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MATHEMATICS

# The different tongues of $q$-calculus

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**Abstract.** In this review paper we summarize the various dialects of $q$-calculus: quantum calculus, time scales, and partitions. The close connection between $\Gamma_q(x)$ functions on the one hand, and elliptic functions and theta functions on the other hand will be shown. The advantages of the Heine notation will be illustrated by the ($q$-)Euler reflection formula, $q$-Appell functions, Carlitz–AlSalam polynomials, and the so-called $q$-addition. We conclude with some short biographies about famous scientists in $q$-calculus.

**Key words:** elliptic functions, theta functions, $q$-Appell functions, $q$-addition, Carlitz–AlSalam polynomial.

## 1. INTRODUCTION

$q$-Calculus is a generalization of many subjects, like hypergeometric series, complex analysis, and particle physics. In short, $q$-calculus is quite a popular subject today. It has developed various dialects like quantum calculus, time scales, partitions, and continued fractions. We will give several aspects of $q$-calculus to enlighten the strong interdisciplinary as well as mathematical character of this subject. The close connection between $q$-calculus on the one hand, and elliptic functions and theta functions on the other hand will be shown. Some new formulas will be given, such as the two restatements of Kummer's ${}_2F_1(-1)$ formula (in $q$-form) (36) and (37).

Several combinatorial formulas, equivalent to expansion formulas for $q$-Appell functions, will be given for the first time. These formulas are not, as usually thought, equivalent to the corresponding Vandermonde formula, etc.

Several examples will be given of $q$-formulas not corresponding to known formulas for $q = 1$ (38), (39), or which are not valid for $q = 1$ (72)–(76), (83). The definitions of $q$-integral (17), (18) also do not seem to be valid for $q = 1$, however a limit process shows that the $q$-integral of a power function gives the same value as for the case $q = 1$. There is also the possibility that the corresponding formulas for $q = 1$ are very simple and well known (78).

Before continuing we will introduce the notation of the author (due to Heine [36]) to be able to give some examples. In an appendix we briefly describe another notation, the so-called Watson [58] notation.

The notation of the author was standard notation in $q$-calculus 1846–1911 [52]. This method is a mixture of Heine [36] and Gasper–Rahman [32]. The advantages of this method have been summarized in [25, p. 495].

Also Jackson [41–43] used a variant of this notation. His biography is found in the appendix. The two notations (Heine and Watson) are equivalent. There is also a third notation, due to Cigler [20], which uses mainly $q$-binomial coefficients. Cigler's notation is equivalent to the Heine and the Watson notation.

In the entire paper, the symbol $\equiv$ will denote definitions, except when we work with congruences.

We now start with the mathematics of the Heine notation. Everywhere $q$ will denote a complex number $|q| < 1$, except for certain cases, when explicitly stated, $q$ will be real and $0 < q < 1$.

**Definition 1.1.** *The power function is defined by* $q^a \equiv e^{a\log(q)}$. *We always use the principal branch of the logarithm.*

*The q-analogues of a complex number a and of the factorial function are defined by*

$$\{a\}_q \equiv \frac{1-q^a}{1-q},\ q \in \mathbb{C}\setminus\{1\}, \tag{1}$$

$$\{n\}_q! \equiv \prod_{k=1}^{n}\{k\}_q,\ \{0\}_q! = 1,\ q \in \mathbb{C}. \tag{2}$$

*The q-shifted factorial is given by*

$$\langle a;q\rangle_n \equiv \begin{cases} 1, & n = 0; \\ \displaystyle\prod_{m=0}^{n-1}(1-q^{a+m}) & n = 1,2,\ldots. \end{cases} \tag{3}$$

*Furthermore,*

$$\langle a;q\rangle_\infty \equiv \prod_{m=0}^{\infty}(1-q^{a+m}),\ 0 < |q| < 1. \tag{4}$$

*Let the Gauss q-binomial coefficient be defined by*

$$\binom{n}{k}_q \equiv \frac{\langle 1;q\rangle_n}{\langle 1;q\rangle_k\langle 1;q\rangle_{n-k}},\ k = 0,1,\ldots,n. \tag{5}$$

**Definition 1.2.** *If* $|q| > 1$, *or*

$$0 < |q| < 1 \ \text{ and } \ |z| < |1-q|^{-1},$$

*the first q-exponential function* $\mathrm{E}_q(z)$ *is given by*

$$\mathrm{E}_q(z) \equiv \sum_{k=0}^{\infty}\frac{1}{\{k\}_q!}z^k. \tag{6}$$

*There is another q-exponential function which is entire when* $0 < |q| < 1$. *To obtain it, we must invert the base in* (6), *i.e.* $q \to \frac{1}{q}$ :

$$\mathrm{E}_{\frac{1}{q}}(z) \equiv \sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}}{\{k\}_q!}z^k. \tag{7}$$

**Definition 1.3.** *The q-gamma function is given by*

$$\Gamma_q(z) \equiv \frac{\langle 1;q\rangle_\infty}{\langle z;q\rangle_\infty}(1-q)^{1-z},\ 0 < |q| < 1. \tag{8}$$

Here we deviate from the usual convention $q < 1$, because we want to work with meromorphic functions of several variables. The $q$-gamma function has simple poles located at $x = -n \pm \frac{2k\pi i}{\log q}$, $n,k \in \mathbb{N}$. Except for this the $q$-gamma function and the gamma function have very similar behaviour.

**Definition 1.4.** *In the following, $\mathbb{C}/\mathbb{Z}$ will denote the space of complex numbers* $\mathrm{mod}\frac{2\pi i}{\log q}$. *This is isomorphic to the cylinder $\mathbb{R}\times e^{2\pi i\theta}$, $\theta\in\mathbb{R}$. The operator*

$$\widetilde{}: \frac{\mathbb{C}}{\mathbb{Z}} \mapsto \frac{\mathbb{C}}{\mathbb{Z}}$$

*is defined by*

$$a \mapsto a+\frac{\pi i}{\log q}. \tag{9}$$

*Furthermore, we define*

$$\widetilde{\langle a;q\rangle_n} \equiv \langle \widetilde{a};q\rangle_n. \tag{10}$$

By (9) it follows that

$$\widetilde{\langle a;q\rangle_n} = \prod_{m=0}^{n-1}(1+q^{a+m}), \tag{11}$$

where this time the tilde denotes an involution which changes a minus sign to a plus sign in all the $n$ factors of $\langle a;q\rangle_n$.

**Definition 1.5.** *Generalizing Heine's ${}_2\phi_1$ series* [39], *we shall define a q-hypergeometric series by* (*compare* [32, p. 4])

$$\begin{aligned} {}_p\phi_r(\hat{a}_1,\ldots,\hat{a}_p;\hat{b}_1,\ldots,\hat{b}_r|q,z) &\equiv {}_p\phi_r\left[\begin{array}{c}\hat{a}_1,\ldots,\hat{a}_p\\ \hat{b}_1,\ldots,\hat{b}_r\end{array}|q,z\right] \\ &\equiv \sum_{n=0}^{\infty}\frac{\langle \hat{a}_1,\ldots,\hat{a}_p;q\rangle_n}{\langle 1,\hat{b}_1,\ldots,\hat{b}_r;q\rangle_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+r-p} z^n, \end{aligned} \tag{12}$$

*where $q\neq 0$ if $p>r+1$, and*

$$\widehat{a} \equiv a \vee \tilde{a}. \tag{13}$$

*The following notation will be convenient*:

$$\mathrm{QE}(x) \equiv q^x. \tag{14}$$

**Definition 1.6.** *Let a and b be any elements with commutative multiplication. Then the Nalli–Ward–AlSalam* (*NWA*) *q-addition is given by*

$$(a\oplus_q b)^n \equiv \sum_{k=0}^{n}\binom{n}{k}_q a^k b^{n-k},\ n=0,1,2,\ldots. \tag{15}$$

*Furthermore, we put*

$$(a\ominus_q b)^n \equiv \sum_{k=0}^{n}\binom{n}{k}_q a^k(-b)^{n-k},\ n=0,1,2,\ldots. \tag{16}$$

**Theorem 1.1.** *The NWA q-addition forms a commutative monoid, i.e. a monoid with commutative q-addition.*

We will come back to this monoid theme later.

**Definition 1.7.** *In 1910 Jackson redefined the general q-integral for a bounded interval* [32,41]

$$\int_0^a f(t,q)\,\mathrm{d}_q(t) \equiv a(1-q)\sum_{n=0}^{\infty} f(aq^n,q)q^n,\ 0<q<1,\ a\in\mathbb{R}. \tag{17}$$

*Following Jackson, we will put*

$$\int_0^{\infty} f(t,q)\,\mathrm{d}_q(t) \equiv (1-q)\sum_{n=-\infty}^{\infty} f(q^n,q)q^n,\ 0<|q|<1, \tag{18}$$

*provided the sum converges absolutely.*

We will now start an exposition of different dialects of $q$-calculus. Quantum calculus is more or less equal to $q$-calculus and started with the book [46]. Here the basic formulas of $q$-calculus are given in a different notation. It was followed by [24], where different $q$-analogues of the Euler integral formula

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\,\mathrm{d}t \tag{19}$$

were discussed.

When we are looking for a $q$-analogue of such an integral, we have a multiple choice. We can use different $q$-exponential functions. We can use the $q$-integral (18) with upper integration limit $+\infty$, or we can use the finite $q$-integral (17) with the upper integration limit $\frac{1}{1-q}$.

Before embarking on this $q$-analogue we are going to mention another formula.

The bilateral summation formula

$${}_1\Psi_1(a;b|q,q^z) = \frac{\langle b-a,a+z,1,1-a-z,q\rangle_\infty}{\langle b,b-a-z,z,1-a,q\rangle_\infty}, \tag{20}$$

an extension of the $q$-binomial theorem, was first stated by S. Ramanujan (see [35]). A proof of (20) was given by Andrews and Askey in 1978 [9] (see the book by Gasper and Rahman [32, pp. 126–127]).

In [24] the formula

$$\Gamma_q(x) = \int_0^{\frac{1}{1-q}} t^{x-1}\mathrm{E}_{\frac{1}{q}}(-qt)\,\mathrm{d}_q(t) \tag{21}$$

was given. This $q$-integral formula is merely a heavily disguised version of (20). The formula (21) was first proved by Nalli [48, p. 337] in a slightly different form. She simply showed that this $q$-integral satisfies the functional equation of the $\Gamma_q$-function. Nalli also introduced a $q$-Riemann zeta function and the so-called $q$-addition. This addition is imperative for obtaining analogues of formulas for hypergeometric functions of one and many variables with function argument $x+y$. These addition formulas are proved in an operational way, so no certain convergence region can be given; in general small function arguments work best. In the footsteps of Jackson, Hahn [33, p. 10] has presented another $q$-analogue of (19) with the upper integration limit $\infty$. Hahn's proof is not as clear as the proof of Nalli. Hahn also does not define the $\Gamma_q$-function explicitly. It looks that Hahn still thought of the Heine $\Omega$-function, which we will define here. Heine, Ashton [11], and Daum [22] used another function without the factor $(1-q)^{1-x}$, which they called the Heine $\Omega$-function. The main difference between the two functions is that $\Omega$ has zeros, in contrast to the $\Gamma_q$ function which has no zeros, and therefore $1/\Gamma_q$ is entire. In his thesis [11], supervised by Lindemann, Ashton showed the connection of the Heine function to elliptic functions. However, Hahn has contributed greatly to the development of $q$-calculus; his reasoning is often on a very high level.

Time scales were introduced by Hilger in his thesis of 1988 [40]. Hilger says that one can generalize differential and integral calculus (for functions of one variable) by replacing the range of definition of the functions under consideration by an arbitrary measure chain (or time scale).

There exists a so-called integral for time scales [1] with inverse operator. However, this integral is a special case of the general $q$-integral. Regular conferences on time scales are held in the US, sometimes also as special sessions of the American Mathematical Society meetings.

Partition theory or additive number theory is a well-established subject with connections to many areas. This is outlined in [47]. The Young frames equivalent to partitions give the representations of the symmetric group. The representations of Lie groups are then obtained as tensor products of these representations. The characters of these representations can be computed by the Weyl determinant formula. Quantum groups started in Santilli's thesis of 1967 [50] as a development from the Lie admissible deformations. Many physical objects have been $q$-deformed, however, a unified connection to the $q$-special functions which form the representations of these quantum groups is still lacking. There is a parallel theory, introduced by Woronowicz [59] in 1987. Woronowicz introduced a compact matrix pseudogroup $(A,\mu)$, where $A$ is a $C^\star$ algebra with unity, and $A$ is an $N\times N$ matrix with entries belonging to $A$. Several interesting formulas have been obtained by this method, however, these results are always in the same spirit as quantum groups.

The mathematical literature is currently flooded with articles about $q$-deformations. Just as an example we mention the highly interesting paper [34], where representations of the $q$-rook monoid are given. A $q$-rook monoid is a $q$-algebra with certain $q$-deformed commutation relations.

## 2. ELLIPTIC FUNCTIONS, THETA FUNCTIONS

The two subjects, elliptic functions and theta functions, developed in the nineteenth century, are intimately connected with each other. They share the beauty and multitude of formulas. As was shown in [28], the Jacobi elliptic functions can be developed into a series of $q$-hypergeometric functions.

We first remind the reader of some elementary facts concerning general elliptic functions, i.e. double-periodic functions in the complex plane taken from the excellent exposition [45]. Just as the rational functions on the Riemann sphere $\Sigma$ form a field denoted $\mathbb{C}(z)$, the meromorphic functions on the torus $\mathbb{C}/\Omega$ are the doubly periodic elliptic functions on $\mathbb{C}$, which form a field, denoted $E(\Omega)$, where $\Omega$ is a fixed lattice. Both $E(\Omega)$ and $E_1(\Omega)$, the field of even elliptic functions, are extension fields of $\mathbb{C}$. About 1850 K. Weierstrass (1815–1897) introduced the Weierstrass sigma-function, $\sigma(z)$, which is $z$ multiplied with the product over all lattice points of those elementary factors which have simple zeros at the lattice points. The Weierstrass zeta-function $\zeta(z)$ is the logarithmic derivative of $\sigma(z)$. The Weierstrass $\wp$-function $\in E_1(\Omega)$ is $-\mathrm{d}\zeta/\mathrm{d}z$.

We now turn to the Jacobi elliptic functions. Here the $\sigma$-functions are called $\theta$-functions; there are four of them. For our purposes it will suffice to study just one of them.

**Definition 2.1.** *The first Jacobi theta function is given by*

$$\theta_1(z,q) \equiv 2\sum_{n=0}^{\infty}(-1)^n \mathrm{QE}\Big(\Big(n+\frac{1}{2}\Big)^2\Big)\sin(2n+1)z\,. \tag{22}$$

*This function has period* $2\pi$ *and quasiperiod* $\frac{-i}{2}\log q$.

The Jacobi elliptic functions are formed as quotients of these four $\theta$-functions. We will now present the $q$-series expansions of the first three Jacobi elliptic functions [28], which were given by Gauss in 1799, Abel in 1828, Jacobi in 1829, Gudermann in 1844, Heine in 1850 [38,39], Durège in 1861, Broch in 1867, Enneper in 1876, Laurent in 1880, Glaisher in 1881, Halphen in 1886, Bellachi in 1894, and Hermite in 1908. It was, however, only Heine who gave the series in the present form. For a historical account on this subject see [28].

**Theorem 2.1.** $2K, 4K$ *and* $2K', 4K'$ *denote the periods of the Jacobi elliptic functions*

$$K = \frac{\pi}{2}\sum_{l=0}^{\infty}\left(\frac{(2l-1)!!}{(2l)!!}\right)^2 k^{2l}, \tag{23}$$

$$q \equiv e^{-\pi\frac{K'}{K}}, \;\; x \equiv \frac{u\pi}{2K}. \tag{24}$$

*Assume that*

$$-\frac{\pi}{2}\frac{K'}{K} < \operatorname{Im}(x) < \frac{\pi}{2}\frac{K'}{K}. \tag{25}$$

*Then* [28]

$$[44,\ (19)]\operatorname{sn} u = \frac{2\pi}{Kk}\operatorname{Im}\left(\frac{q^{\frac{1}{2}}e^{ix}}{1-q}\,{}_2\phi_1(1,\tfrac{1}{2};\tfrac{3}{2}|q^2;qe^{2ix})\right), \tag{26}$$

$$[44,\ (21)]\operatorname{cn} u = \frac{2\pi}{Kk}\operatorname{Re}\left(\frac{q^{\frac{1}{2}}e^{ix}}{1+q}\,{}_2\phi_1(1,\widetilde{\tfrac{1}{2}};\widetilde{\tfrac{3}{2}}|q^2;qe^{2ix})\right), \tag{27}$$

$$[44,\ (25)]\operatorname{dn} u = \frac{\pi}{2K}\operatorname{Re}\left(-1+2\,{}_2\phi_1(1,\widetilde{0};\widetilde{1}|q^2;qe^{2ix})\right). \tag{28}$$

We will now show that the Jacobi theta function plays a central role in $q$-calculus.

**Theorem 2.2.** *The $q$-analogue of Euler's reflection formula for the $\Gamma$ function is* [12, p. 1326], [29]

$$\Gamma_q(z)\Gamma_q(1-z) = \frac{iq^{\frac{1}{8}}(1-q)(\langle 1;q\rangle_\infty)^3}{q^{\frac{z}{2}}\theta_1(\frac{-iz}{2}\log q, \sqrt{q})}. \tag{29}$$

The zeros $z = m\pi$ and $z = \frac{-in}{2}\log q$ of $\theta_1$ correspond to the set of poles $\pm\frac{2m\pi i}{\log q}$ and $-n$ of $\Gamma_q$, respectively.

The formula (29) was known in the literature three times before it was given in English in 2001. It first appeared in a 1869 paper by Thomae [54, p. 262, (6a)]. The second time it appeared in 1873 in a book by Thomae [55, p. 183, (168a)] about, among other things, theta functions. Then it appeared in an Italian paper of 1923 by Nalli [48, p. 338]. The interesting thing is that both Thomae and Nalli used notations for theta functions which are clearly different from the modern notation. The Thomae notation was reminiscent of Riemann theta functions, and the Nalli notation was maybe influenced by nineteenth-century Italian books about elliptic functions.

Thomae [54, p. 262] also claims that his teacher Heine published this equation in [37, p. 310], but it looks like that the equations on page 310 were about something else.

The Jacobi triple product identity (1829) is very important in analytic number theory.

**Theorem 2.3.** *The following formula* [5,56] *for the theta function holds*:

$$\sum_{n=-\infty}^{\infty} q^{n^2}z^n = (q^2, -qz, -qz^{-1}; q^2)_\infty, \tag{30}$$

*where* $z \in \mathbb{C}\backslash\{0\},\ 0 < |q| < 1$.

This relationship generalizes other results, such as the pentagonal number theorem.

There could possibly be some relationship between the Jacobi triple product identity and the $q$-analogue of Euler's reflection formula (29).

The first person to work explicitly with the expression $\Gamma_q(x)\Gamma_q(1-x)$ was Reverend Jackson [41, p. 193], who showed that

$$\Gamma_q(x)\Gamma_q(1-x) = \frac{\Gamma_q(\frac{1}{2})^2}{\sigma_q(x)}, \tag{31}$$

when computing a certain $q$-integral. Here $\sigma_q(x)$ is a certain $\sigma$- or $\theta$-function.

We now come to the definition of one of the zeta functions. Recall the corresponding definition in the Weierstrass elliptic function case:

$$Zs(x) \equiv \frac{\theta_1(x,q)'}{\theta_1(x,q)}. \tag{32}$$

This zeta function has a $q$-series expansion [28]; we have excluded a multiplicative constant:

$$[14,\ \text{p. } 288,\ (43)] Zs(x) = \cot\, x + 4\mathrm{Im}\left(\frac{q^2 e^{2ix}}{1-q^2}\,{}_2\phi_1(1,1;2|q^2;q^2e^{2ix})\right). \tag{33}$$

Brown and Eastham have found two new reformulations of hypergeometric formulas in their recent paper [15].

The first is a restatement of Kummer's ${}_2F_1(-1)$ formula

$${}_2F_1(a,b;1+a-b;-1) = 2\cos\left(\frac{1}{2}\pi a\right)\,\Gamma\left[\begin{array}{c} -a, 1+a-b \\ -\frac{a}{2}, 1+\frac{a}{2}-b \end{array}\right]. \tag{34}$$

Another paper by the author about this is in preparation. Recall that Bailey–Daum's classic $q$-analogue of Kummer's ${}_2F_1(-1)$ theorem has been expressed in the new notation as

**Theorem 2.4.** [26]

$${}_2\phi_1(a,b;1+a-b|q,-q^{1-b}) = \Gamma_q\left[\begin{array}{c} 1+a-b, 1+\frac{a}{2} \\ 1+a, 1+\frac{a}{2}-b \end{array}\right]\frac{\langle \widetilde{1+\frac{a}{2}-b}, \widetilde{1}; q\rangle_\infty}{\langle \widetilde{1+\frac{a}{2}}, \widetilde{1-b}; q\rangle_\infty}. \tag{35}$$

The following two formulas are simply restatements of two equations from [29].

Two $q$-analogues of (34) are given by the following theorems.

**Theorem 2.5.** *A second q-analogue of Kummer's ${}_2F_1(-1)$ formula*

$${}_3\phi_3(a,b,\widetilde{1+\tfrac{a}{2}};1+a-b,\widetilde{\tfrac{a}{2}},\infty|q,q^{1+\frac{a}{2}-b}) = \Gamma_q\left[\begin{array}{c} 1+a-b, -a \\ -\frac{a}{2}, 1+\frac{a}{2}-b \end{array}\right] q^{-\frac{a}{4}}\frac{\theta_1(\frac{ia}{2}\log q, q^{\frac{1}{2}})}{\theta_1(\frac{ia}{4}\log q, q^{\frac{1}{2}})}, \tag{36}$$

*where* $1+a-b \neq 0,-1,-2\ldots$ .

**Theorem 2.6.** *A third q-analogue of Kummer's ${}_2F_1(-1)$ formula*

$${}_4\phi_2(a,b,\widetilde{1+\tfrac{a}{2}},\infty;1+a-b,\widetilde{\tfrac{a}{2}}|q,q^{-\frac{a}{2}-b}) = \Gamma_q\left[\begin{array}{c} 1+a-b, -a \\ -\frac{a}{2}, 1+\frac{a}{2}-b \end{array}\right] q^{-\frac{a}{4}+\frac{ab}{2}}\frac{\theta_1(\frac{ia}{2}\log q, q^{\frac{1}{2}})}{\theta_1(\frac{ia}{4}\log q, q^{\frac{1}{2}})}, \tag{37}$$

*where* $1+a-b \neq 0,-1,-2\ldots$ .

The two intermediary identities in the mock theta function paper by Andrews [6] are examples of $q$-formulas which do not correspond to known formulas for $q = 1$. In the new notation they can be written as follows:

**Theorem 2.7.** *Andrews* [6, p. 68]

$$
{}_3\phi_2\left[\begin{matrix}\frac{c+1-b-t}{2}, b, \widetilde{\frac{c+1-b-t}{2}} \\ c, \widetilde{1}\end{matrix} |q;q^t\right] = \frac{\langle\frac{c+1-b}{2};q^2\rangle_\infty}{\langle c,\widetilde{b};q\rangle_\infty}\frac{\langle\frac{c+b}{2};q^2\rangle_\infty}{\langle\frac{1}{2};q^2\rangle_\infty}\,{}_2\phi_1(\tfrac{1-t}{2},b;\tfrac{c+b}{2}|q^2,q^t)
$$

$$
+\frac{q^b\langle\frac{c-b}{2};q^2\rangle_\infty}{\langle c,\widetilde{b};q\rangle_\infty}\frac{\langle\frac{c+b+1}{2};q^2\rangle_\infty}{\langle\frac{1}{2};q^2\rangle_\infty}\,{}_2\phi_1(\tfrac{2-t}{2},b;\tfrac{c+b+1}{2}|q^2,q^{t+1}). \quad (38)
$$

**Theorem 2.8.** *Andrews* [6, p. 69]

$$
{}_2\phi_1\left[\begin{matrix} b,\widetilde{1-t} \\ c\end{matrix} |q;q^t\right] = \frac{\langle\frac{1+b}{2},\frac{2c-b}{2};q^2\rangle_\infty\langle\widetilde{1};q\rangle_\infty}{\langle c,\widetilde{c-b};q\rangle_\infty}\,{}_3\phi_2\left[\begin{matrix}\widetilde{\frac{c-b-t}{2}},\widetilde{\frac{c+1-b-t}{2}},\frac{b}{2} \\ \frac{2c-b}{2},\frac{1}{2}\end{matrix} |q^2;q^{2t}\right]
$$

$$
+\quad \frac{1+q^{c-b-t}}{1-q}q^t\frac{\langle\frac{b}{2},\frac{2c+1-b}{2};q^2\rangle_\infty\langle\widetilde{1};q\rangle_\infty}{\langle c,\widetilde{c-b};q\rangle_\infty}\,{}_3\phi_2\left[\begin{matrix}\widetilde{\frac{c+1-b-t}{2}},\widetilde{\frac{c+2-b-t}{2}},\frac{b+1}{2} \\ \frac{2c+1-b}{2},\frac{3}{2}\end{matrix} |q^2;q^{2t}\right]. \quad (39)
$$

These two formulas are clearly different from Andrews's $q$-analogues of Kummer's formulas in [8]. For the convenience of the reader we display two of these Kummer's formulas here:

$$
{}_2F_1\left(a,b;\frac{1+a+b}{2};\frac{1}{2}\right) = \Gamma\left[\begin{matrix}\frac{1+a+b}{2},\frac{1}{2} \\ \frac{1+b}{2},\frac{1+a}{2}\end{matrix}\right], \quad (40)
$$

$$
{}_2F_1\left(a,1-a;c;\frac{1}{2}\right) = \Gamma\left[\begin{matrix}\frac{c}{2},\frac{1+c}{2} \\ \frac{1+c-a}{2},\frac{a+c}{2}\end{matrix}\right]. \quad (41)
$$

## 3. $q$-APPELL FUNCTIONS

In this chapter we discuss several cases of $q$-hypergeometric functions of two variables and consider special limit cases to one variable and to combinatorics. In 1880 Paul Emile Appell (1855–1930) [10] introduced some 2-variable hypergeometric series now called Appell functions, which have the following $q$-analogues.

**Definition 3.1.** *The q-Appell functions* [42,43] *have convergence areas in the $x_1$ $x_2$ plane, which are slightly larger than for the corresponding Appell functions. The convergence areas given are those for $q = 1$.*

$$
\Phi_1(a;b,b';c|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty}\frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}}x_1^{m_1}x_2^{m_2}, \quad \max(|x_1|,|x_2|)<1; \quad (42)
$$

$$
\Phi_2(a;b,b';c,c'|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty}\frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1}\langle c';q\rangle_{m_2}}x_1^{m_1}x_2^{m_2}, \quad |x_1|+|x_2|<1; \quad (43)
$$

$$
\Phi_3(a,a';b,b';c|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty}\frac{\langle a;q\rangle_{m_1}\langle a';q\rangle_{m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}}x_1^{m_1}x_2^{m_2}, \quad \max(|x_1|,|x_2|)<1; \quad (44)
$$

$$
\Phi_4(a;b;c,c'|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty}\frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1+m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1}\langle c';q\rangle_{m_2}}x_1^{m_1}x_2^{m_2}, \quad |\sqrt{x_1}|+|\sqrt{x_2}|<1. \quad (45)
$$

**Definition 3.2.** *Let* $\{\theta_i\}_q \equiv x_i D_{q,i}$. *The following inverse pair of symbolic operators defined in* [26,42] *will be used in some of the computations*:

$$\bigtriangledown_q(h) \equiv \Gamma_q \left[ \begin{array}{c} h, h+\{\theta_1\}_q+\{\theta_2\}_q \\ h+\{\theta_1\}_q, h+\{\theta_2\}_q \end{array} \right], \ \triangle_q(h) \equiv \Gamma_q \left[ \begin{array}{c} h+\{\theta_1\}_q, h+\{\theta_2\}_q \\ h+\{\theta_1\}_q+\{\theta_2\}_q, h \end{array} \right]. \tag{46}$$

Then

$$\bigtriangledown_q(h)\langle h;q\rangle_m \langle h;q\rangle_n x_1^m x_2^n = \langle h;q\rangle_{m+n} x_1^m x_2^n. \tag{47}$$

In [26] we found general expansion formulas, which upon specialization of variables lead to 6 expansion formulas for $q$-Appell functions. We write these expansion formulas followed by their companion combinatorial identities.

**Theorem 3.1.** *q-Analogue of* [16, (26)]

$$\begin{aligned}\Phi_2(a;b,b';c,c'|q;x_1,x_2) = &\sum_{r=0}^{\infty} \frac{\langle a,b,b';q\rangle_r}{\langle 1,c,c';q\rangle_r} x_1^r x_2^r q^{r(a+r-1)} \\ &\times {}_2\phi_1(a+r,b+r;c+r|q,x_1)\, {}_2\phi_1(a+r,b'+r;c'+r|q,x_2).\end{aligned} \tag{48}$$

This is equivalent to a form of the first $q$-Vandermonde theorem

$$\frac{\langle a;q\rangle_{m+n}}{\langle 1;q\rangle_m \langle 1;q\rangle_n} = \sum_{r=0}^{\min(m,n)} \frac{\langle a;q\rangle_r}{\langle 1;q\rangle_r} \frac{\langle a+r;q\rangle_{m-r}}{\langle 1;q\rangle_{m-r}} \frac{\langle a+r;q\rangle_{n-r}}{\langle 1;q\rangle_{n-r}} q^{r(a+r-1)}. \tag{49}$$

**Theorem 3.2.** *q-Analogue of* [16, (27)]

$$\begin{aligned}{}_2\phi_1(a,b;c|q,x_1)\, {}_2\phi_1(a,b';c'|q,x_2) = &\sum_{r=0}^{\infty} \frac{(-1)^r \langle a,b,b';q\rangle_r}{\langle 1,c,c';q\rangle_r} x_1^r x_2^r q^{ra+\binom{r}{2}} \\ &\times \Phi_2(a+r;b+r,b'+r;c+r,c'+r|q;x_1,x_2).\end{aligned} \tag{50}$$

This is equivalent to

$$\frac{\langle a;q\rangle_m \langle a;q\rangle_n}{\langle 1;q\rangle_m \langle 1;q\rangle_n} = \sum_{r=0}^{\min(m,n)} \frac{\langle a;q\rangle_{m+n-r}}{\langle 1;q\rangle_r} \frac{(-1)^r}{\langle 1;q\rangle_{m-r} \langle 1;q\rangle_{n-r}} q^{ra+\binom{r}{2}}. \tag{51}$$

**Theorem 3.3.** *q-Analogue of* [16, (28)] (*compare* [2, p. 194])

$$\begin{aligned}\Phi_3(a,a';b,b';c|q;x_1,x_2) = &\sum_{r=0}^{\infty} \frac{(-1)^r \langle a,a',b,b';q\rangle_r}{\langle 1,c+r-1;q\rangle_r \langle c;q\rangle_{2r}} x_1^r x_2^r q^{rc+\frac{3}{2}r(r-1)} \\ &\times {}_2\phi_1(a+r,b+r;c+2r|q,x_1)\, {}_2\phi_1(a'+r,b'+r;c+2r|q,x_2).\end{aligned} \tag{52}$$

This is equivalent to

$$\frac{1}{\langle 1;q\rangle_m\langle 1;q\rangle_n\langle a;q\rangle_{m+n}}=\sum_{r=0}^{\min(m,n)}\frac{(-1)^r}{\langle 1,a+r-1;q\rangle_r\langle 1;q\rangle_{m-r}}\frac{q^{ra+\frac{3}{2}r(r-1)}}{\langle a;q\rangle_{m+r}\langle 1,a+2r;q\rangle_{n-r}}. \tag{53}$$

**Theorem 3.4.** *q-Analogue of* [16, (29)]

$$\begin{aligned}{}_2\phi_1(a,b;c|q,x_1)\,{}_2\phi_1(a',b';c|q,x_2)&=\sum_{r=0}^{\infty}\frac{\langle a,a',b,b';q\rangle_r}{\langle 1,c;q\rangle_r\langle c;q\rangle_{2r}}x_1^r x_2^r q^{rc+r(r-1)}\\&\times\Phi_3(a+r,a'+r;b+r,b'+r;c+2r|q;x_1,x_2).\end{aligned} \tag{54}$$

This is equivalent to

$$\frac{\langle a;q\rangle_{m+n}}{\langle 1,a;q\rangle_m\langle 1,a;q\rangle_n}=\sum_{r=0}^{\min(m,n)}\frac{q^{ra+r(r-1)}}{\langle 1,a;q\rangle_r\langle 1;q\rangle_{m-r}\langle 1;q\rangle_{n-r}}, \tag{55}$$

which for $m=n$ is a special case of the first $q$-Vandermonde theorem.

**Theorem 3.5.** *q-Analogue of* [16, (30)]. *The first version of this equation occurred in* [42, (37), p. 75]. *The same corrected version also occurred in* [2, 6.8, p. 193]:

$$\begin{aligned}\Phi_1(a;b,b';c|q;x_1,x_2)&=\sum_{r=0}^{\infty}\frac{\langle c-a,a,b,b';q\rangle_r}{\langle 1,c+r-1;q\rangle_r\langle c;q\rangle_{2r}}x_1^r x_2^r q^{ra+r(r-1)}\\&\times\,{}_2\phi_1(a+r,b+r;c+2r|q,x_1)\,{}_2\phi_1(a+r,b'+r;c+2r|q,x_2).\end{aligned} \tag{56}$$

*Proof.*

$$\begin{aligned}\Phi_1(a;b,b';c|q;x_1,x_2)=&\sum_{r=0}^{\infty}\frac{\langle -\theta_1,-\theta_2,c-a;q\rangle_r\langle c;q\rangle_{2r}}{\langle 1,a,c+r-1,c+\theta_1,c+\theta_2;q\rangle_r}q^{ra}\varepsilon_1^r\varepsilon_2^r\\&\times\,{}_2\phi_1(a,b;c|q,x_1)\,{}_2\phi_1(a,b';c|q,x_2)=\sum_{r=0}^{\infty}\frac{\langle -\theta_1,-\theta_2,c-a;q\rangle_r\langle c;q\rangle_{2r}}{\langle 1,c+r-1,a,c,c;q\rangle_r}q^{ra}\varepsilon_1^r\varepsilon_2^r\\&\times\,{}_2\phi_1(a,b;c+r|q,x_1)\,{}_2\phi_1(a,b';c+r|q,x_2)=\sum_{r=0}^{\infty}\frac{\langle b,a,b',c-a;q\rangle_r\langle c;q\rangle_{2r}}{\langle 1,c+r-1,c,c,c+r,c+r;q\rangle_r}\\&\times q^{ra+r(r-1)}{x_1}^r{x_2}^r\,{}_2\phi_1(a+r,b+r;c+2r|q,x_1)\,{}_2\phi_1(a+r,b'+r;c+2r|q,x_2)=\ldots.\end{aligned} \tag{57}$$

□

This is equivalent to a form of the $q$-Whipple theorem

$$\frac{\langle a;q\rangle_{m+n}}{\langle 1;q\rangle_m\langle 1;q\rangle_n\langle c;q\rangle_{m+n}}=\sum_{r=0}^{\min(m,n)}\frac{\langle c-a,a;q\rangle_r q^{ra+r(r-1)}}{\langle 1,c+r-1;q\rangle_r\langle 1;q\rangle_{m-r}}\frac{\langle a+r;q\rangle_{m-r}}{\langle c;q\rangle_{m+r}}\frac{\langle a+r;q\rangle_{n-r}}{\langle 1,c+2r;q\rangle_{n-r}}. \tag{58}$$

**Theorem 3.6.** *q-Analogue of* [16, (31)]

$$
{}_2\phi_1(a,b;c|q,x_1)\ {}_2\phi_1(a,b';c|q,x_2)=\sum_{r=0}^{\infty}\frac{(-1)^r\langle a,b,b',c-a;q\rangle_r}{\langle 1,c;q\rangle_r\langle c;q\rangle_{2r}}
\times q^{ra+\binom{r}{2}}{x_1}^r{x_2}^r\Phi_1(a+r;b+r,b'+r;c+2r|q;x_1,x_2). \tag{59}
$$

This is equivalent to a form of the $q$-Pfaff–Saalschütz theorem

$$
\frac{\langle a;q\rangle_m\langle a;q\rangle_n}{\langle 1,c;q\rangle_m\langle 1,c;q\rangle_n}=\sum_{r=0}^{\min(m,n)}\frac{(-1)^r\langle c-a;q\rangle_r}{\langle 1,c;q\rangle_r\langle c;q\rangle_{m+n}}\frac{\langle a;q\rangle_{m+n-r}}{\langle 1;q\rangle_{m-r}\langle 1;q\rangle_{n-r}}q^{ra+\binom{r}{2}}. \tag{60}
$$

As an undergraduate at Cambridge in 1914 W. N. Bailey (1893–1961) was greatly influenced by Ramanujan and wrote the first systematic treatment of hypergeometric series [13]. Lucy J. Slater attended Bailey's lectures on $q$-hypergeometric series in 1947–1950 at London University and wrote many important papers on this subject; among her students was Howard Exton. In 1953 R. P. Agarwal [2] visited Bailey, and made the aforementioned contributions to the subject. In 1966 Slater said [51, p. 234] that there seemed to be no systematic attempt to find summation theorems for basic Appell series, but Andrews [7] managed to prove some summation and transformation formulas for basic Appell series. We are now going to present these formulas in the new notation. Despite their beauty, it turns out that the formulas by Andrews are not very interesting for the special case $q=1$. The reason is that the proofs involve the $q$-binomial theorem; the obtained formulas are only formal.

So we start with $q$-analogues of a couple of formulas by Carlitz [17] to show some Saalschützian theorems for basic double series, which make sense when $q=1$.

Assume that

$$
\gamma+\delta'=\alpha+\beta'-n+1, \tag{61}
$$

$$
\gamma=\beta'+\beta, \tag{62}
$$

$$
\delta=\alpha-\beta'-m+1. \tag{63}
$$

This implies that

$$
\gamma+\delta=\alpha+\beta-m+1. \tag{64}
$$

Then [3, p. 456, (7)]

$$
\begin{aligned}
S&\equiv\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{\langle -m;q\rangle_r\langle -n;q\rangle_s\langle\alpha;q\rangle_{r+s}\langle\beta;q\rangle_r\langle\beta';q\rangle_s q^{r+s}}{\langle 1;q\rangle_r\langle 1;q\rangle_s\langle\gamma;q\rangle_{r+s}\langle\delta;q\rangle_r\langle\delta';q\rangle_s}\\
&=\sum_{r=0}^{m}\frac{\langle -m,\alpha,\beta;q\rangle_r}{\langle 1,\gamma,\delta;q\rangle_r}q^r\sum_{s=0}^{n}\frac{\langle -n,\alpha+r,\beta';q\rangle_s}{\langle 1,\gamma+r,\delta';q\rangle_s}q^s\\
&=\sum_{r=0}^{m}\frac{\langle -m,\alpha,\beta;q\rangle_r}{\langle 1,\gamma,\delta;q\rangle_r}q^r\frac{\langle\gamma-\alpha,\gamma-\beta'+r;q\rangle_n}{\langle\gamma+r,\gamma-\alpha-\beta';q\rangle_n}\\
&=\sum_{r=0}^{m}\frac{\langle -m,\alpha,\gamma-\beta'+n;q\rangle_r}{\langle 1,\gamma+n,\delta;q\rangle_r}q^r\frac{\langle\gamma-\alpha,\gamma-\beta';q\rangle_n}{\langle\gamma,\gamma-\alpha-\beta';q\rangle_n}\\
&=\frac{\langle\gamma-\alpha,\gamma-\beta';q\rangle_n}{\langle\gamma,\gamma-\alpha-\beta';q\rangle_n}\frac{\langle\gamma-\alpha+n,\beta';q\rangle_m}{\langle\gamma+n,\beta'-\alpha;q\rangle_m}\\
&=\frac{\langle\beta+\beta'-\alpha;q\rangle_{m+n}\langle\beta';q\rangle_m\langle\beta;q\rangle_n}{\langle\beta+\beta';q\rangle_{m+n}\langle\beta'-\alpha;q\rangle_m\langle\beta-\alpha;q\rangle_n}.
\end{aligned} \tag{65}
$$

If instead of (62) we assume that

$$\delta' + \delta = 1 + \alpha, \tag{66}$$

(61) implies that

$$\gamma - \beta' + n = \delta. \tag{67}$$

In this case we have [3, p. 456, (8)]

$$\begin{aligned}
S &= \sum_{r=0}^{m} \frac{\langle -m, \alpha, \beta; q\rangle_r}{\langle 1, \gamma, \delta; q\rangle_r} q^r \frac{\langle \gamma - \alpha, \gamma - \beta' + r; q\rangle_n}{\langle \gamma + r, \gamma - \alpha - \beta'; q\rangle_n} \\
&= \frac{\langle \gamma - \alpha, \gamma - \beta'; q\rangle_n}{\langle \gamma, \gamma - \alpha - \beta'; q\rangle_n} \, {}_3\phi_2(-m, \alpha, \beta; \gamma + n, \gamma - \beta' | q, q) \\
&= \frac{\langle \gamma - \alpha, \gamma - \beta'; q\rangle_n}{\langle \gamma, \gamma - \alpha - \beta'; q\rangle_n} \frac{\langle \gamma + n - \alpha, \gamma + n - \beta; q\rangle_m}{\langle \gamma + n, \gamma + n - \alpha - \beta; q\rangle_m} \\
&= \frac{\langle \gamma - \alpha\rangle_{n+m} \langle \gamma - \beta'\rangle_n \langle \gamma - \alpha - \beta'; q\rangle_m q^{m\alpha}}{\langle \gamma\rangle_{n+m} \langle \gamma - \alpha - \beta'; q\rangle_n \langle \gamma - \beta'\rangle_m}.
\end{aligned} \tag{68}$$

Assume again that $\gamma = \beta' + \beta$. We can now use (65) to prove the following transformation formula:

**Theorem 3.7.** *Srivastava* [53, p. 52, (3.3)]

$$\Phi_1(\beta' + \beta - \alpha; \beta', \beta; \beta + \beta' | q; x, y) = (xq^{\beta' - \alpha}; q)_{\alpha - \beta'} (yq^{\beta - \alpha}; q)_{\alpha - \beta} \, \Phi_1(\alpha, \beta, \beta'; \beta + \beta' | q; xq^{\beta' - \alpha}, yq^{\beta - \alpha}). \tag{69}$$

*Proof.*

$$\begin{aligned}
\Phi_1(\beta' + \beta - \alpha; \beta', \beta; \beta + \beta' | q; x, y) &= \sum_{m,n=0}^{\infty} \frac{x^m y^n \langle \beta' - \alpha; q\rangle_m \langle \beta - \alpha; q\rangle_n}{\langle 1; q\rangle_n \langle 1; q\rangle_m} \\
&\times \sum_{r=0}^{m} \sum_{s=0}^{n} \frac{\langle -m; q\rangle_r \langle -n; q\rangle_s}{\langle 1; q\rangle_r \langle 1; q\rangle_s} \frac{q^{r+s} \langle \alpha; q\rangle_{r+s} \langle \beta; q\rangle_r \langle \beta'; q\rangle_s}{\langle \gamma; q\rangle_{r+s} \langle \alpha - \beta' - m + 1; q\rangle_r \langle \alpha - \beta - n + 1; q\rangle_s} \\
&= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{x^r y^s q^{r+s} \langle \alpha; q\rangle_{r+s} \langle \beta; q\rangle_r \langle \beta'; q\rangle_s}{\langle 1; q\rangle_r \langle 1; q\rangle_s \langle \gamma; q\rangle_{r+s}} \sum_{m=r}^{\infty} \sum_{n=s}^{\infty} \frac{\langle -m; q\rangle_r \langle \beta' - \alpha; q\rangle_{m-r}}{\langle 1; q\rangle_m \langle 1; q\rangle_n} \\
&\times (-1)^{r+s} x^{m-r} \langle -n; q\rangle_s \langle \beta - \alpha; q\rangle_{n-s} y^{n-s} q^{-\binom{s}{2} - \binom{r}{2} + s(\beta - \alpha + n - 1) + r(\beta' - \alpha + m - 1)} \\
&= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{x^r y^s \langle \alpha; q\rangle_{r+s} \langle \beta; q\rangle_r \langle \beta'; q\rangle_s}{\langle 1; q\rangle_r \langle 1; q\rangle_s \langle \gamma; q\rangle_{r+s}} q^{s(\beta - \alpha) + r(\beta' - \alpha)} \\
&\times \sum_{m=r}^{\infty} \frac{\langle \beta' - \alpha; q\rangle_{m-r} x^{m-r}}{\langle 1; q\rangle_{m-r}} \sum_{n=s}^{\infty} \frac{\langle \beta - \alpha; q\rangle_{n-s} y^{n-s}}{\langle 1; q\rangle_{n-s}} \\
&= \frac{(xq^{\beta' - \alpha}; q)_\infty}{(x; q)_\infty} \frac{(yq^{\beta - \alpha}; q)_\infty}{(y; q)_\infty} \Phi_1(\alpha, \beta, \beta'; \beta + \beta' | q; xq^{\beta' - \alpha}, yq^{\beta - \alpha}).
\end{aligned} \tag{70}$$

□

**Remark 3.1.** The original paper by Carlitz, which treated the hypergeometric case [17, p. 417, (12)] seems to contain a misprint. Also compare with [18, p. 138, (3)] and [19, p. 1, (1.5)]. In the first $q$-analogue (in Watson's notation) [3, p. 457, eq. 9], another misprint was introduced. This clearly shows the disadvantages of the Watson notation compared to the Heine notation.

Waleed Al-Salam (1926–1996) was a student of Leonard Carlitz (1907–1999) at Duke University in 1958. We will come back to these two people in the next chapter. Waleed Al-Salam had a wife Nadhla Al-Salam, who passed her PHD exam under Carlitz in 1965, and who published on special functions in several respected journals. She had no graduate students.

For the special case $y=0$, (69) becomes a $q$-analogue of Euler's transformation:

**Theorem 3.8.** [32, p. 10, (1.4.3)], [39, p. 115]

$$ {}_2\phi_1(\gamma-\alpha;\beta;\gamma|q;x) = (xq^{\beta-\alpha};q)_{\alpha-\beta}\ {}_2\phi_1(\alpha,\gamma-\beta;\gamma|q;xq^{\beta-\alpha}). \tag{71}$$

We continue with Andrews's formulas, which were originally presented in Watson's notation. We show that although these formulas have no hypergeometric counterpart, they can be presented in the current notation.

**Theorem 3.9.** *The following transformation* [7, p. 618] *holds*:

$$ \Phi_1(\alpha;\beta,\beta';\gamma|q;q^x,q^y) = \frac{\langle\alpha;q\rangle_\infty\langle\beta+x;q\rangle_\infty\langle\beta'+y;q\rangle_\infty}{\langle\gamma;q\rangle_\infty\langle x;q\rangle_\infty\langle y;q\rangle_\infty} \times {}_3\phi_2\left(\gamma-\alpha,x,y;\beta+x,\beta'+y|q,q^\alpha\right). \tag{72}$$

*Proof.* We use the $q$-binomial theorem. Absolute convergence is assumed.

$$\begin{aligned}
\Phi_1(\alpha;\beta,\beta';\gamma|q;q^x,q^y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{q^{xm}q^{yn}\langle\alpha;q\rangle_{m+n}\langle\beta;q\rangle_m\langle\beta';q\rangle_n}{\langle\gamma;q\rangle_{m+n}\langle 1;q\rangle_m\langle 1;q\rangle_n}\\
&= \frac{\langle\alpha;q\rangle_\infty}{\langle\gamma;q\rangle_\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{q^{xm}q^{yn}\langle\gamma+m+n;q\rangle_\infty\langle\beta;q\rangle_m\langle\beta';q\rangle_n}{\langle\alpha+m+n;q\rangle_\infty\langle 1;q\rangle_m\langle 1;q\rangle_n}\\
&= \frac{\langle\alpha;q\rangle_\infty}{\langle\gamma;q\rangle_\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{q^{xm}q^{yn}q^{r(\alpha+m+n)}\langle\gamma-\alpha;q\rangle_r\langle\beta;q\rangle_m\langle\beta';q\rangle_n}{\langle 1;q\rangle_r\langle 1;q\rangle_m\langle 1;q\rangle_n}\\
&= \frac{\langle\alpha;q\rangle_\infty}{\langle\gamma;q\rangle_\infty}\sum_{r=0}^{\infty}\frac{\langle\gamma-\alpha;q\rangle_r}{\langle 1;q\rangle_r}q^{\alpha r}\frac{\langle\beta+x+r;q\rangle_\infty}{\langle x+r;q\rangle_\infty}\frac{\langle\beta'+y+r;q\rangle_\infty}{\langle y+r;q\rangle_\infty}\\
&= \frac{\langle\alpha;q\rangle_\infty}{\langle\gamma;q\rangle_\infty}\frac{\langle\beta+x;q\rangle_\infty}{\langle x;q\rangle_\infty}\frac{\langle\beta'+y;q\rangle_\infty}{\langle y;q\rangle_\infty}\sum_{r=0}^{\infty}\frac{\langle\gamma-\alpha;q\rangle_r}{\langle 1;q\rangle_r}q^{\alpha r}\\
&\quad\times\frac{\langle x;q\rangle_r}{\langle\beta+x;q\rangle_r}\frac{\langle y;q\rangle_r}{\langle\beta'+x;q\rangle_r} = \frac{\langle\alpha;q\rangle_\infty\langle\beta+x;q\rangle_\infty\langle\beta'+y;q\rangle_\infty}{\langle\gamma;q\rangle_\infty\langle x;q\rangle_\infty\langle y;q\rangle_\infty}\\
&\quad\times{}_3\phi_2\left(\gamma-\alpha,x,y;\beta+x,\beta'+y|q,q^\alpha\right).
\end{aligned}\tag{73}$$

□

Furthermore,

**Theorem 3.10.** [7, p. 619]

$$\Phi_1(\beta'-x;\beta,\beta';\beta+\beta'|q;q^x,q^y)=\frac{\langle\beta+x,\beta',\beta'+y-x;q\rangle_\infty}{\langle\beta+\beta',x,y;q\rangle_\infty}. \tag{74}$$

*Proof.*

$$\begin{aligned}\Phi_1(\beta'-x;\beta,\beta';\beta+\beta'|q;q^x,q^y)=&\frac{\langle\beta+x,\beta'-x,\beta'+y;q\rangle_\infty}{\langle\beta+\beta',x,y;q\rangle_\infty}\\ &\times{}_2\phi_1\left(x,y;\beta'+y|q,q^{\beta'-x}\right)=\frac{\langle\beta+x,\beta',\beta'+y-x;q\rangle_\infty}{\langle\beta+\beta',x,y;q\rangle_\infty}.\end{aligned} \tag{75}$$

□

Finally

**Theorem 3.11.** [7, p. 619]

$$\Phi_1(\widetilde{1-y};\beta,x+1-2y;\beta+\widetilde{1+x}-y|q;q^x,q^y)=\frac{\langle\beta+x,\widetilde{1};q\rangle_\infty\langle x+1,2(1-y)+x;q^2\rangle_\infty}{\langle\beta+\widetilde{1+x}-y,x,y;q\rangle_\infty}. \tag{76}$$

*Proof.*

$$\begin{aligned}&\Phi_1(\widetilde{1-y};\beta,x+1-2y;\beta+\widetilde{1+x}-y|q;q^x,q^y)\\ &=\frac{\langle\widetilde{1-y},\beta+x,x+1-y;q\rangle_\infty}{\langle\beta+\widetilde{1+x}-y,x,y;q\rangle_\infty}{}_2\phi_1\left(x,y;x+1-y|q,-q^{1-y}\right)\\ &=\frac{\langle\beta+x,\widetilde{1};q\rangle_\infty\langle x+1,2(1-y)+x;q^2\rangle_\infty}{\langle\beta+\widetilde{1+x}-y,x,y;q\rangle_\infty}.\end{aligned} \tag{77}$$

□

## 4. CARLITZ–ALSALAM POLYNOMIALS AND VARIOUS OTHER $q$-FUNCTIONS

The Carlitz–AlSalam orthogonal polynomials $F_{n,q}(x)$ are examples of $q$-analogues of $x^n$ defined by a generating function made up of $q$-exponential functions.

**Definition 4.1.** [4,23]

$$\sum_{\nu=0}^{\infty}\frac{t^\nu}{\{\nu\}_q!}F_{\nu,q}(x)=\frac{\mathrm{E}_q(xt)}{\mathrm{E}_q(t)\mathrm{E}_q(-t)}. \tag{78}$$

This special type of $x$-dependence in the RHS is characteristic of so-called $q$-Appell polynomials [27]. The vector of $F_{\nu,q}(x)$ can be written as a $q$-Pascal matrix times $F_{\nu,q}(0)$.

The recurrence for $F_{n,q}(x)$ is

$$\begin{aligned} F_{0,q}(x) = 1,\ F_{1,q}(x) = x, \\ F_{n+1,q}(x) = xF_{n,q}(x) - (1-q^n)q^{n-1}F_{n-1,q}(x),\ n \geq 1. \end{aligned} \tag{79}$$

This is an example of a recurrence which has a very simple form for $q = 1$.

The denominator in (78) can be written as $\mathrm{E}_q(t \ominus_q t)$. Already in 1936 Morgan Ward (1901–1963) [57, p. 256] proved the following equations for $q$-subtraction:

$$(x \ominus_q y)^{2n+1} = \sum_{k=0}^{n} (-1)^k \binom{2n+1}{k}_q x^k y^k (x^{2n+1-2k} - y^{2n+1-2k}); \tag{80}$$

$$(x \ominus_q y)^{2n} = (-1)^n \binom{2n}{n}_q x^n y^n + \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k}_q x^k y^k (x^{2n-2k} + y^{2n-2k}). \tag{81}$$

According to a formula of Gauss,

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}_q = (1-q)(1-q^3)\ldots(1-q^{2n-1}). \tag{82}$$

The polynomials $F_{n,q}(x)$ are connected to the Cigler $q$-Hermite polynomials [20] by the following substitution:

$$h_{\nu,q}(x) = \left(\frac{q}{1-q}\right)^{\frac{\nu}{2}} F_{\nu,q}\left(x\sqrt{\frac{1-q}{q}}\right). \tag{83}$$

This substitution is only valid for $q \neq 1$.

We now find the following generating function for the Cigler $q$-Hermite polynomials [20, p. 42]:

$$\frac{\mathrm{E}_q(xt)}{\mathrm{E}_{2,q}(\frac{qt^2}{\{2\}_q})} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} h_{\nu,q}(x). \tag{84}$$

Here we have used a $q$-exponential function adapted to a quadratic function argument, which was first mentioned by Jackson in his articles about $q$-Bessel functions around 1904.

**Definition 4.2.**

$$\mathrm{E}_{2,q}(x) \equiv \sum_{k=0}^{\infty} \frac{x^k}{\{k\}_{q^2}!}. \tag{85}$$

*Exton* [30, p. 168] *has given a third variant of q-exponential function*

$$\mathrm{E}_{\mathrm{Ext},q}(x) \equiv \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{4}}}{\{k\}_q!} x^k. \tag{86}$$

This function is invariant under inversion of basis and entire. No addition theorems and no power series inversion are known. However, the following differentiation formula is obtained:

$$D_q^n \mathrm{E}_{\mathrm{Ext},q}(x) = \mathrm{QE}(\frac{n(n-1)}{4}) \mathrm{E}_{\mathrm{Ext},q}(xq^{\frac{n}{2}}). \tag{87}$$

**Definition 4.3.** *The corresponding q-trigonometric functions are*

$$\sin_{\mathrm{Ext},q}(x) \equiv \sum_{n=0}^{\infty} (-1)^n \mathrm{QE}\left(\frac{n(2n+1)}{2}\right) \frac{x^{2n+1}}{\{2n+1\}_q!}, \tag{88}$$

$$\cos_{\mathrm{Ext},q}(x) \equiv \sum_{n=0}^{\infty} (-1)^n \mathrm{QE}\left(\frac{n(2n-1)}{2}\right) \frac{x^{2n}}{\{2n\}_q!}, \tag{89}$$

*where* $x \in \mathbb{C}$.

As long as no addition formulas for these $q$-trigonometric functions exist, they are not very interesting from an operational point of view. These functions have beautiful graphs, exactly resembling the usual trigonometric graphs. Two very similar $q$-trigonometric functions were given in [31, p. 590]. It should, however, be mentioned that there are $q$-trigonometric functions with $q$-addition theorems and the interested reader could consult the literature for more information about these functions.

In such a complex subject as $q$-calculus, many different definitions have been used. Many physicists prefer another, symmetric definition of a $q$-analogue. Therefore in [21, p. 158] another, similar definition of the NWA $q$-addition (15) was given. Sometimes new concepts in $q$-calculus are developed for the first time by physicists using this other symmetric $q$-analogue. Quantum groups have been around since the early 1980s and many $q$-analogues of matrix Lie groups have been presented, which are often used in string theory. The so-called grand unification theory uses so-called $q$-special functions to represent these quantum groups. Like in $q$-calculus, many dialects of string theory exist, and the physicists are not yet certain which of these dialects is the right one.

### 4.1. Some more facts about Leonard Carlitz

1907 Born in Philadelphia, PA, USA
1927 BA, University of Pennsylvania
1930 PhD, University of Pennsylvania, 1930 under Howard Mitchell, who had studied under Oswald Veblen at Princeton
1930–1931 at CalTech with E. T. Bell
1931 Married Clara Skaler
1931–1932 at Cambridge with G. H. Hardy
1932 Joined the faculty of Duke University where he served for 45 years
1938–1973 Editorial Board of *Duke Mathematics Journal* (Managing Editor from 1945)
1939 Birth of son Michael
1940 Supervision of his first doctoral student E. F. Canaday, degree awarded in 1940
1945 Birth of son Robert
1964 First James B. Duke Professor in Mathematics
1977 Supervised his 44th and last doctoral student, Jo Ann Lutz, degree awarded in 1977
1977 Retired
1990 Death of wife Clara, after 59 years of marriage
1999, 17 September, Died in Pittsburgh, PA

Carlitz moved to San Francisco after 1977, according to Srivastava's wife Rekha. Hari M. Srivastava was a close friend of Carlitz, who said that Carlitz was a reserved person, and a brilliant researcher. Richard Askey said that Carlitz went to Wisconsin in 1977 to meet his graduate student Dennis Stanton and to discuss some mathematics. Askey respected Carlitz highly. A description of Carlitz as a young man is given in [49].

Formula (69) in hypergeometric form is not found elsewhere according to Per Karlsson.

## APPENDIX

We remind the reader that there is also another notation, based on the following $q$-shifted factorial. This is called the Watson notation [58]:

$$(a;q)_n \equiv \begin{cases} 1, & n=0; \\ \prod_{m=0}^{n-1}(1-aq^m), & n=1,2,\ldots\ . \end{cases} \tag{90}$$

The relation between the new and the old notation is

$$\langle a;q\rangle_n = (q^a;q)_n. \tag{91}$$

We conclude with a short biography of F. H. Jackson, one of the greatest heros of $q$-calculus. Jackson used a notation very similar to the notation of the author.

Frank Hilton Jackson (1870–1960) was born in Hull (England) in a family of eleven children. His original interest lay in the classics field, but his father preferred him to study mathematics, and so at a very young age Jackson joined the University of Cambridge. Only 19 years old, he passed the Tripos examination. Later, he received his Doctor of Science degree.

After leaving Cambridge, he was ordained and served for a couple of years as a curate at Bremerton near Salisbury. From here, he entered the Royal Navy, where he served as a chaplain and an instructor for ten years. Serving at the H.M.S. Dido, he experienced the Boxer rebellion and received the China Medal. Subsequently he returned to civil service and worked first as a curate, then as a vicar at different churches in the years 1912–1918. During World War I he became Instructor-Commander in the Royal Navy. After the war he continued his clerical career as rector and later on as rural dean at Chester-Le-Street. In 1930, he was made an honorary canon of Durham Cathedral, in the city of Durham, England, and in 1957, Canon Emeritus. During this long and distinguished clerical career, F. H. Jackson still studied mathematics and wrote about forty papers on different subjects. His main interest lay in basic analogues, or $q$-analogues. He thus wrote on basic hypergeometric functions, including the basic functions of Legendre and Bessel, focusing on their relevance to theta functions and to elliptic functions.

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## $q$-arvutuse eri rakendusvaldkondadest

Thomas Ernst

On antud ülevaade $q$-arvutuse rakendamise võimalustest kvantarvutuses, ajaskaalade ja lahutuste käsitlustes. On näidatud, et on olemas tihe seos ühelt poolt $q$-gammafunktsioonide ja teisalt elliptiliste ning teetafunktsioonide vahel. On illustreeritud Heine sümboolika eeliseid Euleri $q$-sümmeetriavalemi, Appelli $q$-funktsioonide, Carlitzi–AlSalami polünoomide ja nn $q$-liitmise käsitlemisel. Lõpuks on esitatud mõnede tuntud $q$-arvutuse uurijate lühielulugu.