PHYSICS

Thermoelasticity and entropy flow

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Abstract. The well-known theories of thermoelasticity are based on thermal expansion, and recently the theory of heat conduction has been looked into for improvements. The reason is that heat conduction is described with scalar and vector variables but elasticity with second order tensors, and in linear order there is no direct coupling between second order tensors and vectors or scalars if the material is isotropic. The observed deviations from the present theories urge that new tracks for research be sought. Such a new track can be opened by Onsager's thermodynamics supplemented with dynamic degrees of freedom. This theory is usually referred to as extended thermodynamics. The key moment is in the general form of the entropy current out of local equilibrium, which leads to the formal introduction of the transport of the dynamic degrees of freedom. The skeleton of the possible theories is based on the introduction of one or more vectorial dynamic variables. They can be coupled to the current density of the heat flow, while their 'diffusion' intensities are second order tensors coupled directly in linear order to the stress tensor even if the material is isotropic. The possibilities are demonstrated on an example with one dynamic degree of freedom.

Key words: thermodynamic modelling, Onsager–Casimir reciprocal relation, irreversible thermodynamics, thermoelasticity.

1. INTRODUCTION

Non-equilibrium thermodynamics has been applied with success to numerous phenomena even in the form that nowadays is referred to as 'classical irreversible thermodynamics'. This form was accepted just after the first papers of Onsager [1–4] and is presented – more or less strictly – in the classical monographs [5–8]. It concerned the principle of local equilibrium, which turned out to be too tight and was generalized [9–14]. The departure from local equilibrium proved to be very fruitful and opened a broad perspective of applications [15–25]. Here a model for thermal stress is sketched that may be the closest to the original Onsagerian presentation. In the model, thermal stress is invoked by heat conduction through linear Onsager equations.

2. THE MODEL

Assume a moving solid body in which heat conduction happens. The first law reads

$$\rho \dot{u} + \operatorname{div} \vec{J}_q = t : \mathring{d}. \tag{1}$$

Here ρ is the density, u the specific internal energy, $\vec{J}_q$ the current density of heat flow, t Cauchy's stress, and $\mathring{d}$ the symmetric part of the velocity gradient. The $\circ$ above the symbols means co-rotational derivative. Further on the study will be restricted to small deformations for the sake of simpler formulae; with this restriction, the co-rotational derivative of the dilatation tensor d equals the symmetric part of the velocity gradient.

The local state variables are the specific internal energy u, the dilatation tensor d, and a vectorial dynamic variable $\vec{\xi}$ not yet specified. As the variable $\vec{\xi}$ is determined by the other two in equilibrium they can be chosen in a special way so that the entropy function takes the form

$$s = s^e \left(u - \frac{1}{2}\vec{\xi}^{\,2}, d \right). \tag{2}$$

Here $s^e(u,d)$ is the equilibrium entropy function. The choice of the dynamic variables resulting in the above non-equilibrium entropy function is ensured by the Morse lemma [26] and the maximum property of the entropy.

The general form of the entropy balance reads

$$\rho \dot{s} + \operatorname{div} \vec{J}_s = \sigma_s \geq 0. \tag{3}$$

Here $\vec{J}_s$ is the entropy flow density and σ_s the entropy production. The inequality expresses the second law.

The system is not in local equilibrium. The classical formula of the entropy flow is replaced [27] by

$$\vec{J}_s = \frac{1}{T}\left(\vec{J}_q - J_\xi \vec{\xi}\right). \tag{4}$$

Here J_ξ is the current density of the transport of the dynamic variable.

The actual form of the entropy production or better of the energy dissipation function reads

$$T\sigma_s = -\frac{1}{T}\operatorname{grad} T \cdot \left(\vec{J}_q - J_\xi \vec{\xi}\right) - \vec{\xi} \cdot \vec{\sigma}_\xi$$
$$+ (t - t^e) : \mathring{d} - J_\xi : \operatorname{Grad} \vec{\xi}. \tag{5}$$

Here the abbreviating notations

$$t^e = -T\rho\frac{\partial s}{\partial d}$$
$$\vec{\sigma}_\xi = \rho\,\mathring{\vec{\xi}} + Div J_\xi$$

have been introduced. The function t^e is the equilibrium stress belonging to the present specific entropy and deformation, and $\vec{\sigma}_\xi$ is the source density of quantity $\vec{\xi}$.

The four terms in the above formula describe the entropy generation of four processes. The first term belongs to heat conduction, the second to the relaxation of the dynamic variable, the third to the viscous effects, and the fourth to the transport of the dynamic degree of freedom.

The conditions of an equilibrium can be taken from here. According to the first term the temperature has to be uniform, according to the second the dynamic structure has to be relaxed. The third term gives the expression for the stress at an equilibrium. Introducing the notation makes its meaning clear. The last term says that no gradient of $\vec{\xi}$ can be present at an equilibrium.

3. CONSTITUTIVE EQUATIONS

The first two terms belong to vectorial phenomena and the last two to tensorial ones. The Onsager equations fall into two uncoupled sets of equations. One for the vectorial phenomena

$$\left(\vec{J}_q - J_\xi \vec{\xi}\right) = -L_{qq}\frac{1}{T}\operatorname{grad} T - L_{q\xi}\vec{\xi} \tag{6}$$

$$\vec{\sigma}_\xi = -L_{\xi q}\frac{1}{T}\operatorname{grad} T - L_{\xi\xi}\vec{\xi} \tag{7}$$

and one for the tensorial quantities

$$(t - t^e) = L_{dd}\mathring{d} - L_{d\xi}\operatorname{Grad}\vec{\xi}$$
$$J_\xi = L_{\xi d}\mathring{d} - L_{\xi\xi}\operatorname{Grad}\vec{\xi}. \tag{8}$$

Here the quantities $L_{...}$ are isotropic fourth order tensors, the general form of which [28] makes it possible to split the set of equations (8) into three uncoupled sets, one for the traces

$$(p^e - p) = L^0_{dd}\operatorname{div}\vec{v} - L^0_{d\xi}\operatorname{div}\vec{\xi}$$
$$\operatorname{tr} J_\xi = L^0_{\xi d}\operatorname{div}\vec{v} - L^0_{\xi\xi}\operatorname{div}\vec{\xi}, \tag{9}$$

one for the deviatoric parts

$$\left(t^D - t^{eD}\right) = L^D_{dd}\mathring{d}^D - L^D_{d\xi}\left(\operatorname{Grad}\vec{\xi}\right)^D$$
$$J^D_\xi = L^D_{\xi d}\mathring{d}^D - L^D_{\xi\xi}\left(\operatorname{Grad}\vec{\xi}\right)^D, \tag{10}$$

and one for the skew-symmetric parts

$$0 = L^A_{d\xi}\left(\operatorname{Grad}\vec{\xi}\right)^A \tag{11}$$

$$(J_\xi)^A = -L^A_{\xi\xi}\left(\operatorname{Grad}\vec{\xi}\right)^A. \tag{12}$$

Here the $L_{..}$ coefficients are numbers. The superscripts ^{0}, ^{D}, and ^{A} refer to the scalar, the deviatoric, and the skew-symmetric parts, respectively. The coefficient $L^A_{d\xi}$ equals obviously zero as otherwise the equations would be incompatible.

4. DISCUSSION

Nevertheless, to study the detailed mathematics and the consequences of the equations if far beyond the authenticity and the possibilities of a thermodynamicist, I intend to manage with a probable oversimplified case when $\vec{\xi}$ can be neglected in the equation of heat conduction (6), the transport of $\vec{\xi}$ in (7), and the viscous terms in equations (8)–(10), moreover, the relaxation is fast. With these simplifications, the J_ξ loses importance and the equations for heat conduction are not influenced by the dynamic variable even if a theory better than Fourier's is assumed. The $\vec{\xi}$ is proportional to the temperature gradient and the stress tensor is supplemented with a part linearly depending on the second derivatives of the temperature field. Equation (8)$_1$ turns into

$$(t - t^e) = L_{d\xi}\frac{L_{\xi q}}{TL_{\xi\xi}}\operatorname{Grad}\operatorname{grad} T, \tag{13}$$

which may explain the experience that a coat of mortar is hindering the disintegration of the wall in some tunnels.

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Termoelastsus ja entroopiavoog

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Üldtuntud termoelastsusteooriad baseeruvad termilise paisumise nähtusel. Hiljutised täiustused püüavad arvestada soojusjuhtivusteooria elemente. Põhjuseks on see, et soojusjuhtivuse kirjeldamisel käsitletakse skalaarseid ja vektoriaalseid muutujaid, samas kui elastsusteooria vaatleb teist järku tensoreid ning lineaarses seades puudub isotroopse materjali puhul otsene seos teist järku tensorite ja vektorite või skalaarite vahel. Kirjeldatud kõrvalekalded olemasolevatest teooriatest suunavad teadlasi uutele radadele. Sellise uue raja võib avada Onsageri termodünaamika, milles on lisana arvestatud dünaamilisi vabadusastmeid. Seda teooriat refereeritakse kui laiendatud termodünaamikat. Otsustav on siin lokaalsest tasakaalust väljuva entroopiavoo üldine kirjeldus, mis formaalselt arvestab dünaamiliste vabadusastmete transporti. Võimalike teooriate raamistik baseerub ühe või mitme vektoriaalse dünaamilise muutuja sissetoomisel. Need võivad olla seotud soojusvoo hetketihedusega, kuna nende "difusiooni" intensiivsused on teist järku tensorid, mis on lineaarsel juhul otseselt seotud pingetensoriga isegi siis, kui materjal on isotroopne. Teooria võimalusi on demonstreeritud näitega, kus on arvestatud ühe dünaamilise vabadusastmega.