Proceedings of the Estonian Academy of Sciences,
2008, **57**, 2, 61–69
doi: 10.3176/proc.2008.2.01
Available online at www.eap.ee/proceedings

MATHEMATICS

On chaotic and stable behaviour of the von Foerster–Lasota equation in some Orlicz spaces

Antoni Leon Dawidowicz[a] and Anna Poskrobko[b*]

[a] Institute of Mathematics, Jagiellonian University, ul. Reymonta 4, 30-059 Kraków, Poland;
Antoni.Leon.Dawidowicz@im.uj.edu.pl
[b] Faculty of Computer Science, Białystok Technical University, ul. Wiejska 45A, 15-351 Białystok, Poland

Received 15 March 2007, in revised form 6 March 2008

Abstract. We study the chaotic and stable behaviour of the von Foerster–Lasota equation in Orlicz spaces with homogeneous φ-function of any positive degree. This work is, in particular, the generalization of the asymptotic properties of the von Foerster–Lasota equation in integrable spaces with exponent p greater than or equal to 1.

Key words: von Foerster–Lasota equation, chaos, stability, Orlicz spaces.

1. INTRODUCTION

In 1926 McKendrick [11] proposed the first age-dependent model of the dynamics of a population. He assumed that the state of a population in time t is described by a function $u(.,t)$. The number of individuals in age from the interval $[x_1,x_2]$ equals $\int_{x_1}^{x_2} u(x,t)\mathrm{d}x$. From this paper the equation

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=\lambda(x)u$$

follows, called in the literature as *McKendrick equation* or more often as *von Foerster equation*. McKendrick's model was generalized in many ways, among others by Gurtin and MacCamy [5] or by the authors [3]. This equation is part of the mathematical description of a particular population, as the population of red blood cells is (see [14]). It is the model with a feedback, because a circulatory system controls the global number of erythrocytes to some quantity optimum which can change. It happens, for example, during mountain trips or in a case of any disease of the respiratory system.

In their next paper [7] the authors of [14] introduced a new model of precursor cells. There the main assumption is the fact that cells mature with different intensity. The form of the equation is the following:

$$\frac{\partial u}{\partial t}+c(x)\frac{\partial u}{\partial x}=f(x,u),$$

where $c:[0,1]\to\mathbb{R}$ and $f:[0,1]\times[0,\infty)\to\mathbb{R}$ are the given functions fulfilling suitable conditions. In this model x denotes the degree of cell differentiation (maturity) and $\int_{x_0}^{x_1} u(t_0,x)\mathrm{d}x$ is the number of cells having

* Corresponding author, aposkrobko@wi.pb.edu.pl

at time $t = t_0$ the value x in the interval $[x_0, x_1]$. The coefficient c is the velocity of cell differentiation. Because of biological application the above equation is still the matter of interest for many mathematicians: Lasota and Pianigiani [8], Rudnicki [13], Łoskot [10], and Lasota and Szarek [9]. In this paper we consider a simpler case of the equation, that is

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = \gamma u.$$

Study of periodic or chaotic solutions of the von Foerster–Lasota equation is interesting from a medical point of view. In this paper we consider the semidynamical system $(T_t)_{t\geqslant 0}$ that is connected with the presented equation and fix our attention on the existence of its periodic and chaotic solutions in some function spaces. Such behaviour is already well described in the space of continuously differentiable functions [6], in the L^p space $(p > 1)$ or in the space of Hölder continuous functions [2]. This paper is the generalization of the results referring to the asymptotic properties of the above equation in the integrable spaces L^p, but in our case with the exponent $p < 1$. In order to do that, we present the definitions of Orlicz spaces with Luxemburg and F-norm in Section 2. We mention also some basic properties of such spaces (see [12]). In Section 3 we define the dynamical system connected with the von Foerster–Lasota equation. We consider the conditions of chaos and stability of such a semidynamical system and use there Devaney's definition of chaos (see [4]). Let us remind here that according to Devaney, a dynamical system $(F_t)_{t\geqslant 0}$ defined in a metric space (V, d) is chaotic as

- $(F_t)_{t\geqslant 0}$ has a property of *sensitive dependence on initial conditions in the sense of Guckenhaimer*, i.e. there is a positive real number M (a sensitivity constant) such that for every point $v \in V$ and every $\varepsilon > 0$ there exist $w \in B(v, \varepsilon)$ and $t > 0$, such that $d(F_t v, F_t w) \geqslant M$;
- $(F_t)_{t\geqslant 0}$ is *transitive*, that is for all nonempty open subsets $U_1, U_2 \subset V$ there exists $t > 0$ such that $F_t(U_1) \cap U_2 \neq \emptyset$;
- the set of periodic points of the system $(F_t)_{t\geqslant 0}$ is dense in V.

Section 4 contains the generalization of the von Foerster–Lasota equation and describes common asymptotic properties of two dynamical systems: basic and generalized. We will see that all properties of these two systems depend on one common value of the parameter γ.

2. ORLICZ SPACES AND LUXEMBURG NORM

Definition 2.1. *Let X be a real vector space. A functional $\rho : X \to [0, \infty]$ is called a modular if there holds for arbitrary $x, y \in X$*
1. $\rho(x) = 0$ *iff* $x = 0$,
2. $\rho(-x) = \rho(x)$,
3. $\rho(\alpha x + \beta y) \leqslant \rho(x) + \rho(y)$ *for* $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$.

If in place of 3. *there holds*
3′. $\rho(\alpha x + \beta y) \leqslant \alpha^s \rho(x) + \beta^s \rho(y)$ *for* $\alpha, \beta \geqslant 0$, $\alpha^s + \beta^s = 1$,
then the modular ρ is called s-convex. 1-*Convex modulars are called convex.*

Definition 2.2. *Let X be a vector space. A functional $x \to |x|$ is called F-norm if there holds for arbitrary $x, y \in X$*
1. $|x| = 0$ *iff* $x = 0$,
2. $|x + y| \leqslant |x| + |y|$,
3. *for each scalar* a, $|ax| = |x|$ *when* $|a| = 1$,
4. *for each scalar* a_k *and* a *if* $a_k \to a$ *and* $|x_k - x| \to 0$, $|a_k x_k - ax| \to 0$.

Let (Ω, Σ, μ) be a measure space, where Ω is a nonempty set, Σ is a σ-algebra of a subset of Ω, and μ is a nonnegative, complete measure not vanishing identically. A real function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, where $\mathbb{R}_+ = [0, \infty)$, is called φ-function if it is nondecreasing and continuous and such that $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$, $\varphi(u) \to \infty$ as $u \to \infty$.

Let X be the set of all real-valued, Σ-measurable, and finite μ-almost everywhere functions on Ω, with equality μ-almost everywhere. Then for every $x \in X$

$$\rho(x) = \int_\Omega \varphi(|x(t)|)\mathrm{d}\mu$$

is a modular in X. Moreover, if φ is convex φ-function, then ρ is a convex modular in X.

Definition 2.3. *The modular space X_ρ will be called Orlicz space and denoted by $L^\varphi(\Omega,\Sigma,\mu)$ (or briefly L^φ):*

$$L^\varphi = \left\{ x \in X : \int_\Omega \varphi(\beta|x(t)|)\mathrm{d}\mu \to 0 \text{ as } \beta \to 0^+ \right\}.$$

In a modular space X_ρ

$$|x|^F = \inf\left\{ s > 0 : \int_\Omega \varphi\left(\left|\frac{x(t)}{s}\right|\right)\mathrm{d}\mu \leqslant s \right\}$$

is a F-norm. If φ is convex, then the functional

$$\|x\|^L = \inf\left\{ s > 0 : \int_\Omega \varphi\left(\left|\frac{x(t)}{s}\right|\right)\mathrm{d}\mu \leqslant 1 \right\}$$

is a norm in L^φ, called the Luxemburg norm. It is known that the space L^φ with the norm $\|x\|^L$ is a Banach space.

Example 2.4. The L^p spaces over the interval $[a,b]$ are examples of the Orlicz spaces with the modular $\rho(x) = \int_a^b |x(t)|^p \mathrm{d}t$ which is convex for $p \geqslant 1$. The paper [2] considers the properties of the von Foerster–Lasota equation in such a space. All of them depend on the critical value of the coefficient γ which equals $-\frac{1}{p}$. It means that the solution of the equation displays chaotic behaviour in the sense of Devaney for $\gamma > -\frac{1}{p}$ and is strongly stable for $\gamma \leqslant -\frac{1}{p}$. For $0 < p < 1$ the modular $\rho(x) = \int_a^b |x(t)|^p \mathrm{d}t$ is p-convex modular, and such an Orlicz space is only the Fréchet space with the F-norm $|x|^F = \int_a^b |x(t)|^p \mathrm{d}t$. This paper is an attempt at the generalization of the earlier results concerning the asymptotic properties of the von Foerster–Lasota equation in the $L^p(0,1)$ $(p \geqslant 1)$ space. Therefore, our intention is to consider the Orlicz space X_ρ with the φ-function homogeneous of any degree, i.e. for all real $k > 0$, $\varphi(kx) = k^\alpha \varphi(x)$, where α is a real number (a degree).

In this section we introduce also some definitions and notations which will appear in the subsequent chapters.

Definition 2.5. *A function $v_0 \in V$ is a* periodic point *of the semigroup $(T_t)_{t\geqslant 0}$ with a period $t_0 \geqslant 0$ if and only if $T_{t_0} v_0 = v_0$. A number $t_0 > 0$ is called a* principal period *of a periodic point v_0 if and only if the set of all periods of v_0 is equal to $\mathbb{N}t_0$.*

Definition 2.6. *The semigroup $(T_t)_{t\geqslant 0}$ is* strongly stable *in V if and only if for every $v \in V$,*

$$\lim_{t\to\infty} T_t v = 0 \text{ in } V.$$

3. CHAOTIC AND STABLE SOLUTIONS OF THE VON FOERSTER–LASOTA EQUATION

We consider the partial differential equation

$$\frac{\partial u}{\partial t}+x\frac{\partial u}{\partial x}=\gamma u,\ \ t\geqslant 0,\ 0\leqslant x\leqslant 1,\ \gamma\in\mathbb{R} \tag{3.1}$$

with the initial condition

$$u(0,x)=v(x),\ \ 0\leqslant x\leqslant 1, \tag{3.2}$$

where v belongs to some normed vector space V of functions defined on $[0,1]$. Define the function T_t by the formula

$$(T_t v)(x)=u(t,x)=\mathrm{e}^{\gamma t}v(x\mathrm{e}^{-t}),\ \ x\in[0,1], \tag{3.3}$$

where u is the unique solution of (3.1) and (3.2), see [6]. If for every $v\in V$ and $t\geqslant 0$ the function T_t belongs to V, then the family $(T_t)_{t\geqslant 0}$ is a semigroup on the space V.

Now we will formulate some theorems describing the chaotic and stable behaviour of the above dynamical system. We will consider these properties in some Orlicz spaces with the φ-function fulfilling the following

Assumption. The φ-function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is homogeneous of the degree $0<p<1$.

So, under this assumption we consider the φ-function $\varphi(x)=Cx^p$, where $C>0$ and $0<p<1$. It is the only possible form of the φ-function fulfilling the above assumption.

Theorem 3.1. *If $\gamma>-\frac{1}{p}$, then for any $t_0>0$ there exists a periodic point $v_0\in L^{\varphi}$ of the dynamical system $(T_t)_{t\geqslant 0}$.*

Proof. Let w be an arbitrary function belonging to $L^{\varphi}(\mathrm{e}^{-t_0},1)$. We can define a function v_0 on the interval $(0,1]=\bigcup_{n=0}^{\infty}(\mathrm{e}^{-(n+1)t_0},\mathrm{e}^{-nt_0}]$ by squeezing the graph of the function w into the intervals $(\mathrm{e}^{-(n+1)t_0},\mathrm{e}^{-nt_0}]$. We put

$$v_0(x)=\begin{cases}\mathrm{e}^{-n\gamma t_0}w(x\mathrm{e}^{nt_0}) & \text{for}\quad x\in(\mathrm{e}^{-(n+1)t_0},\mathrm{e}^{-nt_0}]\\ 0 & \text{for}\quad x=0\end{cases}. \tag{3.4}$$

It is sufficient to prove that v_0 belongs to the $L^{\varphi}(0,1)$ space:

$$\begin{aligned}\rho_{[0,1]}(\beta v_0) &= \int_0^1\varphi(\beta|v_0(x)|)\mathrm{d}x=\sum_{n=0}^{\infty}\int_{\mathrm{e}^{-(n+1)t_0}}^{\mathrm{e}^{-nt_0}}\varphi(\beta|v_0(x)|)\mathrm{d}x\\ &= \sum_{n=0}^{\infty}\int_{\mathrm{e}^{-(n+1)t_0}}^{\mathrm{e}^{-nt_0}}\varphi(\beta\mathrm{e}^{-n\gamma t_0}|w(x\mathrm{e}^{nt_0})|)\mathrm{d}x\\ &= \sum_{n=0}^{\infty}\mathrm{e}^{-nt_0}\int_{\mathrm{e}^{-t_0}}^{1}\varphi(\beta\mathrm{e}^{-n\gamma t_0}|w(x)|)\mathrm{d}x\\ &= \sum_{n=0}^{\infty}\mathrm{e}^{-nt_0(1+\gamma p)}\int_{\mathrm{e}^{-t_0}}^{1}\varphi(\beta|w(x)|)\mathrm{d}x\\ &= \sum_{n=0}^{\infty}\mathrm{e}^{-nt_0(1+\gamma p)}\rho_{[\mathrm{e}^{-t_0},1]}(\beta w).\end{aligned}$$

Since $\mathrm{e}^{-t_0(1+\gamma p)}<1$, we get the equality

$$\rho_{[0,1]}(\beta v_0)=\frac{1}{1-\mathrm{e}^{-t_0(1+\gamma p)}}\rho_{[\mathrm{e}^{-t_0},1]}(\beta w).$$

This gives the conclusion that $\rho_{[0,1]}(\beta v_0)\to 0$ as $\beta\to 0^+$ because of the assumption $w\in L^{\varphi}$. □

Theorem 3.2. *If $\gamma > -\frac{1}{p}$, then the set of periodic points of* (3.1) *is dense in the $L^\varphi(0,1)$ space.*

Proof. Let w be an arbitrary function from the $L^\varphi(0,1)$ space and let $\varepsilon > 0$. Define v by the formula (3.4). Fix t_0 so large that $|w|^F_{[0,\mathrm{e}^{-t_0}]} < \frac{\varepsilon}{2}$ and $|v|^F_{[0,\mathrm{e}^{-t_0}]} < \frac{\varepsilon}{2}$. For $x \in [\mathrm{e}^{-t_0}, 1]$ $v(x) = w(x)$ we finally have

$$|v-w|^F_{[0,1]} = |v-w|^F_{[0,\mathrm{e}^{-t_0}]} \leqslant |v|^F_{[0,\mathrm{e}^{-t_0}]} + |w|^F_{[0,\mathrm{e}^{-t_0}]} < \varepsilon.$$

This completes the proof. □

Theorem 3.3. *If $\gamma > -\frac{1}{p}$, then the dynamical system $(T_t)_{t\geqslant 0}$ is transitive in the $L^\varphi(0,1)$ space.*

Proof. Let

$$B(v_1,\varepsilon_1) = \{\sigma \in L^\varphi(0,1) : |v_1 - \sigma|^F_{[0,1]} < \varepsilon_1\}$$

and

$$B(v_2,\varepsilon_2) = \{\sigma \in L^\varphi(0,1) : |v_2 - \sigma|^F_{[0,1]} < \varepsilon_2\}$$

be two open balls with centres in $v_1, v_2 \in L^\varphi(0,1)$. Let us define the function

$$w(x) = \begin{cases} \mathrm{e}^{-\gamma t} v_2(x\mathrm{e}^t) & \text{for} \quad x < \mathrm{e}^{-t} \\ v_1(x) & \text{for} \quad x \geqslant \mathrm{e}^{-t} \end{cases}$$

at the suitable choice of t. We should show that the above function w belongs to the space $L^\varphi(0,1)$.

$$\begin{aligned} \rho_{[0,\mathrm{e}^{-t}]}(\beta w) &= \int_0^{\mathrm{e}^{-t}} \varphi\left(\beta\left|\mathrm{e}^{-\gamma t} v_2(x\mathrm{e}^t)\right|\right)\mathrm{d}x = \mathrm{e}^{-t}\int_0^1 \varphi\left(\beta\left|\mathrm{e}^{-\gamma t} v_2(x)\right|\right)\mathrm{d}x \\ &= \mathrm{e}^{-t(\gamma p+1)}\int_0^1 \varphi\left(\beta|v_2(x)|\right)\mathrm{d}x = \mathrm{e}^{-t(\gamma p+1)}\rho_{[0,1]}(\beta v_2), \end{aligned}$$

hence

$$\rho_{[0,1]}(\beta w) \leqslant \rho_{[0,\mathrm{e}^{-t}]}(\beta w) + \rho_{[\mathrm{e}^{-t},1]}(\beta w) \leqslant \mathrm{e}^{-t(\gamma p+1)}\rho_{[0,1]}(\beta v_2) + \rho_{[0,1]}(\beta v_1).$$

From the assumption we have $\mathrm{e}^{-t(\gamma p+1)} < 1$. Therefore $\rho_{[0,1]}(\beta w) \to 0$, as $\beta \to 0^+$. It turns out from this fact that $v_1, v_2 \in L^\varphi(0,1)$. So $w \in L^\varphi(0,1)$. Besides, from the above equality we can draw the following conclusion: $|w|^F_{[0,\mathrm{e}^{-t}]} = \mathrm{e}^{-t(\gamma p+1)}|v_2|^F_{[0,1]}$. Then

$$\begin{aligned} |v_1 - w|^F_{[0,1]} &= |v_1 - w|^F_{[0,\mathrm{e}^{-t}]} \leqslant |v_1|^F_{[0,\mathrm{e}^{-t}]} + |w|^F_{[0,\mathrm{e}^{-t}]} \\ &= |v_1|^F_{[0,\mathrm{e}^{-t}]} + \mathrm{e}^{-t(\gamma p+1)}|v_2|^F_{[0,1]}. \end{aligned}$$

From the estimation it turns out that for t large enough we obtain $|v_1 - w|^F_{[0,1]} < \varepsilon_1$, hence $w \in B(v_1,\varepsilon_1)$. Therefore $T_t w \in T_t(B(v_1,\varepsilon_1))$ and $v_2 = T_t w \in B(v_2,\varepsilon_2)$. We learn from the above that the intersection of two sets $B(v_2,\varepsilon_2)$ and $T_t(B(v_1,\varepsilon_1))$ is not empty. So we conclude that the dynamical system $(T_t)_{t\geqslant 0}$ is transitive in the space $L^\varphi(0,1)$. □

As proved in the paper of Banks et al. [1] the sensitive dependence of the dynamical system on initial conditions in the sense of Guckenhaimer appears immediately from its transitivity and density of the set of its periodic points. This is expressed by the following corollary:

Corollary 3.4. *If $\gamma > -\frac{1}{p}$, then the dynamical system $(T_t)_{t\geqslant 0}$ is chaotic in the sense of Devaney in the $L^\varphi(0,1)$ space.*

Theorem 3.5. *If $\gamma \leqslant -\frac{1}{p}$, then the semigroup $(T_t)_{t\geqslant 0}$ is strongly stable in the $L^\varphi(0,1)$ space.*

Proof. Let $v \in L^{\varphi}(0,1)$ be an arbitrary function. For $s > 0$ we obtain

$$\begin{aligned}\rho_{[0,1]}\left(\frac{T_t v}{s}\right) &= \int_0^1 \varphi\left(\left|\frac{T_t v(x)}{s}\right|\right) \mathrm{d}x = \int_0^1 \varphi\left(\left|\frac{\mathrm{e}^{\gamma t} v(x\mathrm{e}^{-t})}{s}\right|\right) \mathrm{d}x \\ &= \mathrm{e}^t \int_0^{\mathrm{e}^{-t}} \varphi\left(\left|\frac{\mathrm{e}^{\gamma t} v(x)}{s}\right|\right) \mathrm{d}x = \mathrm{e}^{t(1+\gamma p)} \int_0^{\mathrm{e}^{-t}} \varphi\left(\left|\frac{v(x)}{s}\right|\right) \mathrm{d}x \\ &= \mathrm{e}^{t(1+\gamma p)} \rho_{[0,\mathrm{e}^{-t}]}\left(\frac{v}{s}\right).\end{aligned}$$

From the above we get

$$|T_t v|^F_{[0,1]} = \inf\left\{s > 0 : \int_0^1 \varphi\left(\left|\frac{T_t v(x)}{s}\right|\right) \mathrm{d}x \leqslant s\right\} = \mathrm{e}^{t(1+\gamma p)} |v|^F_{[0,\mathrm{e}^{-t}]}.$$

Due to $\mathrm{e}^{1+\gamma p} \leqslant 1$ one has $|v|^F_{[0,\mathrm{e}^{-t}]} \to 0$ as $t \to \infty$. This proves the strong stability of the system $(T_t)_{t \geqslant 0}$ in the $L^{\varphi}(0,1)$ space. □

4. GENERALIZATION OF THE VON FOERSTER–LASOTA EQUATION

Let us consider a more general form of the equation

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = \lambda(x)u, \quad t \geqslant 0,\ 0 \leqslant x \leqslant 1 \tag{4.1}$$

with the initial condition

$$u(0,x) = v(x), \quad 0 \leqslant x \leqslant 1, \tag{4.2}$$

where v belongs to some normed vector space V of functions defined on $[0,1]$ and $\lambda : [0,1] \to \mathbb{R}$ is a given continuous function. Let a semidynamical system $\widetilde{T}_t$ be given by the formula

$$(\widetilde{T}_t v)(x) = \widetilde{u}(t,x),$$

where $\widetilde{u}(t,x)$ is the unique solution of (4.1), (4.2) and is given by the formula

$$(\widetilde{T}_t v)(x) = \widetilde{u}(t,x) = \mathrm{e}^{g(x)} \mathrm{e}^{-g(x\mathrm{e}^{-t})} v(x\mathrm{e}^{-t}) \tag{4.3}$$

for $x \in [0,1]$, where

$$g(x) = -\int_x^1 \frac{\lambda(s)}{s} \mathrm{d}s.$$

We are interested in finding a connection between two equations: (3.1), presented in Section 3, and (4.1). It is easy to check that if u and $\widetilde{u}$ are the solutions of equations (3.1) and (4.1), respectively, we have the equality

$$u(t,x) = \kappa(x)\widetilde{u}(t,x), \tag{4.4}$$

where

$$\kappa(x) = \mathrm{e}^{\int_0^x \frac{\lambda(0)-\lambda(s)}{s} \mathrm{d}s} \quad \text{and} \quad \gamma = \lambda(0). \tag{4.5}$$

We assume that the above integral is convergent.

Definition 4.1. *Two dynamical systems $(F_t)_{t\geqslant 0}$ and $(G_t)_{t\geqslant 0}$, where $F_t : X \to X$ and $G_t : Y \to Y$, are topologically equivalent if there exists homeomorphism $h : X \to Y$, such that the following diagram is commutative*:

$$\begin{array}{ccc} X & \xrightarrow{F_t} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow[G_t]{} & Y. \end{array}$$

It means that $h \circ F_t = G_t \circ h$.

Remark 4.2. Many dynamical properties transfer by topological equivalence, for example, stability, the density of the set of periodic points, the existence of fixed points of the equation or its periodic orbits and many others.

The connection between the dynamical systems (3.1) and (4.1) described above can be illustrated by the following diagram:

$$\begin{array}{ccc} V & \xrightarrow{T_t} & V \\ {\scriptstyle m_\kappa}\downarrow & & \downarrow{\scriptstyle m_\kappa} \\ V & \xrightarrow[\widetilde{T}_t]{} & V. \end{array}$$

We can show that this diagram is commutative. Let $(m_\kappa v)(x) = \frac{1}{\kappa(x)} v(x)$, then

$$\begin{aligned}
\left(\widetilde{T}_t(m_\kappa v)\right)(x) &= \mathrm{e}^{g(x)}\mathrm{e}^{-g(x\mathrm{e}^{-t})} m_\kappa v(x\mathrm{e}^{-t}) = \mathrm{e}^{g(x)}\mathrm{e}^{-g(x\mathrm{e}^{-t})} \frac{1}{\kappa(x\mathrm{e}^{-t})} v(x\mathrm{e}^{-t}) \\
&= \exp\left(-\int_x^1 \frac{\lambda(s)}{s}\mathrm{d}s + \int_{x\mathrm{e}^{-t}}^1 \frac{\lambda(s)}{s}\mathrm{d}s - \int_0^{x\mathrm{e}^{-t}} \frac{\lambda(0)-\lambda(s)}{s}\mathrm{d}s\right) v(x\mathrm{e}^{-t}) \\
&= \exp\left(\int_{x\mathrm{e}^{-t}}^x \frac{\lambda(s)}{s}\mathrm{d}s - \int_0^x \frac{\lambda(0)-\lambda(s)}{s}\mathrm{d}s + \int_{x\mathrm{e}^{-t}}^x \frac{\lambda(0)-\lambda(s)}{s}\mathrm{d}s\right) v(x\mathrm{e}^{-t}) \\
&= \exp\left(-\int_0^x \frac{\lambda(0)-\lambda(s)}{s}\mathrm{d}s\right) \exp\left(\int_{x\mathrm{e}^{-t}}^x \frac{\lambda(s)}{s}\mathrm{d}s + \int_{x\mathrm{e}^{-t}}^x \frac{\lambda(0)-\lambda(s)}{s}\mathrm{d}s\right) v(x\mathrm{e}^{-t}) \\
&= \exp\left(-\int_0^x \frac{\lambda(0)-\lambda(s)}{s}\mathrm{d}s\right) \exp\left(\int_{x\mathrm{e}^{-t}}^x \frac{\gamma}{s}\mathrm{d}s\right) v(x\mathrm{e}^{-t}) \\
&= \frac{1}{\kappa(x)}\mathrm{e}^{\gamma t} v(x\mathrm{e}^{-t}) = \frac{1}{\kappa(x)}(T_t v)(x) = (m_\kappa(T_t v))(x).
\end{aligned}$$

Theorem 4.3. *Assume that*

$$\exists C>0 \quad \exists q>0 \quad \forall x\in[0,1] \quad |\lambda(0)-\lambda(x)| \leqslant Cx^q \tag{4.6}$$

holds. Then we have the following equivalence: *the function $\widetilde{u}$ belongs to the space $L^\varphi(0,1)$ if and only if $u \in L^\varphi(0,1)$.*

Proof. Assuming that $\widetilde{u} \in L^{\varphi}(0,1)$, we deduce

$$\begin{aligned}
\rho_{[0,1]}(\beta u) &= \int_0^1 \varphi(\beta|u(t,x)|)\mathrm{d}x = \int_0^1 \varphi(\beta|\kappa(x)\widetilde{u}(t,x)|)\mathrm{d}x \\
&\leqslant \int_0^1 \varphi\left(\beta \mathrm{e}^{\int_0^x \frac{|\lambda(0)-\lambda(s)|}{s}\mathrm{d}s}|\widetilde{u}(t,x)|\right)\mathrm{d}x \\
&\leqslant \int_0^1 \varphi\left(\beta \mathrm{e}^{\frac{C}{q}x^q}|\widetilde{u}(t,x)|\right)\mathrm{d}x \leqslant \int_0^1 \mathrm{e}^{\frac{Cp}{q}x^q}\varphi(\beta|\widetilde{u}(t,x)|)\,\mathrm{d}x \\
&\leqslant \mathrm{e}^{\frac{Cp}{q}}\int_0^1 \varphi(\beta|\widetilde{u}(t,x)|)\,\mathrm{d}x = \mathrm{e}^{\frac{Cp}{q}}\rho_{[0,1]}(\beta\widetilde{u}).
\end{aligned}$$

So $\rho_{[0,1]}(\beta u) \to 0$ as $\beta \to 0^+$. In the same manner we can establish the inverse implication. □

Due to the commutativity of the diagram we have the topological equivalence of the systems $(T_t)_{t\geqslant 0}$ and $(\widetilde{T}_t)_{t\geqslant 0}$. Therefore the properties of the system $(T_t)_{t\geqslant 0}$ described in Section 3 (the existence of the periodic points for any time, the density of the set of periodic points, transitivity and also stability) transfer on the system $(\widetilde{T}_t)_{t\geqslant 0}$. All these properties depend on the value $\gamma = \lambda(0)$. Moreover, Theorem 4.3 shows that the solutions of equations (3.1) and (4.1) stay in the same space $L^{\varphi}(0,1)$ under some assumptions concerning the function λ. It provides the property of sensitive dependence on initial conditions for the system $(\widetilde{T}_t)_{t\geqslant 0}$, which is a metric, not topological property. So if, by appropriate assumptions, the system $(T_t)_{t\geqslant 0}$ is chaotic or stable, or has got a dense subset of its periodic points, then the system $(\widetilde{T}_t)_{t\geqslant 0}$ has got exactly the same properties. Thus all properties of the system $(\widetilde{T}_t)_{t\geqslant 0}$ depend on the value $\gamma = \lambda(0)$. If $\lambda(0) > -\frac{1}{p}$, then for any t_0 the periodic solution exists and the set of periodic points (4.1) is dense in the $L^{\varphi}(0,1)$ space. If $\lambda(0) \leqslant -\frac{1}{p}$, then the system is strongly stable.

5. CONCLUSIONS

This work is the generalization of the results included in paper [2], which treat the existence of periodic solutions, problem of chaos, and stability of the von Foerster–Lasota equation in integrable spaces with the exponent p, where $1 \leqslant p < \infty$. The value of the coefficient γ is decisive. If $\gamma > -\frac{1}{p}$, then for any t_0 the periodic solution exists for which t_0 is its principal period. Under the same inequality the set of periodic points (3.1) is dense in the $L^p(0,1)$ space. If $\gamma \leqslant -\frac{1}{p}$, then the system is strongly stable. So all properties of the system $(T_t)_{t\geqslant 0}$, in this Banach space, are similar as in the Fréchet space presented above. Thus the results of this paper allow us to judge about asymptotic properties of the von Foerster–Lasota equation in any L^p space irrespective of the value of the exponent p.

ACKNOWLEDGEMENT

The second author acknowledges the support from Białystok Technical University (grant No. W/WI/8/07).

REFERENCES

1. Banks, J., Brooks, J., Cairns, G., Davis, G. and Stacey, P. On Devaney's definition of chaos. *Amer. Math. Monthly*, 1992, **99**, 332–334.
2. Brzeźniak, Z. and Dawidowicz, A. L. On the periodic solution of the Lasota equation. *Semigroup Forum* (submitted).
3. Dawidowicz, A. L. and Poskrobko, A. Age-dependent population dynamics with the delayed argument. *Progr. Nonlinear Differential Equations Appl.*, 2007, **75**, 165–174.
4. Devaney, R. L. *An Introduction to Chaotic Dynamical Systems*. Addison-Wesley, New York, 1989.

5. Gurtin, M. E. and MacCamy, R. C. Non-linear age-dependent population dynamics. *Arch. Rat. Mech. Anal.*, 1974, **54**, 281–300.
6. Lasota, A. Stable and chaotic solutions of a first order partial differential equation. *Nonlinear Anal.*, 1981, **5**, 1181–1193.
7. Lasota, A., Mackey, M. C and Ważewska-Czyżewska, M. Minimizing therapeutically induced anemia. *J. Math. Biology*, 1981, **13**, 149–158.
8. Lasota, A. and Pianigiani, G. Invariant measures on topological spaces. *Boll. Un. Mat. Ital.*, 1977, **15-B**, 592–603.
9. Lasota, A. and Szarek, T. Dimension of measures invariant with respect to Ważewska partial differential equation. *J. Differential Equations*, 2004, **196**, 448–465.
10. Łoskot, K. Turbulent solutions of first order partial differential equation. *J. Differential Equations*, 1985, **58**, 1–14.
11. McKendrick, A. G. Application of mathematics to medical problems. *Proc. Edin. Math. Soc.*, 1926, **44**, 98–130.
12. Musielak, J. *Orlicz Spaces and Modular Spaces*. Lecture Notes in Math., Vol. 1034, Springer-Verlag, Berlin, 1983.
13. Rudnicki, R. Invariant measures for the flow of a first order partial differential equation. *Ergodic Theory Dynam. Systems*, 1985, **5**, 437–443.
14. Ważewska-Czyżewska, M. and Lasota, A. Matematyczne problemy dynamiki układu krwinek czerwonych. *Roczniki PTM, Mat. Stos.*, 1976, **VI**, 23–40.

Von Foersteri–Lasota võrrandite kaootilisest ja stabiilsest käitumisest Orliczi ruumides

Antoni Leon Dawidowicz ja Anna Poskrobko

On uuritud von Foersteri–Lasota võrrandi kaootilist ja stabiilset käitumist Orliczi ruumides homogeense positiivset järku φ-funktsiooniga. On üldistatud von Foersteri–Lasota võrrandi asümptootilisi omadusi integreeruvates ruumides, kus eksponent p on suurem või võrdne ühega.