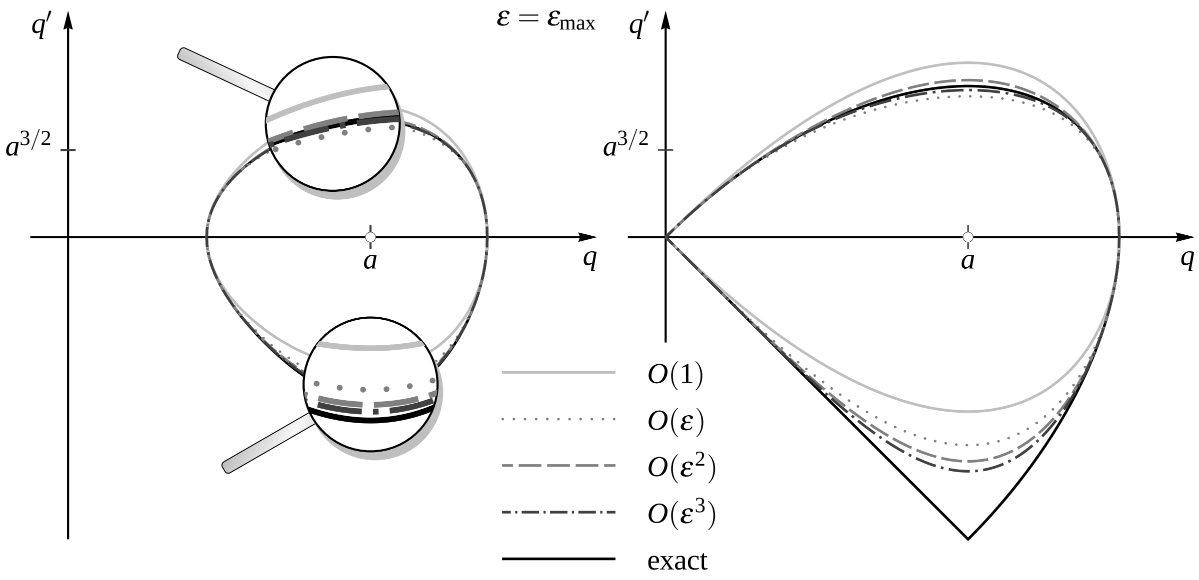

Fig. 2. Approximate phase curves of the extended KdV equation with the maximal micrononlinearity parameter. Left: periodic wave (with magnified areas). Right: limiting solitary wave.

4. ASYMMETRIC PERIODIC WAVES

The final integration will be performed using the approximation of q' by the power series (5). Without loss of generality one may assume that q attains its maximum value q_3 at $\theta = 0$. Using this as the initial condition for the definite integration, the values of q will decrease as θ increases. Therefore the lower signs in (5) are chosen. For performing the integration one needs the reciprocal value $1/q'$ which is obtained as

$$\frac{\mathrm{d}\theta}{\mathrm{d}q} = \frac{1}{q'} = -\frac{1}{\sqrt{f(q)}}\left[1 - 2\varepsilon\sqrt{f(q)}\right] + O(\varepsilon^2) = -\frac{1}{\sqrt{f(q)}} + 2\varepsilon + O(\varepsilon^2). \tag{8}$$

The analysis is restricted here to the $O(\varepsilon)$ approximation but can easily be extended to higher orders. With the use of the initial condition $q(0) = q_3$, the integration yields

$$\theta = \int_{q_3}^{q} \left[\frac{-1}{\sqrt{f(q)}} + 2\varepsilon\right] \mathrm{d}q. \tag{9}$$

The integral can be evaluated explicitly by using the substitution

$$q = q_2 + (q_3 - q_2)\cos^2\varphi \tag{10}$$

of the integration variable. Performing the integration gives the result

$$\theta = \frac{1}{\eta} F(\varphi; k) - 2\varepsilon(q_3 - q), \tag{11}$$

where F denotes the incomplete elliptic integral of the first kind and the constants

$$\eta = \sqrt{\frac{q_3 - q_1}{2}}, \qquad k = \sqrt{\frac{q_3 - q_2}{q_3 - q_1}} \tag{12}$$

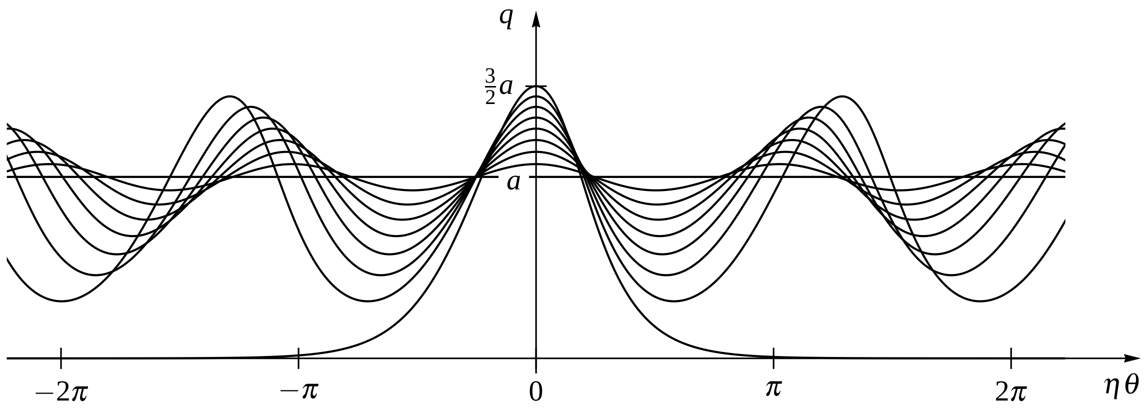

Fig. 3. Periodic waves and solitary wave governed by the extended KdV equation ($\varepsilon = \varepsilon_{\max}$).

have been introduced. Solving (11) for the auxiliary variable φ and resubstituting this into the transformation formula (10) yields

$$q = q_2 + (q_3 - q_2)\,\mathrm{cn}^2\,\eta\,[\theta + 2\varepsilon(q_3 - q)]\,. \tag{13}$$

This is an implicit representation of the periodic wave solutions of the extended KdV equation (1), though only in a first approximation. For $\varepsilon = 0$ it passes into the cnoidal wave solution of the KdV equation. Figure 3 shows a family of periodic waves together with their limiting solitary wave, as described by (13). The waves look very much like the corresponding cnoidal waves, but are inclined to the right.

5. CONCLUDING REMARKS

As known from previous studies [4,5], the propagation of one-dimensional deformation waves in a nonlinear microstructured solid leads to an evolution equation which has the form of an extended Korteweg–de Vries equation. Janno and Engelbrecht [3] have demonstrated that, due to the nonlinearity of the microscale, the solitary wave profile becomes asymmetric. The same effect appears in the case of the respective evolution equation which has been solved approximately by Randrüüt and Braun [4]. Although solitary waves constitute the most interesting type of solutions, the same procedure is applied here to a more general case. Solitary waves can be considered as the long-wave limit of periodic solutions which, in the KdV case, have the form of cnoidal waves.

It is shown that, due to the nonlinearity in microscale, cnoidal waves stay periodic but become inclined in the same manner as solitary waves. Compared with the classical cnoidal waves ($\varepsilon = 0$), the periodic waves for $\varepsilon > 0$ have a steeper slope at the leading flank, while the trailing flank falls off gentler. Qualitatively the behaviour is as expected from the solitary-wave limit.

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Üldistatud Kortewegi-de Vriesi võrrandi perioodilistest lahenditest

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Üksiklainete levi mikrostruktuursetes tahkistes on leidnud põhjalikku käsitlemist Janno ja Engelbrechti [3] poolt. Selle artikli autorid on näidanud, et niisuguste lainete evolutsioonivõrrand on kõrgemat järku lisaliikmega Kortewegi-de Vriesi võrrand (üldistatud KdV-võrrand), kusjuures lisaliige kirjeldab mikrostruktuuri mittelineaarsust. On teada, et mikrostruktuuri mittelineaarsuse tõttu on üksiklaine ebasümmeetriline.

Üksiklaineid võib vaadelda kui lõpmatusele läheneva lainepikkusega perioodilisi laineid, mida KdV juhtumil nimetatakse knoidaalseteks laineteks. Selles artiklis on uuritud üldistatud KdV-võrrandi perioodilisi lahendeid. Vastavaid faasidiagramme kirjeldab kuupvõrrand, mille lahendite analüütiline integreerimine osutub tõenäoliselt võimatuks. Seetõttu lahendatakse see kuupvõrrand ligikaudselt mikrostruktuuri mittelineaarsuse parameetri väikeste väärtuste korral, mis võimaldab saada perioodilisi lahendeid ilmutamata kujul.

Knoidaalsete lainetega võrreldes on üldistatud KdV-võrrandi perioodilistel lahenditel vastavalt liikumise suunale esikülg järsema kaldega kui tagumine. Mittelineaarne lisaliige ei mõjuta laine amplituudi, perioodi ja levimiskiiruse vahelisi seoseid. Selgub, et lisaliikme mõju perioodilistele lahenditele on samasugune kui üksiklainelisele lahendile, põhjustades laineprofiili ebasümmeetria.