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MATHEMATICS

Application of operator monotone functions in economics

Frank Hansen

Department of Economics, University of Copenhagen, Øster Farimagsgade 5, building 26, DK-1353 Copenhagen K, Denmark; Frank.Hansen@econ.ku.dk

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Abstract. Operator monotone functions play an important role in economics. We show that 2-monotonicity is equivalent to decreasing relative risk premium, a notion recently introduced in microeconomics. We also show that an operator monotone function is risk vulnerable, a notion introduced by Gollier and Pratt.

Key words: mathematical economics, operator monotone function, relative risk premium, risk vulnerability, utility function.

1. INTRODUCTION

Daniel Bernoulli [1] gave in 1738 an explanation of the Saint Petersburg paradox stated in 1713 by his cousin, Nicolas Bernoulli, in a letter to Pierre Raymond de Montmort. The resolution of the paradox led to the explicit introduction of the notion of utility of money, the expected utility hypothesis, and the principle of diminishing marginal utility. The idea is that a decision maker's utility of wealth x is given by a (utility) function $u(x)$. If the wealth is a stochastic variable with possible outcomes $x=(x_1,\dots,x_n)$ and probabilities $p=(p_1,\dots,p_n)$, then the expected utility is given by

$$E_u[x]=p_1u(x_1)+\cdots+p_nu(x_n).$$

The decision maker is said to be rational if his decisions maximize the expected utility. The assumptions of greed and risk aversion are then characterized by the requirements that u be non-decreasing and concave.

The probabilities may be objective, that is given by the mechanics of the setup. This is the case if for example the wealth results from the outcome of a well-specified lottery. But probabilities may also be subjective and formed by the decision maker as a result of his expectations of the unknown future. In his seminal work on the foundation of statistics, Savage [7] gave an axiomatic description of preferences such that both the utility function (up to the composition with a strictly increasing affine transformation) and the subjective probabilities can be uniquely derived from the decision maker's preferences, provided they satisfy a few natural conditions.

There is empirical evidence for the claim that some decision makers may have preferences that cannot be represented by expected utility [2]. This notion goes back to Knight [5], who introduced the distinction between risk and uncertainty, where risky events have assigned probabilities while others in Knight's words are 'ambiguous events for which ordinary probabilities cannot be defined'. In this paper we only deal with decision problems under risk and thus assume that decision makers are 'probabilistic sophisticated'.

In the next section we demonstrate that a broad class of preferences introduced in [4] are represented by matrix monotone functions of order 2. In Section 3 we consider the notion of risk vulnerability introduced by Gollier and Pratt [3] and show that operator monotone utility functions are risk vulnerable.

2. RELATIVE RISK PREMIUM

A decision maker is in general guided by his preferences. This applies for example when deciding the composition of a portfolio of assets. There is a vast literature in decision theory beginning with Savage [7] that discusses how preferences are represented by utility functions. We shall demonstrate that a natural class of preferences are represented by the 2-monotone functions.

Consider a decision maker with wealth x who is given the choice between obtaining an alternative level of wealth y_1 with certainty, or to participate in a lottery that will result in either unchanged wealth x or a level of wealth y_2 strictly bigger than y_1. Equivalently, one could consider the choice to be between participating in a lottery with outcomes x and y_2 or receive the sure level of wealth y_1. Let $a = a(x, y_1, y_2)$ be the expected outcome of the lottery between x and y_2 that gives the same expected utility as obtaining y_1 with certainty. The risk premium π demanded by the agent for being indifferent between the lottery and the sure outcome is thus $\pi = a - y_1$.

We notice that the risk premium is positive when the utility function is concave. This is common in economic theory. An agent is said to be risk averse if he prefers the expected outcome of a lottery instead of participating in the lottery, and this is by Jensen's inequality equivalent to concavity of the utility function. This is a reflection of most people's preferences. We demand a higher return of a risky investment in stocks than in a sure investment in a government bond. In the slightly different experiment described in Fig. 1 risk aversion entails that the risk premium is positive.

The relative risk premium

$$\lambda(x) = \frac{a - y_1}{y_1 - x} \qquad x \neq y_1$$

is introduced in [4] as the quotient between the risk premium $\pi = a - y_1$ and the opportunity cost $y_1 - x$ that the agent puts on the line by choosing the lottery in place of y_1 with certainty. It is not unreasonable to assume that most people would demand not only a smaller risk premium π but also a smaller relative risk premium $\lambda(x)$ as a function of wealth x when everything else is kept unchanged. The question is what restrictions this assumption puts on the utility function.

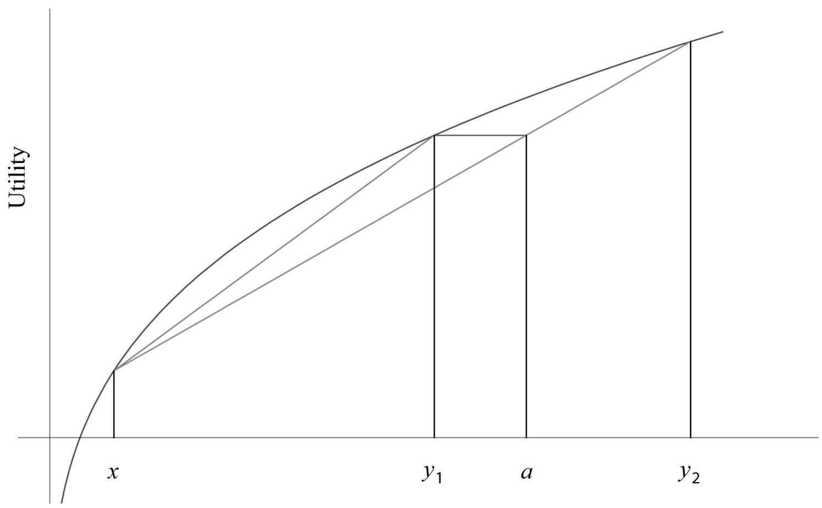


Fig. 1. Utility versus wealth.

If we work out the geometry in Fig. 1 it is not difficult to derive that the relative risk premium $\lambda(x)$ is given by the expression

$$\lambda(x) = \frac{[x, y_1]_u}{[x, y_2]_u} - 1,$$

where the divided difference $[t, s]_u$ is defined by setting

$$[t, s]_u = \frac{u(t) - u(s)}{t - s} \qquad t \neq s.$$

Decreasing relative risk premium is therefore equivalent to the condition

$$\det \begin{pmatrix} [x_1, y_1]_u & [x_1, y_2]_u \\ [x_2, y_1]_u & [x_2, y_2]_u \end{pmatrix} \geq 0$$

for $x_1 < x_2$ and $y_1 < y_2$. But this condition is by Löwner's theory [6] equivalent to 2-monotonicy of u. This has several surprising consequences. It for example entails that the sum (and composition when applicable) of two utility functions with decreasing relative risk premium also has decreasing relative risk premium. This is virtually impossible to derive directly from the geometrical condition as described in Fig. 1. It also means that a utility function with decreasing relative risk premium is continuously differentiable. In conclusion we get the following characterization [4]:

Theorem 1. *Let u be a strictly increasing three times continuously differentiable function defined in an open interval I. The following conditions are equivalent*:

(i) *u has decreasing relative risk premium*;

(ii) *the matrix*

$$\begin{pmatrix} u'(x) & \dfrac{u''(x)}{2} \\ \dfrac{u''(x)}{2} & \dfrac{u'''(x)}{6} \end{pmatrix}$$

is positive semi-definite for each $x \in I$;

(iii) *the derivative u' can be written in the form*

$$u'(x) = \frac{1}{c(x)^2} \qquad x \in I,$$

where c is a positive concave function.

The Pratt and Arrow measure of absolute risk aversion $r(x)$ of a decision maker with (strictly increasing and twice differentiable) utility function $u(x)$ is given by

$$r(x) = -\frac{u''(x)}{u'(x)}.$$

The particular form of the measure is independent of affine transformations of u in accordance with Savage's theory. The condition (iii) in Theorem 1 entails the following result.

Theorem 2. *Let $u\colon I \to \mathbf{R}$ be a strictly increasing and three times continuously differentiable utility function with decreasing relative risk premium defined in an interval of the form $I = (x_0, \infty)$. The absolute risk aversion $r(x)$ is then a decreasing function in wealth x and tends to zero as x approaches infinity.*

3. RISK VULNERABILITY

The notion of risk vulnerability introduced by Gollier and Pratt [3] refers to the very plausible assumption that people who are not quite sure whether their funds are intact tend to take less risky positions than those who do not have this worry. Surprisingly, many increasing and concave utility functions do not have this property. This applies for example to piecewise affine functions. It is also noteworthy that a sum of two risk vulnerable functions need not be risk vulnerable.

Suppose a decision maker with utility function u is subject to an unfair risk to wealth. We represent the risk by a lottery $\tilde{z} = (z_1, \dots, z_n)$ with probabilities $p = (p_1, \dots, p_n)$ and non-positive expected value $E(\tilde{z}, p)$. The indirect utility function v is defined by setting

$$v(x) = E_u(x + \tilde{z}) = \sum_{i=1}^{n} p_i u(x + z_i),$$

where the domain is chosen so as to be meaningful contingent on $\tilde{z}$. The decision maker is risk vulnerable if the absolute risk aversion increases in the presence of an unfair risk to wealth, that is if

$$-\frac{v''(x)}{v'(x)} \geq -\frac{u''(x)}{u'(x)}.$$

The main result in this section is that operator monotone functions are risk vulnerable. Gollier and Pratt [3] introduced a difficult-to-apply bivariate inequality that characterizes risk vulnerability. We recently introduced [4] the so-called auxiliary function

$$\varphi(t) = -u'(x)u''(x+t) + u''(x)u'(x+t) \qquad t \in I - x$$

with base point $x \in I$ (notice that $\varphi(0) = 0$) and obtained the following characterization:

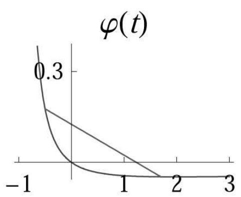
Theorem 3. *Let $u\colon I \to \mathbf{R}$ be a twice continuously differentiable strictly increasing concave function with decreasing absolute risk aversion defined in an open interval I. Then the following statements are equivalent*:
(i) *u is risk vulnerable*;
(ii) *the auxiliary function φ with base point $x \in I$ satisfies*

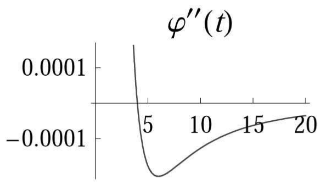
$$\frac{\varphi(y_1)}{y_1} + \frac{\varphi(-y_2)}{y_2} \geq 0$$

for all $y_1, y_2 > 0$ such that $x + y_1 \in I$ and $x - y_2 \in I$.

Since a risk vulnerable function has decreasing absolute risk aversion, we are not making any restrictions by imposing the assumption in the theorem.

The condition in Theorem 3 is equivalent to non-negativity of the second divided difference $[y_1, -y_2, 0]_\varphi$ and has thus an interesting geometrical interpretation. It means that any chord of φ across zero has non-negative value in zero. This condition is considerably weaker than convexity. We can visualize the condition by considering the risk vulnerable function $u(x) = x^{1/2}$ and calculate the auxiliary function $\varphi(t)$ with base point $x = 1$,





Any chord of φ across zero has a non-negative value in zero, but the auxiliary function is not convex.

Since the auxiliary function is a quadratic form in u, we have in Theorem 3 a tool that is more suitable for the analysis of sums of functions. In general, we still have the problem that the absolute risk aversion of a sum of functions is difficult to calculate. But if we consider functions of the form

$$u(x) = -\sum_{i=1}^{n} \frac{a_i}{x+\lambda_i} \qquad x > 0, \tag{3.1}$$

where $\lambda_1, \ldots, \lambda_n$ and $a_1, \ldots, a_n$ are positive real numbers, then we realize that they are operator monotone and therefore have decreasing relative risk premium. It follows by the result in the last section that such functions also have decreasing absolute risk aversion, so the assumption in Theorem 3 is satisfied. We therefore only have to analyse the condition in the theorem and this is a quadratic form in u. The result, which requires quite long and tedious calculations, is that any function of the form (3.1) is risk vulnerable. By standard approximation methods we therefore conclude:

Corollary 1. *Any function of the form*

$$u(x) = \alpha x + \beta + \int_0^\infty \left(\frac{\lambda}{\lambda^2+1} - \frac{1}{x+\lambda} \right) d\mu(\lambda),$$

where $\alpha \geq 0$ *and* μ *is a positive Borel measure with support in* $[0,\infty)$ *such that the integral*

$$\int (\lambda^2+1)^{-1}\, d\mu(\lambda) < \infty,$$

is risk vulnerable and has decreasing relative risk premium.

However, the functions characterized in the corollary are recognized to be exactly the set of operator monotone functions defined in the positive half-axis.

4. CONCLUSIONS

We have demonstrated that two notions in microeconomics, decreasing relative risk premium and risk vulnerability, are related to algebraic properties of the agent's utility function. The notion of decreasing relative risk premium is simply equivalent to 2-monotonicity of the utility function, and it implies that the agent's absolute risk aversion is a decreasing function of wealth and tends to zero in infinity. The notion of risk vulnerability is implied by operator monotonicity of the agent's utility function. The significance of the result is that it provides a large domain of risk vulnerable functions invariant under convex combinations and composition. It should be noted that there exist risk vulnerable functions that are not operator monotone.

The results in this paper were obtained by applying characterizations of 2-monotonicity and operator monotonicity, respectively. It would be interesting to obtain the results by purely algebraic methods; that is by explicitly applying 2-monotonicity or operator monotonicity.

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Operaator-monotoonsete funktsioonide kasutamine majandusteaduses

Frank Hansen

Operaator-monotoonse funktsiooni mõiste on seostatud mitmete majandusteaduslike mõistetega. On näidatud, et 2-monotoonsus on ekvivalentne kahaneva suhtelise riskipreemiaga ja operaator-monotoonsed funktsioonid on Gollier' ning Pratti mõttes riskihaavatavad.