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MATHEMATICS

Some modern developments in the theory of real division algebras

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Abstract. The study of real division algebras was initiated by the construction of the quaternion and the octonion algebras in the mid-19th century. In spite of its long history, the problem of classifying all finite-dimensional real division algebras is still unsolved. We review the theory of this problem, with focus on recent contributions.

Key words: real division algebra, classification.

1. INTRODUCTION

An *algebra* over a field k is understood to be a vector space A over k equipped with a bilinear multiplication map $A \times A \to A$, $(a,b) \mapsto ab$. If $A \neq 0$, and the linear maps $L_a : A \to A$, $x \mapsto ax$ and $R_a : A \to A$, $x \mapsto xa$ are invertible for all $a \in A \setminus \{0\}$, then A is called a *division algebra*. In case A is finite-dimensional, L_a and R_a are invertible if and only if they are injective, which happens precisely when A has no non-trivial divisors of zero (i.e., $xy = 0$ only if $x = 0$ or $y = 0$).

Despite the simple definition, division algebras are highly non-trivial objects. Even when restricting attention to finite-dimensional division algebras over a fixed field k, a classification is known only when k is algebraically closed, in which case every such algebra is isomorphic to k itself.

Over the real number field, $\mathbb{R}$ itself and the complex numbers $\mathbb{C}$ are immediate examples of finite-dimensional division algebras. For pairs of complex number, one may define a multiplication map $\mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2$ by

$$((x_1,x_2),(y_1,y_2)) \mapsto (x_1,x_2)(y_1,y_2) = (x_1y_1 - \bar{y}_2x_2\,,\, x_2\bar{y}_1 + y_2x_1). \tag{1}$$

This multiplication is bilinear over $\mathbb{R}$ (though obviously not over $\mathbb{C}$), and it turns out that the real algebra defined has no non-trivial zero divisors. This is the *quaternion* algebra $\mathbb{H}$, first considered by Hamilton in 1843 (it may be remarked, however, that the formulae for calculating with quaternions appeared earlier in works by Euler in 1748 and Gauß in 1819). The map

$$x = (x_1,x_2) \mapsto \bar{x} = (\bar{x}_1, -x_2) \tag{2}$$

is an involution on $\mathbb{H}$.

Taking x_1,x_2,y_1,y_2 to be quaternions, (1) defines a multiplication on $\mathbb{H}^2$. The resulting algebra is again a division algebra, the *octonions* $\mathbb{O}$, constructed independently by Graves in 1843 and Cayley in 1845. Also on $\mathbb{O}$, the map defined by (2) is an involution. Both $\mathbb{H}$ and $\mathbb{O}$ are equipped with scalar products, in each case given by $\langle (x_1,x_2),(y_1,y_2)\rangle = \langle x_1,y_1\rangle + \langle x_2,y_2\rangle$ and satisfying $\langle x,x\rangle = x\bar{x}$.

Just as $\mathbb{O}$ is constructed from $\mathbb{H}$, and $\mathbb{H}$ from $\mathbb{C}$, complex numbers may be seen as pairs of real numbers with multiplication given by (1) (the involution $x \mapsto \bar{x}$ being the identity map on $\mathbb{R}$). The construction carried out in each of these cases is called the *doubling*, or the *Cayley–Dickson process*. As we have seen, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are all constructed from $\mathbb{R}$ via iteration of this process. However, the 16-dimensional algebra obtained from $\mathbb{O}$ in this manner is no longer a division algebra.

The four algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are often referred to as the classical real division algebras and they share several important properties. They all have identity elements, and are *absolute valued* in the sense that they possess a norm with respect to which $\|xy\| = \|x\|\|y\|$ for arbitrary elements x and y. A theorem by Albert [2] asserts that these are, up to isomorphism, the only finite-dimensional absolute-valued algebras[1] combining these two properties. Urbanik and Wright [32] extended this result to hold also without the assumption of finite-dimensionality.

Every associative finite-dimensional real division algebra is isomorphic to one of $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ [20]. Although the octonions fail to be associative, they satisfy the weaker condition of *alternativity*, meaning that any subalgebra generated by two elements is associative. Moreover, according to a theorem by Zorn from 1931 [33], $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ classify all finite-dimensional real division algebras that are alternative.

A new era in the theory of real division algebras was launched by Hopf [22] in 1940, when he proved that a finite-dimensional *commutative* real division algebra has either dimension one or two, and furthermore, that the dimension of any real division algebra is either a power of two or infinite. Hopf's methods were topological, and inspired other topologists in a development which culminated in 1958 with the (1,2,4,8)-theorem [8,24], which asserts that any finite-dimensional real division algebra has dimension 1, 2, 4, or 8.

The aim of the present article is to give an overview of some of the developments in the theory of finite-dimensional real division algebras that have occurred in the last 50 years. The field is still far from fully discovered. Although the (1,2,4,8)-theorem reduces the number of possible dimensions to four, classifications of all n-dimensional real division algebras are known only for $n \in \{1,2\}$. By a classification we mean a cross-section for the isomorphism classes, i.e., a set of objects in which precisely one representative for each isomorphism class occurs.

The classification problem for one-dimensional real division algebras is trivial, every such algebra being isomorphic to $\mathbb{R}$. In Section 2 we outline the solution in the two-dimensional case. Whereas general classifications are not known for $n > 2$, there are certain subclasses where the problem has been solved.

Seeking to generalize the theorem of Zorn on real alternative division algebras, one is naturally led to consider *power associative* algebras, defined by the property that every subalgebra generated by a single element is associative. A real finite-dimensional division algebra is power associative if and only if it is *quadratic*, that is, if it has an identity element $1 \neq 0$ and the set $\{1, x, x^2\}$ is linearly dependent for all x (this follows from the fact that every finite-dimensional power-associative division algebra has an identity element [30, Lemma 5.3]). Thus, in Section 3 we treat quadratic division algebras over $\mathbb{R}$, describing their classification in dimension four.

Another property that generalizes alternativity is *flexibility*. An algebra A is called flexible if any two elements $x, y \in A$ satisfy the identity $x(yx) = (xy)x$. The classification of all finite-dimensional real flexible division algebras, which was completed just recently, is discussed in Section 4.

Our survey is partial. For example, we are not going into the theory of absolute valued algebras, which in finite dimension are division algebras. One reason for this is that an extensive and up-to-date review [29] already exists, another is that the field is too large to be fairly treated within the scope of a short article as the present.

Benkart and Osborn [4] determined all Lie algebras that arise as derivation algebras of real finite-dimensional division algebras. Results towards the classification of all real finite-dimensional division algebras with a fixed derivation algebra type are found in [5,19,28]. Again, these important contributions are left out, since they would require more space than what could be afforded here.

To abbreviate notation, from here on we use the word algebra to mean 'finite-dimensional real algebra'.

[1] An absolute-valued algebra is a normed vector space with an algebra structure satisfying the above-mentioned identity.

2. DIVISION ALGEBRAS OF DIMENSION TWO

The first classification of the two-dimensional division algebras was given by Burdujan [9] in 1985. His approach is the classical one with multiplication tables and is based on the number of non-zero idempotents in the algebra in question, which is between one and three [31]. Burdujan's article contains no proofs. A similar approach was taken two years earlier in an article by Althoen and Kugler [3], who however failed to provide a classification in our sense. More recently, a classification along these lines was presented by Gottschling [21].

Another solution to the classification problem for two-dimensional division algebras is based on the concept of isotopy, first introduced by Albert [1] in 1942. This approach was pursued by Hübner and Petersson [23] (using the more general theory for arbitrary two-dimensional algebras developed in [27]), and independently by Dieterich [16]. Below we shall outline the basic idea of their solution.

Let A be an arbitrary algebra, and S, T, and U invertible linear transformations of A. The algebra $B = (A,*)$, with the same underlying vector space as A and multiplication defined by $x * y = U((Sx)(Ty))$, is called the *isotope* of A given by S, T, and U. If U is the identity transformation on A, then B is called a *principal isotope* of A. Every isotope of A is isomorphic to a principal isotope, and isotopy defines an equivalence relation between algebra structures on a given vector space. Clearly, the property of being a division algebra is preserved under isotopy.

Now let A be a division algebra, and $a \in A$ an arbitrary non-zero element. The linear maps of respectively left and right multiplication with a are invertible, and one may form the isotope $B = (A, \circ)$ of A with multiplication $x \circ y = (R_a^{-1}x)(L_a^{-1}y)$. Since $a^2 \circ x = (R_a^{-1}a^2)(L_a^{-1}x) = a(L_a^{-1}x) = L_a L_a^{-1}x = x$ and similarly $x \circ a^2 = x$ for all $x \in B$, a^2 is an identity element in B. Hence every division algebra has an isotope which is unital.

A two-dimensional unital division algebra B is spanned by the identity element 1 and some element $b \in B \setminus \mathbb{R}1$. Since the commutative and associative laws clearly hold for products of the basis elements, B is commutative and associative. Thus it is a field extension of $\mathbb{R}$, and as such isomorphic to the complex numbers.

The group of invertible $\mathbb{R}$-linear transformations of $\mathbb{C}$ is denoted by $GL(\mathbb{C})$. For $S, T \in GL(\mathbb{C})$, let $\mathbb{C}_{ST}$ be the principal isotope of $\mathbb{C}$ given by S and T, i.e. $\mathbb{C}_{ST} = (\mathbb{C}, *)$, with $x * y = (Sx)(Ty)$. The previous discussion implies that every division algebra of dimension two is isomorphic to $\mathbb{C}_{ST}$ for some $S, T \in GL(\mathbb{C})$. Moreover, $F \in GL(\mathbb{C})$ is an algebra isomorphism $\mathbb{C}_{ST} \to \mathbb{C}_{S'T'}$ if and only if $F((Sx)(Ty)) = (S'Fx)(T'Fy)$ for all $x, y \in \mathbb{C}$.

Next, one can show that the pair $(\mathrm{sign}(\det S), \mathrm{sign}(\det T))$ is an isomorphism invariant for $\mathbb{C}_{ST}$. Therefore, the classification problem decomposes into four subproblems, each of which can be further reduced and solved by means of elementary Euclidean geometry. For details, see [16].

The classification of all two-dimensional division algebras immediately gives rise to a classification of all commutative division algebras: the algebra $\mathbb{C}_{ST}$ is commutative if and only if $S = T$. In view of Hopf's theorem this, together with the fact that $\mathbb{R}$ is the only one-dimensional division algebra, renders a classification in the commutative case. An independent treatment of the commutative division algebras is given in [13].[2]

3. QUADRATIC DIVISION ALGEBRAS

An extensive study of quadratic division algebras was carried out by Osborn in [26]. Here we shall recapitulate his main result, and describe the classification, due to Dieterich[3] [14], of all quadratic division algebras of dimension four.

[2] Also [6] deals with commutative division algebras, using isotopy. However, due to a technical mistake, the alleged classifying list presented in the article misses some isomorphism classes. See [13] for a detailed account.

[3] Osborn's claim to have classified all four-dimensional quadratic division algebras over an arbitrary field F of characteristic different from two "modulo the theory of quadratic forms over F", is not accurate. See [15].

Let V be a finite-dimensional Euclidean space. A *dissident map* on V is a linear map $\eta : V \wedge V \to V$ with the property that $u, v, \eta(u \wedge v) \in V$ are linearly independent whenever $u, v \in V$ are. If in addition $\xi : V \wedge V \to \mathbb{R}$ is a linear form, (V, ξ, η) is called a *dissident triple*. By a *morphism* $(V, \xi, \eta) \to (V', \xi', \eta')$ is meant an orthogonal map $\sigma : V \to V'$ such that $\eta'(\sigma \wedge \sigma) = \sigma\eta$ and $\xi'(\sigma \wedge \sigma) = \xi$. Denote by $\mathscr{Q}$ the category of quadratic division algebras, and by $\mathscr{D}$ the category of dissident triples. If $(V, \xi, \eta) \in \mathscr{D}$, then the algebra $\mathscr{H}(V, \xi, \eta) = \mathbb{R} \times V$ with multiplication defined by

$$(\alpha, u)(\beta, v) = (\alpha\beta - \langle u, v\rangle + \xi(u \wedge v)\,,\, \alpha v + \beta u + \eta(u \wedge v))$$

is a quadratic division algebra. For a morphism $\sigma : (V, \xi, \eta) \to (V', \xi', \eta')$ in $\mathscr{D}$, set $\mathscr{H}(\sigma) = \mathbb{I}_{\mathbb{R}} \times \sigma : \mathscr{H}(V, \xi, \eta) \to \mathscr{H}(V', \xi', \eta')$. This establishes $\mathscr{H}$ as a functor from $\mathscr{D}$ to $\mathscr{Q}$.

Proposition 3.1 [17,26]. *The functor $\mathscr{H} : \mathscr{D} \to \mathscr{Q}$ is an equivalence of categories.*

In view of the above proposition, classifying all four-dimensional quadratic division algebras amounts to classifying all dissident triples (V, ξ, η) with $\dim V = 3$. In doing this, a key ingredient is the construction of all dissident maps on three-dimensional Euclidean space.

Let V be a Euclidean space of dimension three, and π a vector product[4] on V. Consider a dissident map η on V. Since $\dim(V \wedge V) = 3 = \dim V$, both η and π are bijective linear maps. Now the linear endomorphism $\eta\pi^{-1} : V \to V$ is definite, because otherwise there would exist an element $u \in V \setminus \{0\}$ such that $\langle \eta\pi^{-1}(u), u\rangle = 0$. Then taking $v, w \in V$ such that $\pi(v \wedge w) = u$ would give $\langle \eta(v \wedge w), u\rangle = 0$, which implies $\eta(v \wedge w) \in \mathrm{span}\{v, w\}$, contradicting the dissidence property of η. This proves that every dissident map η on V factors as $\eta = \varepsilon\pi$, where $\varepsilon : V \to V$ is linear and definite. Conversely, given any such $\varepsilon : V \to V$, the composed map $\varepsilon\pi : V \wedge V \to V$ is dissident. In case $\varepsilon : V \to V$ is negative definite, one may write $\eta = \varepsilon\pi = \varepsilon'\pi'$, where $\varepsilon' = -\varepsilon$ is positive definite and $\pi' = -\pi$ again is a vector product, isomorphic to π. This means that every dissident map η on V factors uniquely into a positive definite linear endomorphism and a vector product on V.

Given an antisymmetric linear endomorphism δ of V, let $\xi_\delta : V \wedge V \to \mathbb{R}$ be the form defined by $\xi_\delta(u \wedge v) = \langle \delta(u), v\rangle$. Clearly, $\delta \mapsto \xi_\delta$ defines a bijection from the set of antisymmetric linear endomorphisms of V to the set of linear forms $V \wedge V \to \mathbb{R}$. Denote by $\mathscr{P}$ the set of pairs (δ, ε) of linear endomorphisms on V, in which δ is antisymmetric and ε positive definite. The group $\mathrm{SO}(V)$ acts on $\mathscr{P}$ by

$$\sigma \cdot (\delta, \varepsilon) = (\sigma\delta\sigma^{-1}, \sigma\varepsilon\sigma^{-1}). \tag{3}$$

Fix a vector product π on V, and define $\mathscr{I}(\delta, \varepsilon) = (V, \xi_\delta, \varepsilon\pi)$ for $(\delta, \varepsilon) \in \mathscr{P}$. Let $\mathscr{D}_3$ be the category of dissident triples (W, ξ, η) satisfying $\dim W = 3$. Taking into account the above discussion, one can verify the following proposition.

Proposition 3.2. *The assignment $\mathscr{I}$ defines an equivalence from the category of the group action*[5] (3) *to $\mathscr{D}_3$, acting as identity on morphisms.*

Propositions 3.1 and 3.2 reduce the classification problem for the quadratic division algebras of dimension four to the problem of finding a normal form for $\mathscr{P}$ under the action of $\mathrm{SO}(V)$ given by (3). A solution to this problem is described in [14].

As for dissident triples in seven-dimensional space (or equivalently, eight-dimensional quadratic division algebras), one is still far from a solution of the classification problem. Factorization of a dissident map into a vector product and a definite linear endomorphism is no longer possible in general. The most notable result here is the classification of all dissident triples corresponding to flexible algebras, which is described in the next section. Other contributions are [17,18,25].

[4] A *vector product* is a dissident map π with the property that $u, v, \pi(u \wedge v) \in V$ are orthonormal whenever $u, v \in V$ are. Vector products exist in dimension 0, 1, 3, and 7 only, and are unique up to isomorphism in each dimension.

[5] Given a group G acting on a set X, the category of the action has object set X, and morphism sets $\mathrm{Mor}(x, y) = \{g \in G \mid g \cdot x = y\}$.

4. FLEXIBLE DIVISION ALGEBRAS

The first major contribution to the theory of flexible division algebras was the article [7] by Benkart, Britten, and Osborn in 1982. It separates the class of such algebras into three disjoint subclasses. To state their main theorem, some additional notation is required.

Let $B = \mathscr{H}(V,\xi,\eta) \in \mathscr{Q}$, and $\lambda \in \mathbb{R} \smallsetminus \{0\}$. We define f_λ to be the linear endomorphism of $B = \mathbb{R} \times V$ defined by $f_\lambda(\alpha, v) = (\alpha, \lambda v)$. Now the *scalar isotope* of B determined by λ, denoted ${}_\lambda B$, is the principal isotope of B given by f_λ in both arguments, that is, ${}_\lambda B = (B,*)$ with multiplication $x * y = f_\lambda(x) f_\lambda(y)$. This is a flexible division algebra whenever B is.

The real vector space $\mathfrak{su}_3\mathbb{C}$ of anti-hermitean complex 3×3-matrices with trace zero is a Lie algebra under the commutator multiplication $[\,,]$. For each $\delta \in \mathbb{R} \smallsetminus \{0\}$, $\mathfrak{su}_3\mathbb{C}$ with multiplication $x \star y = \delta[x,y] + \frac{i}{2}\left(xy + yx - \frac{2}{3}\mathrm{tr}(xy)\mathbb{I}_3\right)$ is a flexible division algebra of dimension eight, denoted O_δ. The algebras of this type are called *generalized pseudo-octonions*.

Theorem 4.1 [7, Theorem 1.4]**.** *Every flexible division algebra is either*
1. *commutative,*
2. *a scalar isotope of a quadratic flexible division algebra of dimension four or eight, or*
3. *a generalized pseudo-octonion algebra.*

The different cases are mutually exclusive.

In [11] it was proved that two scalar isotopes ${}_\lambda A$ and ${}_\mu B$ of quadratic flexible division algebras are isomorphic if and only if $\mu = \lambda$ and $A \simeq B$, and that $O_\delta \simeq O_{\delta'}$ if and only if $\delta = \pm\delta'$. As we saw in Section 2, the commutative division algebras have already been classified. Hence, to classify all flexible division algebras, it remains to consider the ones which are both flexible and quadratic.

Let $\mathscr{Q}^{\mathrm{fl}}$ be the category of flexible quadratic division algebras, and $\mathscr{D}^{\mathrm{fl}}$ the corresponding (under $\mathscr{H}$) dissident triples. A dissident triple (V,ξ,η) is in $\mathscr{D}^{\mathrm{fl}}$ precisely when $\xi = 0$ and $\langle \eta(u \wedge v), u \rangle = 0$ for all $u, v \in V$ [26, p. 203]. Using the theory developed in Section 3, it is not difficult to show that $\{(V, 0, \lambda\pi)\}_{\lambda>0}$, where π is a vector product on a three-dimensional space V that classifies $\mathscr{D}_3^{\mathrm{fl}} = \mathscr{D}_3 \cap \mathscr{D}^{\mathrm{fl}}$. Thus $\{\mathscr{H}(V,0,\lambda\pi)\}_{\lambda>0}$ classifies the four-dimensional flexible quadratic division algebras.

The eight-dimensional case has been studied by Cuenca Mira et al. [10]. In their deeply technical paper, they construct all flexible quadratic division algebras of dimension eight, and establish a necessary and sufficient criterion for when two such algebras are isomorphic. We shall formulate their principal theorem in the language of dissident triples.

Let π be a vector product on a seven-dimensional Euclidean space V, and denote by $\mathrm{Pds}(V)$ the set of positive definite symmetric linear endomorphisms of V. The automorphism group $\mathrm{Aut}(\pi) = \mathrm{Aut}(V,0,\pi)$ of the dissident triple $(V,0,\pi)$ is isomorphic to the exceptional Lie group $\mathscr{G}_2$. It is a subgroup of $\mathrm{O}(V)$, and hence conjugation with elements in $\mathrm{Aut}(\pi)$ defines an action on $\mathrm{Pds}(V)$:

$$\mathrm{Aut}(\pi) \times \mathrm{Pds}(V) \to \mathrm{Pds}(V),\ (\sigma,\delta) \mapsto \sigma \cdot \delta = \sigma\delta\sigma^{-1}. \tag{4}$$

Denote $\mathscr{D}_7^{\mathrm{fl}} = \mathscr{D}_7 \cap \mathscr{D}^{\mathrm{fl}}$, where $\mathscr{D}_7$ is the category of dissident triples (W,ξ,η) for which $\dim W = 7$.

Theorem 4.2 (cf. [10, Theorem 5.7])**.** *The category $\mathscr{D}_7^{\mathrm{fl}}$ is equivalent to the category $\mathscr{A}$ of the group action* (4). An equivalence $\mathscr{J} : \mathscr{A} \to \mathscr{D}_7^{\mathrm{fl}}$ is given by $\mathscr{J}(\delta) = (V, 0, \delta\pi(\delta \wedge \delta))$ for objects $\delta \in \mathrm{Pds}(V)$ and $\mathscr{J}(\sigma) = \sigma$ for morphisms $\sigma \in \mathrm{Aut}(\pi)$.

Thus, if $\mathscr{N} \subset \mathrm{Pds}(V)$ is a cross-section for the orbit set of (4), then $\mathscr{H}\mathscr{J}(\mathscr{N}) \subset \mathscr{Q}^{\mathrm{fl}}$ is a cross-section for the isomorphism classes of the eight-dimensional flexible quadratic division algebras. The normal form problem posed here requires, in contrast to the ones encountered in previous sections, additional theory and technical considerations.

A *Cayley triple* in V is a triple $(u,v,z) \in V^3$ such that $u, v, \pi(u \wedge v), z \in V$ are orthonormal. Every Cayley triple $c = (u,v,z)$ determines an orthonormal basis $\underline{b}_c = (u,\ v,\ \pi(u \wedge v),\ z,\ \pi(u \wedge z),\ \pi(v \wedge z),\ \pi(\pi(u \wedge v) \wedge z))$ to V. Denote by $\mathscr{C}$ the set of Cayley triples in V, and by $[T]_c$ the matrix of a linear endomorphism T of

V with respect to the basis $\underline{b}_c$ determined by $c \in \mathscr{C}$. The group $\mathrm{Aut}(\pi)$ acts simply transitively on $\mathscr{C}$ via $\sigma \cdot (u,v,z) = (\sigma(u), \sigma(v), \sigma(z))$, i.e., given two Cayley triples c and c', there exists a unique $\sigma \in \mathrm{Aut}(\pi)$ such that $\sigma \cdot c = c'$. Hence, fixing a triple $s \in \mathscr{C}$ gives a bijection $\mathrm{Aut}(\pi) \to \mathscr{C}$, $\sigma \mapsto \sigma \cdot s$.

For all $\delta \in \mathrm{Pds}(V)$, the task now is to choose Cayley triples c in V in such a way that the resulting matrices $[\delta]_c$ are the same within the orbits of action (4). Then the normal form of δ is the endomorphism which, in the basis $\underline{b}_s$, is given by the matrix $[\delta]_c$.

The set $\mathfrak{p}(\delta) = \{(\lambda, \dim\ker(\delta - \lambda \mathbb{I}_V)) \mid \ker(\delta - \lambda \mathbb{I}_V) \neq 0)\}$ is an invariant for the orbit of δ under $\mathrm{Aut}(\pi)$. This yields a decomposition of the normal form problem for (4) into 15 different subproblems, determined by the dimensions of the eigenspaces of δ. These subproblems are treated in [12], and the classification of all flexible division algebras thereby obtained.

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Reaalsete jagamisega algebrate teooria tänapäevastest arengutest

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Reaalsete jagamisega algebrate uurimine algas 19. sajandi keskel seoses kvatern- ja oktonioonsete algebrate kasutuselevõtuga. Pikale ajaloole vaatamata on kõigi lõplikumõõtmeliste reaalsete jagamisega algebrate klassifitseerimine senini lõpetamata. Artiklis on antud sellest probleemist ülevaade, keskendudes viimastele tulemustele.