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MATHEMATICS

Complex interpolation of compact operators mapping into lattice couples

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Abstract. After 44 years it is still not known whether an operator mapping one Banach couple boundedly into another and acting compactly on one (or even both) of the "endpoint" spaces also acts compactly between the complex interpolation spaces generated by these couples. We answer this question affirmatively in certain cases where the "range" Banach couple is a couple of lattices on the same measure space.

Key words: functional analysis, complex interpolation, compact operator, Banach lattice.

1. INTRODUCTION

All Banach spaces in this paper will be over the complex field. The closed unit ball of a Banach space A will be denoted by $\mathscr{B}_A$. For any two Banach spaces A and B, the notation $T : A \xrightarrow{b} B$ will mean, just like the usual notation $T : A \to B$, that T is a linear operator T defined on A (and also possibly defined on a larger space) and it maps A into B boundedly. The notation $T : A \xrightarrow{c} B$ will mean that $T : A \xrightarrow{b} B$ with the additional condition that T maps A into B compactly.

We will write $A \overset{1}{\subset} B$ when A is continuously embedded with norm 1 into B, and $A \overset{1}{=} B$ when A and B coincide with equality of norms.

For each ***Banach couple*** (or ***interpolation pair***) $\vec{A} = (A_0, A_1)$ and each $\theta \in [0,1]$, we will let $[A_0, A_1]_\theta$ denote the complex interpolation space of Alberto Calderón [3]. We also let A_j° denote the closure of $A_0 \cap A_1$ in A_j for $j = 0,1$. The couple (A_0, A_1) is called ***regular*** if $A_j^\circ = A_j$ for $j = 0,1$. The spaces $A_0 \cap A_1$ and $A_0 + A_1$ are Banach spaces when they are equipped with their usual norms (as e.g., on p. 114 of [3]).

For any two fixed Banach couples $\vec{A} = (A_0, A_1)$ and $\vec{B} = (B_0, B_1)$, the notation $T : \vec{A} \xrightarrow{c,b} \vec{B}$ will mean that the linear operator $T : A_0 + A_1 \to B_0 + B_1$ satisfies $T : A_0 \xrightarrow{c} B_0$ and $T : A_1 \xrightarrow{b} B_1$. The notation $\vec{A} \blacktriangleright \vec{B}$ will mean that every linear operator $T : A_0 + A_1 \to B_0 + B_1$ which satisfies $T : \vec{A} \xrightarrow{c,b} \vec{B}$ also satisfies $T : [A_0, A_1]_\theta \xrightarrow{c} [B_0, B_1]_\theta$ for every $\theta \in (0,1)$. The notation $(*.*) \blacktriangleright \vec{B}$ for some fixed Banach couple $\vec{B}$ will mean that $\vec{A} \blacktriangleright \vec{B}$ for every Banach couple $\vec{A}$. Analogously, the notation $\vec{A} \blacktriangleright (*.*)$ for some fixed Banach couple $\vec{A}$ will mean that $\vec{A} \blacktriangleright \vec{B}$ for every Banach couple $\vec{B}$.

Some 44 years ago, Calderón [3] proved that $(*.*) \blacktriangleright \vec{B}$ for all Banach couples $\vec{B}$ which satisfy a certain approximation condition. Since then it has been established that $\vec{A} \blacktriangleright \vec{B}$ for a large variety of other different choices of $\vec{A}$ and $\vec{B}$. (See, e.g., the 12 papers and website referred to on p. 72 of [7], and [7] itself.) However, we still do not know whether $\vec{A} \blacktriangleright \vec{B}$ holds for *all* choices of $\vec{A}$ and $\vec{B}$, i.e., whether "$(*.*) \blacktriangleright (*.*)$".

In this paper we shall add to the library of known examples of couples $\vec{A}$ and $\vec{B}$ satisfying $\vec{A} \blacktriangleright \vec{B}$ in the context of spaces of measurable functions. We shall use the terminology ***lattice couple*** to mean a Banach couple $\vec{A} = (A_0, A_1)$ where both A_0 and A_1 are complexified Banach lattices of measurable functions defined on the same σ-finite measure space.

Cobos et al. [4, Theorem 3.2 p. 289] proved that $\vec{A} \blacktriangleright \vec{B}$ whenever *both* $\vec{A}$ and $\vec{B}$ are lattice couples, provided that B_0 and B_1 both have the Fatou property, or that at least one of B_0 and B_1 has absolutely continuous norm. Subsequently, Cwikel and Kalton [8, Corollary 7 part (c) on p. 270] generalized this result by showing that $\vec{A} \blacktriangleright (*.*)$ for any lattice couple $\vec{A}$.

In this paper we shall obtain a different generalization of the above-mentioned result of [4], namely we will show that $(*.*) \blacktriangleright \vec{B}$ for every lattice couple $\vec{B}$ satisfying one or the other of the same conditions imposed in [4]. In fact, some other weaker conditions on $\vec{B}$ are also sufficient. Roughly speaking, as indeed the reader might naturally guess, our approach is to take the "adjoint" of the above-mentioned result $\vec{A} \blacktriangleright (*.*)$ of [8], using arguments in the style of Schauder's classical theorem about adjoints of operators. But this is apparently not quite as simple to do as one might at first expect.

In forthcoming papers we plan to extend our main result $(*,*) \blacktriangleright \vec{B}$ to more general lattice couples and non-lattice couples $\vec{B}$, including some which are rather close in some sense to the couple $(\ell^\infty(FL^\infty), \ell^\infty(FL^\infty_1))$. We recall (see [9], or [5]) that $(*.*) \blacktriangleright (\ell^\infty(FL^\infty), \ell^\infty(FL^\infty_1))$ if and only if $(*.*) \blacktriangleright (*.*)$.

Pustylnik [21] recently obtained a very general compactness theorem which has some overlap with our result here.

2. A RATHER GENERAL ARZELÀ-ASCOLI–SCHAUDER THEOREM

In this section we describe the result which will play the role of Schauder's theorem for the proof of our main result.

Let us recall that a ***semimetric space*** (X,d), also often referred to as a ***pseudometric space***, is defined exactly like a metric space, except that the condition $d(x,y) = 0$ for a pair of points $x, y \in X$ does not imply that $x = y$. (However, $d(x,x) = 0$ for all $x \in X$.) Each semimetric space (X,d) gives rise to a metric space $\left(\widetilde{X}, \vec{d}\right)$ in an obvious way, where $\widetilde{X}$ is the set of equivalence classes of X defined by the relation $x \sim y \Longleftrightarrow d(x,y) = 0$.

Here are three definitions and three propositions concerning an arbitrary semimetric space (X,d). The definitions are exactly analogous to standard definitions for metric spaces, and the propositions are proved exactly analogously to the standard proofs of the corresponding standard propositions in the case of metric spaces, or by invoking those standard propositions for the particular metric space $\left(\widetilde{X}, \vec{d}\right)$.

Definition 2.1. *Let $B(x,r)$ denote the* ***ball of radius** r **centred at*** x, *i.e., for each $x \in X$ and $r > 0$, we set $B(x,r) = \{y \in X : d(x,y) \le r\}$.*

Definition 2.2. *The semimetric space (X,d) is said to be* ***totally bounded*** *if, for each $r > 0$, there exists a finite set $F_r \subset X$ such that $X = \bigcup_{x \in F_r} B(x,r)$.*

Definition 2.3. *The semimetric space (X,d) is said to be* ***separable*** *if there exists a countable set $Y \subset X$ such that $\inf_{y \in Y} d(x,y) = 0$ for each $x \in X$.*

Proposition 2.4. *If (X,d) is totally bounded, then it is separable.*

Proposition 2.5. *(X,d) is not totally bounded if and only if for some $r > 0$ there exists an infinite set $E \subset X$ such that $d(x,y) > r$ for all $x, y \in E$ with $x \neq y$.*

Proposition 2.6. *(X,d) is totally bounded if and only if every sequence $\{x_n\}_{n \in \mathbb{N}}$ in X has a* ***Cauchy subsequence***, *i.e., a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ which satisfies $\lim_{N \to \infty} \sup \left\{ d(x_{n_p}, x_{n_q}) : p, q > N \right\} = 0$.*

The following theorem obviously contains the classical theorem of Schauder, and it is a simple exercise to show that it also contains the classical theorem of Arzelà-Ascoli. After obtaining it we learned that, even though it generalizes these two very important theorems, it is itself merely a special case, a "lite" version, of considerably more abstract results presented by Bartle in [1] (cf. also e.g., [19]) and which, as explained in [1], have their roots in earlier work, mainly of R. S. Phillips [20], Šmulian [23], and Kakutani [14]. However, it seems easier to give a direct proof of this theorem than to deduce it from [1]. Furthermore, we learned that essentially the same theorem had also been obtained independently, apparently slightly before us, by Eliahu Levy. His proof in [16] is perhaps better than the one to be given here, and Dr. Levy and I have since refined it to obtain a quantitative result [10].

Theorem 2.7. *Let A and B be two sets and let $h : A \times B \to \mathbb{C}$ be a function with the properties that*

$$\sup_{a\in A} |h(a,b)| < \infty \text{ for each fixed } b \in B \tag{1}$$

and

$$\sup_{b\in B} |h(a,b)| < \infty \text{ for each fixed } a \in A. \tag{2}$$

Define $d_A(a_1,a_2) := \sup_{b\in B} |h(a_1,b) - h(a_2,b)|$ for each pair of elements a_1 and a_2 in A. Define $d_B(b_1,b_2) = \sup_{a\in A} |h(a,b_1) - h(a,b_2)|$ for each pair of elements b_1 and b_2 in B. Then (A,d_A) and (B,d_B) are semimetric spaces and

$$(A,d_A) \text{ is totally bounded if and only if } (B,d_B) \text{ is totally bounded.} \tag{3}$$

Proof. It is obvious that (A,d_A) and (B,d_B) are semimetric spaces. For the proof of (3), because of the symmetrical roles of A and B, we only have to prove one of the two implications. Suppose then that (A,d_A) is totally bounded. By Proposition 2.4, there exists a countable subset Y of A which is dense in A. Let us show that

$$d_B(b_1,b_2) = \sup_{y\in Y} |h(y,b_1) - h(y,b_2)| \text{ for all } b_1,b_2 \in B. \tag{4}$$

The inequality "$\geq$" in (4) is obvious. For the reverse inequality, given any b_1 and b_2 in B and any arbitrarily small positive ε, we choose $a \in A$ such that

$$d_B(b_1,b_2) \leq |h(a,b_1) - h(a,b_2)| + \varepsilon/3. \tag{5}$$

Then we choose $z \in Y$ such that

$$d_A(z,a) < \varepsilon/3. \tag{6}$$

We have that $|h(a,b_1) - h(a,b_2)|$ is bounded above by

$$\begin{aligned} & |h(a,b_1) - h(z,b_1)| + |h(z,b_1) - h(z,b_2)| + |h(z,b_2) - h(a,b_2)| \\ \leq \; & 2d_A(a,z) + \sup_{y\in Y} |h(y,b_1) - h(y,b_2)|. \end{aligned}$$

This, combined with (5) and (6), completes the proof of (4).

We shall now assume that (B,d_B) is not totally bounded and show that this leads to a contradiction. By this assumption and by Proposition 2.5, there exists some positive number r and some infinite sequence $\{b_n\}_{n\in\mathbb{N}}$ of elements of B such that

$$d_B(b_m,b_n) > r \text{ for each } m,n \in \mathbb{N} \text{ with } m \neq n. \tag{7}$$

For each fixed $y \in Y$ it follows from (2) that the numerical sequence $\{h(y,b_n)\}_{n\in\mathbb{N}}$ is bounded and thus has a convergent subsequence. Since Y is countable we can apply a standard Cantor "diagonalization" argument to obtain a subsequence $\{b_{\gamma_n}\}_{n\in\mathbb{N}}$ of $\{b_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} h(y,b_{\gamma_n})$ exists for each $y \in Y$. Therefore, after simply changing our notation, we can assume the existence of an infinite sequence $\{b_n\}_{n\in\mathbb{N}}$ in B which satisfies (7) and also

$$\lim_{n\to\infty} h(y,b_n) \text{ exists and is finite for each } y \in Y. \tag{8}$$

In view of (4) and (7), for each pair of integers m and n with $0 < m < n$ there exists an element $y_{m,n} \in Y$ such that $|h(y_{m,n},b_m) - h(y_{m,n},b_n)| > r$, and so, in particular,

$$|h(y_{m,m+1},b_m) - h(y_{m,m+1},b_{m+1})| > r \text{ for all } m \in \mathbb{N}. \tag{9}$$

Our assumption that (A,d_A) is totally bounded ensures, by Proposition 2.6, that there exists a strictly increasing sequence of positive integers $\{m_k\}_{k\in\mathbb{N}}$ such that $\{y_{m_k,m_k+1}\}_{k\in\mathbb{N}}$ is a Cauchy sequence in (A,d_A). Now we set $z_k = y_{m_k,m_k+1}$ for each k. We choose some sufficiently large integer N for which

$$d_A(z_N,z_k) < r/4 \text{ for all } k \geq N. \tag{10}$$

Now we combine (9) and (10) to obtain that, for each $k \geq N$,

$$\begin{aligned} r &< |h(z_k,b_{m_k}) - h(z_k,b_{m_k+1})| \\ &\leq |h(z_k,b_{m_k}) - h(z_N,b_{m_k})| + |h(z_N,b_{m_k}) - h(z_N,b_{m_k+1})| \\ &\quad + |h(z_N,b_{m_k+1}) - h(z_k,b_{m_k+1})| \\ &< \frac{r}{4} + |h(z_N,b_{m_k}) - h(z_N,b_{m_k+1})| + \frac{r}{4}. \end{aligned}$$

In view of (8), we obtain that $\lim_{k\to\infty} |h(z_N,b_{m_k}) - h(z_N,b_{m_k+1})| = 0$. So the inequalities on the preceding lines would imply that $r \leq r/2$. This contradiction shows that (B,d_B) must be totally bounded, and so completes the proof of the theorem. □

3. PRELIMINARIES ABOUT LATTICES AND LATTICE COUPLES

Let (Ω,Σ,μ) be a σ-finite measure space here and in the sequel. (Some of the assertions which we will be making here are simply false if (Ω,Σ,μ) is not σ-finite.)

Definition 3.1. *We say that a Banach space X is a **CBL**, or a **complexified Banach lattice of measurable functions on** Ω if*

(i) *all the elements of X are (equivalence classes of a.e. equal) measurable functions $f : \Omega \to \mathbb{C}$ and*

(ii) *for any measurable functions $f : \Omega \to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$, if $f \in X$ and $|g| \leq |f|$ a.e., then $g \in X$ and $\|g\|_X \leq \|f\|_X$.*

We will now recall a number of definitions and basic facts about CBLs. In several cases the relevant proofs of these facts in the literature to which we refer are given for Banach lattices of *real* valued functions. But in all those cases it is an obvious and easy exercise to adapt those proofs to our case here.

Any two CBLs X_0 and X_1 on the same underlying measure space always form a Banach couple. See e.g., [3, p. 122 and p. 161], [15, Corollary 1, p. 42], or [12, Remark 1.41, pp. 34–35]. (As explicitly stated and shown in [12] this is also true for non σ-finite measure spaces.)

For each CBL X on (Ω,Σ,μ), there exists a measurable subset Ω_X of Ω, which may be called the ***support*** of X, such that, for every function $g \in X$, we have $g(\omega) = 0$ for a.e. $\omega \in \Omega\backslash\Omega_X$. Furthermore, there exists a function $f_X \in X$ such that $f_X(\omega) > 0$ for a.e. $\omega \in \Omega_X$. (Cf. e.g., Remarks 1.3 and 1.4 on p. 14 of [12].) Obviously the set Ω_X is unique to within a set of measure zero. (Of course, on the other hand, the function f_X certainly is *not* unique.) If $\Omega_X = \Omega$ (at least to within a set of measure zero) then we say that X is ***saturated***.

The set Ω_X has an additional useful property: There exists a sequence of sets $\{E_n\}_{n\in\mathbb{N}}$ in Σ such that

$$\Omega_X = \bigcup_{n\in\mathbb{N}} E_n \text{ with } E_n \subset E_{n+1}, \mu(E_n) < \infty, \text{ and } \chi_{E_n} \in X \text{ for each } n \in \mathbb{N}. \tag{11}$$

The actual construction of Ω_X and of the sequence $\{E_n\}_{n\in\mathbb{N}}$ can be performed by an "exhaustion" process described in the proof of Theorem 3 on pp. 455–456 of [24] and also described (perhaps slightly more explicitly for our purposes here) in the first part of the proof of Proposition 4.1 on p. 58 of [11]. (Note however that there is a small misprint in [11], the omission of "$\mu(E)$", in the third line of this latter proof, i.e., the numbers α_k must of course be defined by $\alpha_k = \sup\{\mu(E) : E \in \Sigma, E \subset F_k, \chi_E \in X\}$.) For one possible (very easy and of course not unique) way to construct a function $f_X \in X$ with the above-mentioned property see, e.g., [12, p. 14 Remark 1.4].

Lemma 3.2. *If X_0 and X_1 are both saturated CBLs on the same measure space (Ω,Σ,μ), then $X_0 \cap X_1$ is saturated, and $[X_0,X_1]_\theta$ is saturated for each $\theta \in (0,1)$.*

Proof. The function $\min\{f_{X_0}, f_{X_1}\}$ is in $X_0 \cap X_1$ and therefore it is also in $[X_0,X_1]_\theta$. It is strictly positive a.e. on Ω. So neither of the sets $\Omega\backslash\Omega_{X_0\cap X_1}$ and $\Omega\backslash\Omega_{[X_0,X_1]_\theta}$ can have positive measure. □

Given an arbitrary CBL X on (Ω,Σ,μ) we define the functional $\|\cdot\|_{X'}$ by

$$\|f\|_{X'} := \sup\left\{\left|\int_\Omega fgd\mu\right| : g \in X, \|g\|_X \le 1\right\} \tag{12}$$

for each measurable function $f : \Omega \to \mathbb{C}$.

Remark 3.3. Obviously we can replace $|\int_\Omega fgd\mu|$ by $\int_\Omega |fg|d\mu$ in the formula (12).

Let X' be the set of all measurable functions $f : \Omega \to \mathbb{C}$ for which $\|f\|_{X'} < \infty$. Clearly X' is a linear space and $\|\cdot\|_{X'}$ is a seminorm on X' satisfying

$$\left|\int_\Omega fgd\mu\right| \le \|f\|_{X'}\|g\|_X \text{ for all } f \in X' \text{ and all } g \in X. \tag{13}$$

The space X' is customarily referred to as the ***Köthe dual*** or the ***associate space*** of X.

If $\mu(\Omega_X) > 0$, then, via a series of theorems, including one (Theorem 1, p. 470 in [24]) which uses Hilbert space techniques, it can be shown that X' is non-trivial, i.e., it contains elements which do not vanish a.e. on Ω_X. If, furthermore, X is saturated, then $\|\cdot\|_{X'}$ is a norm with respect to which X' is a saturated CBL on (Ω,Σ,μ). (See e.g., [24, p. 472, Theorem 4].)

Of course X' can be identified with a subspace of X^*, the dual space of X, and in some, but not all, cases it is also a ***norming subspace*** of X^*, i.e., it satisfies

$$\|g\|_X = \sup\left\{\left|\int_\Omega fgd\mu\right| : f \in X', \|f\|_{X'} \le 1\right\} \text{ for each } g \in X. \tag{14}$$

A result of Lorentz and Luxemburg, which appears as Proposition 1.b.18 on p. 29 of [17] (stated and proved there only for Köthe function spaces), gives necessary and sufficient conditions on X for (14) to hold. In particular the σ-order continuity of X is a sufficient condition. So is the Fatou property. (To extend the proof of Proposition 1.b.18 to general CBLs it is necessary to make a small modification on lines 6 and 7 on p. 30 using Remark 1.3 on p. 14 of [12].)

The associate space $(X')'$ of X', i.e. the *second associate* of X, is usually denoted by X''. Obviously $X \subset X''$ and $\|x\|_{X''} \le \|x\|_X$ for each $x \in X$. Obviously X'' is a CBL whenever X (and therefore also X') is saturated.

As in e.g., [15], we say that the CBL X has ***absolutely continuous norm*** if $\lim_{n\to\infty}\|f\chi_{E_n}\|_X = 0$ for every $f \in X$ and every sequence $\{E_n\}_{n\in\mathbb{N}}$ of measurable sets satisfying $E_{n+1} \subset E_n$ for all n and $\bigcap_{n\in\mathbb{N}} E_n = \emptyset$.

As in e.g., [17], we say that the CBL X is ***σ-order continuous*** if $\lim_{n\to\infty}\|f_n\|_X = 0$ for every sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions in X satisfying $0 \le f_{n+1} \le f_n$ and $\lim_{n\to\infty} f_n = 0$ a.e. It is easy to see that these two properties of X are in fact equivalent.

A CBL X is said to have the ***Fatou property*** if whenever $\{f_n\}_{n\in\mathbb{N}}$ is a norm bounded a.e. monotonically non-decreasing sequence of nonnegative functions in X, its a.e. pointwise limit f is also in X with $\|f\|_X = \lim_{n\to\infty}\|f_n\|_X$. If X is saturated, then X has the Fatou property if and only if $X = X''$ isometrically. (See [24, p. 472]. Cf. also [17, p. 30], but recall that there extra hypotheses are imposed.)

We remark that obvious counterexamples (see e.g., [12, Remark 7.3 p. 92]) show that the above claims about X' and X'' are false for certain non σ-finite measure spaces.

Given a pair of CBLs X_0 and X_1 on (Ω,Σ,μ) and a number $\theta \in (0,1)$, we define the space $X_0^{1-\theta}X_1^{\theta}$, analogously to the definition in [3, Section 13.5 p. 123], to be the set of all measurable functions $f : \Omega \to \mathbb{C}$ of the form

$$f = u f_0^{1-\theta} f_1^{\theta}, \tag{15}$$

where $u \in L^\infty(\mu)$ and f_j is a nonnegative function in $\mathscr{B}_{X_j}$ for $j = 0,1$. For each $f \in X_0^{1-\theta}X_1^{\theta}$ we define $\|f\|_{X_0^{1-\theta}X_1^{\theta}} = \inf\|u\|_{L^\infty(\mu)}$, where the infimum is taken over all representations of f of the form (15) with the stated properties. It can be shown that this is in fact a norm on $X_0^{1-\theta}X_1^{\theta}$, with respect to which $X_0^{1-\theta}X_1^{\theta}$ is a CBL. This is proved in Section 33.5 on pp. 164–165 of [3].

The norm 1 inclusions

$$[X_0,X_1]_\theta \overset{1}{\subset} X_0^{1-\theta}X_1^{\theta} \overset{1}{\subset} [X_0,X_1]^\theta \tag{16}$$

are special cases (set $B_0 = B_1 = \mathbb{C}$) of the results (i) and (ii) of Section 13.6 on p. 125 of [3] (proved in [3, Section 33.6 on pp. 171–180]). Furthermore, with the help of Bergh's theorem [2], (16) can be strengthened to tell us that

$$\|x\|_{[X_0,X_1]_\theta} = \|x\|_{X_0^{1-\theta}X_1^{\theta}} = \|x\|_{[X_0,X_1]^\theta} \text{ for all } x \in [X_0,X_1]_\theta. \tag{17}$$

We will need to use the formula

$$\left(X_0^{1-\theta}X_1^{\theta}\right)' = (X_0')^{1-\theta}(X_1')^{\theta}, \tag{18}$$

which holds with equality of norms (or seminorms when Ω_{X_0} or Ω_{X_1} is strictly smaller than Ω) for all pairs of CBLs X_0 and X_1 on (Ω,Σ,μ). This formula was originally stated and proved by Lozanovskii [18] under certain hypotheses, then by Reisner [22] under other hypotheses. The general version stated here is proved in [12, Section 7, pp. 91–97] using Reisner's proof and a remark of N. J. Kalton [pers. comm.].

4. THE MAIN RESULT

Our main result is a corollary of the following theorem.

Theorem 4.1. *Let $\vec{G} = (G_0,G_1)$ be an arbitrary Banach couple and let $\vec{X} = (X_0,X_1)$ be an arbitrary couple of saturated CBLs on an arbitrary σ-finite measure space (Ω,Σ,μ). Then every linear operator T which satisfies $T : \vec{G} \overset{c,b}{\to} \vec{X}$ has the compactness property*

$$T : [G_0,G_1]_\theta \overset{c}{\to} [X_0'',X_1'']_\theta$$

for each $\theta \in (0,1)$.

Corollary 4.2. *Let $\vec{X} = (X_0,X_1)$ be an arbitrary couple of saturated CBLs on a σ-finite measure space. Suppose that either*

(i) *X_0 and X_1 both have the Fatou property*, *or*

(ii) *at least one of the spaces X_0 and X_1 is σ-order continuous.*

Then

$$(*.*) \blacktriangleright \vec{X}.$$

Remark 4.3. The requirement that X_0 and X_1 are both saturated is merely a technical convenience which makes the formulation and proof of Theorem 4.1 simpler and shorter. In fact, it is entirely unnecessary for Corollary 4.2. The easy and rather obvious extension of the proof of Corollary 4.2 to the nonsaturated case uses the easily checked fact that $\Omega_{X_0\cap X_1} = \Omega_{[X_0,X_1]_\theta}$ and replaces the spaces X_0 and X_1 in an appropriate way by their "restrictions" to the smaller measure space $\Omega_{X_0\cap X_1}$.

Via an examination of the proofs of Theorem 4.1 and Corollary 4.2 it is clear that other conditions on the couple $\vec{X}$, weaker than those stated in Corollary 4.2, are also sufficient to ensure that $(*,*) \blacktriangleright \vec{X}$.

Proof of Theorem 4.1. Since $[G_0^\circ, G_1^\circ]_\theta = [G_0, G_1]_\theta$ [3, Sections 9.3 (p. 116) and 29.3 (pp. 113–114)] we can clearly suppose without loss of generality that $\vec{G}$ is a regular couple. Let $\langle \cdot, \cdot \rangle$ denote the duality between $G_0 \cap G_1$ and $(G_0 \cap G_1)^*$. Let G be any one of the spaces G_0, G_1 or $[G_0, G_1]_\theta$ and define $G^\#$ to be the subspace of elements $\gamma \in (G_0 \cap G_1)^*$ for which the norm $\|\gamma\|_{G^\#} := \sup\{|\langle g, \gamma\rangle| : g \in \mathscr{B}_G \cap G_0 \cap G_1\}$ is finite. Of course $G^\#$, when equipped with this norm, is a Banach space which is continuously embedded in $(G_0 \cap G_1)^*$. So $(G_0^\#, G_1^\#)$ is a Banach couple.

We could of course identify $G^\#$ with the dual of G, but it is more convenient to use the above definition. Note also that in fact $G_0^\# + G_1^\# \stackrel{1}{=} (G_0 \cap G_1)^*$. Calderón's remarkable duality theorem [3, Section 12.1 p. 121 and Section 32.1 pp. 148–156] can be expressed by the formula $([G_0, G_1]_\theta)^\# \stackrel{1}{=} [G_0^\#, G_1^\#]^\theta$. For a more detailed discussion of all these issues we refer to [6].

Let T be an arbitrary linear operator satisfying $T : \vec{G} \xrightarrow{c,b} \vec{X}$. We may suppose, without loss of generality, that $\|T\|_{\vec{G}\to\vec{X}} := \max_{j=0,1} \|T\|_{G_j\to X_j} = 1$. For $j = 0,1$, let X_j' be the associate space of X_j. For each $g \in G_0 \cap G_1$ and each $z \in X_0' + X_1'$ define $h(g,z) = \int_\Omega zTg\,d\mu$. Of course (cf. (13)) the function h satisfies

$$|h(g,z)| \le \|z\|_{X_j'} \|Tg\|_{X_j} \le \|z\|_{X_j'} \|g\|_{G_j} \tag{19}$$

for $j = 0,1$ and all $g \in G_0 \cap G_1$ and $z \in X_j'$. Therefore h also satisfies

$$|h(g,z)| \le \|z\|_{X_0'+X_1'} \|g\|_{G_0\cap G_1} \text{ for all } g \in G_0 \cap G_1 \text{ and } z \in X_0' + X_1'. \tag{20}$$

For each fixed $z \in X_0' + X_1'$ we define the linear functional Sz on $G_0 \cap G_1$ by $\langle g, Sz\rangle = h(g,z)$. Of course Sz depends linearly on z and it is clear from (20) that we have thus defined a bounded linear operator $S : X_0' + X_1' \to (G_0 \cap G_1)^*$. For $j = 0,1$, in view of (19), we see that, for each $z \in X_j'$, we have $Sz \in G_j^\#$ with $\|Sz\|_{G_j^\#} \le \|z\|_{X_j'}$, i.e., $S : X_j' \xrightarrow{b} G_j^\#$. (Note, cf. [6], that we do not have to consider the extension of Sz to a space larger than $G_0 \cap G_1$.)

We now wish to show that S satisfies the compactness condition

$$S : X_0' \xrightarrow{c} G_0^\#. \tag{21}$$

We will do this by applying Theorem 2.7. We consider the restriction of the function $h(g,y)$ to the set $A \times B$ where $A = \mathscr{B}_{G_0} \cap G_1$ and $B = \mathscr{B}_{X_0'}$. Given any sequence $\{g_n\}_{n\in\mathbb{N}}$ in A, we of course have (cf. (13) and (19)) that $d_A(g_m, g_n) = \sup\{|h(g_m,z) - h(g_n,z)| : z \in B\} \le \|Tg_m - Tg_n\|_{X_0}$. So the fact that $T : G_0 \xrightarrow{c} X_0$ implies that (A, d_A) is totally bounded. (Cf., e.g., Proposition 2.5 or Theorem 15 on p. 22 of [13].) Consequently, in view of Theorem 2.7 and Proposition 2.6, if $\{z_n\}_{n\in\mathbb{N}}$ is an arbitrary sequence in B, then it has a subsequence which is Cauchy with respect to the semimetric

$$\begin{aligned} d_B(y,z) &= \sup\{|h(g,y) - h(g,z)| : g \in A\} \\ &= \sup\left\{|\langle g, S(y-z)\rangle| : g \in G_0 \cap G_1, \|g\|_{G_0} \le 1\right\} \\ &= \|S(y-z)\|_{G_0^\#}. \end{aligned}$$

This is exactly the condition (21).

Since X_0' and X_1' are both CBLs of measurable functions on the measure space (Ω,Σ,μ), we can use (21) and $S : X_1' \xrightarrow{b} G_1^\#$ and apply part (c) of Corollary 7 on p. 270 of [8] to deduce that

$$S : [X_0',X_1']_\theta \xrightarrow{c} [G_0^\#,G_1^\#]_\theta. \tag{22}$$

We are now ready for a second application of Theorem 2.7. Once more we will use the same function h defined above and restricted to a set $A\times B$, where this time we choose $A=\mathscr{B}_{[G_0,G_1]_\theta}\cap G_0\cap G_1$ and $B=\mathscr{B}_{[X_0',X_1']_\theta}$. This time, for each $y,z\in B$, we of course have $S(y-z)\in[G_0^\#,G_1^\#]_\theta$. So, using the isometry $([G_0,G_1]_\theta)^\# \overset{1}{=} [G_0^\#,G_1^\#]^\theta$ mentioned above, and then Bergh's theorem [2], we obtain that

$$\begin{aligned} d_B(y,z) &= \sup\left\{|\langle g,S(y-z)\rangle| : g\in\mathscr{B}_{[G_0,G_1]_\theta}\cap G_0\cap G_1\right\} = \|S(y-z)\|_{([G_0,G_1]_\theta)^\#} \\ &= \|S(y-z)\|_{[G_0^\#,G_1^\#]^\theta} = \|S(y-z)\|_{[G_0^\#,G_1^\#]_\theta}. \end{aligned}$$

The compactness property (22) of S implies that (B,d_B) is totally bounded. Consequently, by Theorem 2.7, (A,d_A) is also totally bounded. In view of Proposition 2.6 and the fact that $G_0\cap G_1$ is dense in $[G_0,G_1]_\theta$ [3, Section 9.3 (p. 116) and Section 29.3 (pp. 113–114)], this means that the proof of Theorem 4.1 will be complete once we have shown that

$$d_A(g_1,g_2)=\|Tg_1-Tg_2\|_{[X_0'',X_1'']_\theta} \text{ for all } g_1,g_2\in A. \tag{23}$$

By definition, for each g_1 and g_2 in A we have

$$d_A(g_1,g_2)=\sup_{y\in\mathscr{B}_{[X_0',X_1']_\theta}}\left|\int_\Omega yT(g_1-g_2)d\mu\right|.$$

At this stage we need not consider the particular form of the element Tg_1-Tg_2. We know that it is an element of $X_0\cap X_1$. So, to obtain (23) it suffices to show that

$$\sup_{y\in\mathscr{B}_{[X_0',X_1']_\theta}}\left|\int_\Omega xyd\mu\right| = \|x\|_{[X_0'',X_1'']_\theta} \text{ for each } x\in X_0\cap X_1. \tag{24}$$

Since $X_0\cap X_1\subset X_0''\cap X_1''\subset[X_0'',X_1'']_\theta$, we have, from (17) applied to the couple (X_0'',X_1''), that the right side of (24) equals $\|x\|_{(X_0'')^{1-\theta}(X_1'')^\theta}$ and this in turn, in view of (18) applied to the couple (X_0',X_1'), equals $\|x\|_{\left((X_0')^{1-\theta}(X_1')^\theta\right)'}$. It follows that (24) is equivalent to the formula

$$\sup_{y\in\mathscr{B}_{[X_0',X_1']_\theta}}\left|\int_\Omega xyd\mu\right| = \sup_{y\in\mathscr{B}_{(X_0')^{1-\theta}(X_1')^\theta}}\left|\int_\Omega xyd\mu\right| \text{ for each } x\in X_0\cap X_1. \tag{25}$$

Applying (17) to the couple (X_0',X_1'), we of course obtain the inequality " $\leq$ " in (25). To show the reverse inequality " $\geq$ ", we fix some $x\in X_0\cap X_1$ and $y\in\mathscr{B}_{(X_0')^{1-\theta}(X_1')^\theta}$ and we shall construct a sequence $\{y_n\}_{n\in\mathbb{N}}$ in $\mathscr{B}_{[X_0',X_1']_\theta}$ for which

$$\lim_{n\to\infty}\int_\Omega xy_nd\mu=\int_\Omega xyd\mu. \tag{26}$$

By Lemma 3.2, since X_0' and X_1' are both saturated, so is $[X_0',X_1']_\theta$. Consequently (cf. (11)) there exists an expanding sequence $\{E_n\}_{n\in\mathbb{N}}$ of sets in Σ such that $\Omega=\bigcup_{n\in\mathbb{N}}E_n$ and $\chi_{E_n}\in[X_0',X_1']_\theta$ for each $n\in\mathbb{N}$. Let $y_n=y\chi_{E_n\cap\{\omega\in\Omega:|y(\omega)|\leq n\}}$. Then $y_n\in[X_0',X_1']_\theta$ and we have $|xy_n|\leq|xy|$ and $\lim_{n\to\infty}x(\omega)y_n(\omega)=x(\omega)y(\omega)$ for all $\omega\in\Omega$. The function xy is integrable, since $X_0\cap X_1\overset{1}{\subset}[X_0,X_1]_\theta\overset{1}{\subset}X_0^{1-\theta}X_1^\theta$ and $(X_0')^{1-\theta}(X_1')^\theta\overset{1}{=}\left(X_0^{1-\theta}X_1^\theta\right)'$

(cf. (17) and (18) and Remark 3.3). So (26) follows from the Lebesgue dominated convergence theorem. As already explained, this implies (25) and therefore also (24) and (23), and so completes the proof of the theorem. □

Proof of Corollary 4.2. This uses all steps of the preceding proof up to (23). Then it is required to establish a variant of (23) or of (24) where $\|\cdot\|_{[X_0'',X_1'']_\theta}$ is replaced by $\|\cdot\|_{[X_0,X_1]_\theta}$ or (cf. (17)) by $\|\cdot\|_{X_0^{1-\theta}X_1^\theta}$. If X_0 and X_1 both have the Fatou property then $X_0''=X_0$ and $X_1''=X_1$ and we are done. Otherwise, in view of (25) and (18), it will suffice if we show that $\sup\{|\int_\Omega xyd\mu| : y\in\mathscr{B}_{\left(X_0^{1-\theta}X_1^\theta\right)'}\} = \|x\|_{X_0^{1-\theta}X_1^\theta}$ for each $x\in X_0\cap X_1$.

This will hold whenever $\left(X_0^{1-\theta}X_1^\theta\right)'$ is a norming subspace of the dual of $X_0^{1-\theta}X_1^\theta$ (and possibly also under a weaker assumption than that, since we are only considering elements x in $X_0\cap X_1$). As already observed above (just after (14)), one sufficient condition for this to happen is when $X_0^{1-\theta}X_1^\theta$ is σ-order continuous. The σ-order continuity of $X_0^{1-\theta}X_1^\theta$ can be ensured by requiring that at least one of the spaces X_0 and X_1 is σ-order continuous (cf. Proposition 4 on p. 80 of [22] or Theorem 1.29 on p. 27 of [12]). □

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Võrepaaridesse kujutavate kompaktsete operaatorite kompleksne interpolatsioon

Michael Cwikel

Küsimus, kas Banachi (ruumide) paaride vahel tegutsev tõkestatud lineaarne operaator, mis tegutseb ühel (või isegi mõlemal) lähteruumil kompaktselt, tegutseb kompaktselt ka nende ruumide poolt genereeritud komplekssete interpolatsiooniruumide vahel, on püsinud lahtisena juba 44 aastat. Artiklis on vastatud sellele küsimusele jaatavalt juhul, kui operaatori sihtpaar on teatavaid loomulikke eeldusi rahuldav (ühel ja samal mõõduga ruumil tegutsevate) Banachi võrede paar.