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MECHANICS

# Energy changes in elastic plates due to holes and cracks

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**Abstract.** The formation of cavities in stressed elastic plates causes changes in their energy. In particular, the change in energy due to the presence of a crack has been extensively studied over the past decades. The paper adds some comments on the old Griffith controversy and calculates then energy changes due to circular and elliptical flaws and, as a limiting case, due to a crack.

**Key words:** applied mechanics, Griffith controversy, energy changes, circular hole, elliptical hole, crack, $M$ integral.

## 1. INTRODUCTION

Originally, the motivation of the present author to study energy changes due to the formation of a circular hole in an elastic plate arose from a biological problem. It has been observed that crystals may form holes as they appear in biological systems such as the skeletal structure in echinoderms, e.g., sea urchins [1]. When the results [2] were presented at the EUROMECH Colloquium 478 on “Non-equilibrium Dynamical Phenomena in Inhomogeneous Solids” in Tallinn, 2006, a discussion ignited on a rather old problem. Briefly, using the stress analysis of Inglis [3] for an elliptical hole in an infinite plate under all-around tension $\sigma_0$, Griffith [4] calculated in 1920 the change in the strain energy $\Delta U$ in a cracked plate with crack length $2a$ as

$$\Delta U = \frac{\sigma_0^2 \pi a^2}{8G}(3-\kappa), \qquad (1920) \qquad (1)$$

with shear modulus $G$, $\kappa = 3-4\nu$ for plane strain, $\kappa = (3-\nu)/(1+\nu)$ for plane stress and Poisson’s ratio $\nu$.

In 1924, Griffith [5] revised this formula as

$$\Delta U = \frac{\sigma_0^2 \pi a^2}{8G}(1+\kappa) \qquad (1924) \qquad (2)$$

without giving any details of the calculation. It may be interesting to note that G. I. Taylor, who communicated Griffith’s first paper [4], already knew that Eq. (1) required correction. Nevertheless, he recommended publication since “the correction affects the numerical value …, but not their order of magnitude. The main argument is therefore not impaired …” ([4], p. 198, Note).

From these two pioneering papers a long discussion started, focussing not only on the question whether formula (1) or (2) is correct but also, more interestingly, *how* Griffith may have obtained the modification. A detailed historical review of various arguments on this controversy may be found in [6], especially in the article of Cotterell [7].

Sih and Liebowitz [8] found out that the resemblance of the expressions for the excess of strain energy due to a crack and a circular hole was remarkable. In order to explain the trap that Griffith fell into in his 1920 paper, they examined two problems. Firstly, they calculated the strain-energy density in an infinite plate with a circular hole of radius $r_0$, subjected to biaxial tensions $\sigma_{\mathrm{I}}$ and $\sigma_{\mathrm{II}}$ at infinity using Clapeyron's theorem (cf., e.g., [9]), subtracted from the result the strain-energy density of a plate without a hole under the same load, and arrived for uniform tension $\sigma_0$ at

$$\Delta U = \frac{\sigma_0^2 \pi r_0^2}{4G}(3-\kappa). \qquad (1920) \qquad (3)$$

Secondly, they treated the concentric-annulus problem under prescribed biaxial tractions at the outer boundary $r = R$ and calculated the strain energy, again with the use of Clapeyron's theorem. Then they subtracted the strain-energy density of a circular plate with the radius $R$. Passing in turn to the limit $R \to \infty$, they found for uniform tension

$$\Delta U = \frac{\sigma_0^2 \pi r_0^2}{4G}(1+\kappa). \qquad (1924) \qquad (4)$$

Equations (1) and (3), and (2) and (4) differ merely by a factor 2. The difference between (1) and (2), and (3) and (4) seem to result from the sequence of taking the limit $R \to \infty$ first and calculating the energy change in a second step or vice versa. However, the discrepancy arises from the fact that Sih and Liebowitz [8] applied Clapeyron's theorem [9] to a finite body (outer radius $R$), and then $R$ tended to infinity. Using this procedure they missed finite energy which is located exactly at infinity (L. Truskinowski, pers. comm. 2007). It is shown in [2] that Clapeyron's theorem is not applicable when the difference of energies involves different domains, i.e., infinite plates with and without a hole $r_0 \le r \le \infty$ and $0 \le r \le \infty$, respectively, are considered.

Several authors ([7,10]) have developed different but somehow related strategies to overcome the problem by correcting the stresses and displacements at infinity. In this contribution, we give a further variant that Griffith may have used and provide an elegant alternative to calculating energy changes due to cavities and cracks.

## 2. A CIRCULAR HOLE IN AN INFINITE PLATE

Let us consider an infinite elastic plate with a hole of radius $r_0$ under biaxial tensions $\sigma_{\mathrm{I}}$ and $\sigma_{\mathrm{II}}$ (Kirsch problem). In polar coordinates $(r, \varphi)$, with the origin at the centre of the hole (Fig. 1), the

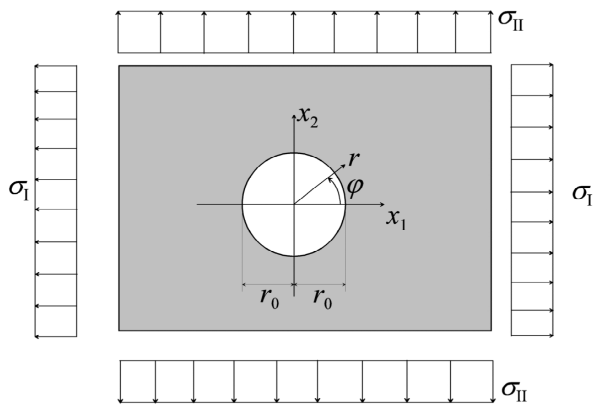


**Fig. 1.** Plate with a hole subjected to biaxial tension.

distribution of stresses and displacements (cf., e.g., [11]) are given for $r \geq r_0$ as

$$\sigma_{rr}^{(K)} = \frac{1}{2}(\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}})\left(1 - \frac{r_0^2}{r^2}\right) + \frac{1}{2}(\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})\left(1 - 4\frac{r_0^2}{r^2} + 3\frac{r_0^4}{r^4}\right)\cos 2\varphi, \tag{5a}$$

$$\sigma_{\varphi\varphi}^{(K)} = \frac{1}{2}(\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}})\left(1 + \frac{r_0^2}{r^2}\right) - \frac{1}{2}(\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})\left(1 + 3\frac{r_0^4}{r^4}\right)\cos 2\varphi, \tag{5b}$$

$$\sigma_{r\varphi}^{(K)} = -\frac{1}{2}(\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})\left(1 + 2\frac{r_0^2}{r^2} - 3\frac{r_0^4}{r^4}\right)\sin 2\varphi; \tag{5c}$$

$$u_r^{(K)} = +\frac{r}{8G}\left[(\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}})\left(\kappa - 1 + 2\frac{r_0^2}{r^2}\right) + 2(\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})\left(1 + (\kappa+1)\frac{r_0^2}{r^2} - \frac{r_0^4}{r^4}\right)\cos 2\varphi\right], \tag{6a}$$

$$u_\varphi^{(K)} = -\frac{r}{8G}2(\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})\left(1 + (\kappa-1)\frac{r_0^2}{r^2} + \frac{r_0^4}{r^4}\right)\sin 2\varphi. \tag{6b}$$

In the absence of the hole $(r_0 = 0)$, the undisturbed biaxial stress and displacement fields are for $r \geq 0$

$$\sigma_{rr}^{(0)} = \frac{1}{2}(\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}}) + \frac{1}{2}(\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})\cos 2\varphi, \tag{7a}$$

$$\sigma_{\varphi\varphi}^{(0)} = \frac{1}{2}(\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}}) - \frac{1}{2}(\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})\cos 2\varphi, \tag{7b}$$

$$\sigma_{r\varphi}^{(0)} = \sigma_{\varphi r}^{(0)} = -\frac{1}{2}(\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})\sin 2\varphi; \tag{7c}$$

$$u_r^{(0)} = \frac{r}{8G}[(\kappa-1)(\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}}) + 2(\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})\cos 2\varphi], \tag{8a}$$

$$u_\varphi^{(0)} = -\frac{r}{8G}2(\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})\sin 2\varphi. \tag{8b}$$

The (complementary) strain-energy density is given by

$$W^* = \frac{1}{16G}[(\kappa+1)(\sigma_{rr}^2 + \sigma_{\varphi\varphi}^2) + 2(\kappa-3)\sigma_{rr}\sigma_{\varphi\varphi} + 8\sigma_{r\varphi}^2], \tag{9}$$

and we obtain $W^{*(0)}$ by replacing $\sigma_{ij}$ by $\sigma_{ij}^{(0)}$ and $W^{*(k)}$ by replacing $\sigma_{ij}$ by $\sigma_{ij}^{(K)}$ $(i, j = r, \varphi)$. The strain-energy density $W$ written in terms of the strain tensor and the complementary strain-energy density $W^*$ written in terms of the stress tensor are numerically identical within the framework of the linear theory of elasticity and can be easily transferred into each other by Hooke's law without applying a Legendre transformation [9]. Therefore, in the following we will not distinguish between $W$ and $W^*$.

For later use we introduce the difference between (6a), (6b) and (8a), (8b) as

$$\Delta u_r = u_r^{(K)} - u_r^{(0)} = +\frac{r_0^2}{4Gr}\left[(\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}}) + (\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})\left(\kappa + 1 - \frac{r_0^2}{r^2}\right)\cos 2\varphi\right], \tag{10a}$$

$$\Delta u_\varphi = u_\varphi^{(K)} - u_\varphi^{(0)} = -\frac{r_0^2}{4Gr}(\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})\left(\kappa - 1 + \frac{r_0^2}{r^2}\right)\sin 2\varphi, \tag{10b}$$

and the hoop stress along the rim of the hole $r = r_0$ (Eq. 5b) as

$$\sigma_{\varphi\varphi}^{(KR)} = \sigma_{\rm I} + \sigma_{\rm II} - 2(\sigma_{\rm I} - \sigma_{\rm II})\cos 2\varphi. \qquad (11)$$

As has already been pointed out by Eshelby [12–14], the change in the total energy $\Delta\Pi$, i.e., the sum of the change in the strain energy $\Delta U$ and the change in the potential energy of the loading $\Delta P$ governs the process, and the change in the total energy $\Delta\Pi$ is always negative when holes or cracks are formed independently on whether the loading mechanism is fixed-load, fixed-grip or in between. ("It is therefore wrong to say, as it is often done, that introducing a crack into a body or lengthening an existing one decreases the elastic energy (here the strain energy), though, …this can hardly lead to anything worse than an error in sign, which is usually silently corrected by common sense" [14], p. 142.)

In calculating the change in the total energy $\Delta\Pi$ when a circular hole is formed under load we consider first a perfect plate, and apply the stresses $\sigma_{\rm I}$ and $\sigma_{\rm II}$ at infinity. During the whole process, $\sigma_{\rm I}$ and $\sigma_{\rm II}$ will not be changed (fixed-load condition). Now we cut out the hole and calculate the change in the strain energy $\Delta U$ (cf. [2,8]):

$$\Delta U = \int_{r_0}^{\infty}\int_0^{2\pi} W^{(K)} r\mathrm{d}r\mathrm{d}\varphi - \int_0^{\infty}\int_0^{2\pi} W^{(0)} r\mathrm{d}r\mathrm{d}\varphi = \frac{\pi r_0^2}{16G}[(\sigma_{\rm I} + \sigma_{\rm II})^2(3-\kappa) + 2(\sigma_{\rm I} - \sigma_{\rm II})^2(\kappa - 1)]. \quad (1920) \qquad (12)$$

It may be noted that the calculation is straightforward but rather cumbersome.

For a rotational symmetric stress state $\sigma_{\rm I} = \sigma_{\rm II} = \sigma_0$, relation (3) is recovered as expected. Due to the removal of material, the plate is weakened and the load application point (even though at infinity) moves by an amount $\Delta u_i\,(i = r, \varphi)$. With (10a), (10b) the potential of the external forces $P$ thus changes to (cf. [2])

$$\Delta P = -\lim_{r\to\infty}\int_0^{2\pi}(\sigma_{rr}^{(0)}\Delta u_r + \sigma_{r\varphi}^{(0)}\Delta u_\varphi)\, r\mathrm{d}\varphi = -\frac{\pi r_0^2}{4G}[(\sigma_{\rm I} + \sigma_{\rm II})^2 + (\sigma_{\rm I} - \sigma_{\rm II})^2\kappa]. \qquad (13)$$

On adding (12) and (13) we arrive at the correct result for the change in the total energy $\Delta\Pi$:

$$\Delta\Pi = -\frac{\pi r_0^2(\kappa+1)}{16E}[(\sigma_{\rm I} + \sigma_{\rm II})^2 + 2(\sigma_{\rm I} - \sigma_{\rm II})^2], \quad (1924) \qquad (14)$$

and recover Eq. (4) for $\sigma_{\rm I} = \sigma_{\rm II} = \sigma_0$ (with the "silently corrected" sign).

It might be that Griffith thought along these lines when he corrected his result in the paper published in 1924.

## 3. PATH-INDEPENDENT INTEGRALS AND ENERGY CHANGES

The first correct published derivation of Eq. (2) is probably due to Sneddon [15]. He considered first a plate without a crack and calculated constant tractions along a line where the crack would happen to occur. Next he evaluated the work of these tractions as they would relax during the formation of an elliptical crack opening. This method was also applied in [2] for the formation of a circular hole leading to a correct result (14). It turned out that the algebra is less involved in comparison with Griffith's method, and instead of knowing the complete state of stresses and displacements in the plate with a hole, merely the displacements along the rim of the hole $u_i^{(K)}(r_0, \varphi)$ must be given.

An even more elegant derivation uses the energy-release rate for a self-similar expansion of a defect calculated from the path-independent $M$ integral [16–18]:

$$\mathrm{M} = \oint_\Gamma x_\alpha(W\delta_{\beta\alpha} - \sigma_{\beta\gamma}u_{\gamma,\alpha})n_\beta \mathrm{d}s, \qquad (15)$$

with an arbitrary integration path $\Gamma$ (arc length $\mathrm{d}s,$ unit outward normal vector $n_\beta$) surrounding the defect, Kronecker's tensor of unity $\delta_{\alpha\beta},$ and displacement gradient $u_{\gamma,\alpha}.$ The summation convention $(\alpha, \beta, \gamma = 1, 2)$ is implied for repeated indices. Due to the self-similar expansion of the hole $r_0 \to \alpha r_0,$ the energy of the system changes and the energy release rate is calculated as

$$\frac{\partial \Pi}{\partial \alpha} = r_0 \frac{\partial \Pi}{\partial r_0} = -M. \tag{16}$$

It follows that the change in energy due to the growth of the hole radius from zero to $r_0$ is

$$\Delta\Pi = -\int_0^{r_0} \frac{M(\bar{r}_0)}{\bar{r}_0} \mathrm{d}\bar{r}_0. \tag{17}$$

If the integration path $\Gamma$ coincides with the contour of a traction-free circular hole, Eq. (15) is simplified to

$$M = \frac{r_0^2(\kappa+1)}{16G} \int_0^{2\pi} \sigma_{\varphi\varphi}^2 \mathrm{d}\varphi. \tag{18}$$

Before starting the analysis it can be mentioned that Bilby and Eshelby [14] used a different approach by employing path-independent integrals based on Betti's reciprocity theorem.

By replacing in Eq. (18) $\sigma_{\varphi\varphi}$ by $\sigma_{\varphi\varphi}^{(KR)}$ (Eq. (11)), it is rather easy to calculate the $M$ integral for the Kirsch problem

$$M = \frac{\pi r_0^2(\kappa+1)}{8G}[(\sigma_\mathrm{I} + \sigma_\mathrm{II})^2 + 2(\sigma_\mathrm{I} - \sigma_\mathrm{II})^2], \tag{19}$$

and by evaluating (17) it follows immediately that

$$\Delta\Pi = -\frac{\pi r_0^2(\kappa+1)}{16G}[(\sigma_\mathrm{I} + \sigma_\mathrm{II})^2 + 2(\sigma_\mathrm{I} - \sigma_\mathrm{II})^2], \tag{20}$$

which is identical to Eq. (14). Only the knowledge of the hoop stress along the rim of the hole is required to calculate the "energy of the hole".

Similarly we treat the elliptical hole problem. The solution involves either a formulation in elliptical coordinates [3,19] and/or complex potentials combined with the conformal mapping technique [20]. The formulae of the complete solution are rather intricate and unwieldy, the expression of the stress distribution along the rim of the cavity, however, is rather handy and may be found in textbooks on elasticity or fracture (e.g., [21,22]).

Consider an infinite elastic sheet with an elliptical cavity with semi-axes $a$ and $b$ subjected to a state of biaxial stresses $\sigma_\mathrm{I}$ and $\sigma_\mathrm{II}$ at infinity as depicted in Fig. 2. With

$$c = \frac{a-b}{a+b}, \tag{21a}$$

$$\tan\vartheta = \frac{1-c}{1+c}\tan\varphi, \tag{21b}$$

the boundary stress $\sigma_{\vartheta\vartheta}$ as function of the angle $\varphi$ is [22]

$$\sigma_{\vartheta\vartheta} = \frac{(1-c^2)(\sigma_\mathrm{I} + \sigma_\mathrm{II}) + 2(c - \cos 2\varphi)(\sigma_\mathrm{I} - \sigma_\mathrm{II})}{1 + c^2 - 2c\cos 2\varphi}, \tag{22}$$

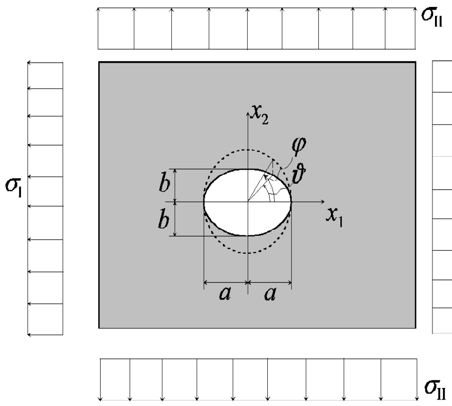


**Fig. 2.** A plate with an elliptical hole subjected to biaxial tension.

and in order to evaluate $M$ from Eq. (18), we have with (21b)

$$M = \frac{a^2(\kappa+1)}{16G}\int_0^{2\pi}\sigma_{\vartheta\vartheta}^2 \mathrm{d}\vartheta = \frac{a^2(\kappa+1)}{16G}\frac{1-c}{1+c}\int_0^{2\pi}\sigma_{\vartheta\vartheta}^2 \mathrm{d}\varphi$$

$$= \frac{a^2(\kappa+1)}{16G}\frac{1-c}{1+c}\int_0^{2\pi}\left[\frac{A}{(\delta-\cos 2\varphi)^2}+\frac{B}{(\delta-\cos 2\varphi)}+C\right]\mathrm{d}\varphi, \quad (23)$$

with the abbreviations

$$\delta = \frac{1+c^2}{2c}, \quad (24a)$$

$$A = \left(\frac{1-c^2}{2c}\right)^2\left[(\sigma_I+\sigma_{II})^2 - \frac{2}{c}(\sigma_I^2-\sigma_{II}^2)+\frac{1}{c^2}(\sigma_I-\sigma_{II})^2\right], \quad (24b)$$

$$B = \frac{1-c^2}{c^2}\left[(\sigma_I^2-\sigma_{II}^2)-\frac{1}{c}(\sigma_I-\sigma_{II})^2\right], \quad (24c)$$

$$C = \frac{1}{c^2}(\sigma_I-\sigma_{II})^2. \quad (24d)$$

The integrals occurring in (23) are tabulated (e.g., [23]), and the result of the integration is

$$M = \frac{2\pi a^2(\kappa+1)}{16G(1+c)^2}[(1+c^2)(\sigma_I+\sigma_{II})^2 - 4c(\sigma_I^2-\sigma_{II}^2)+2(\sigma_I-\sigma_{II})^2]. \quad (25)$$

To arrive at the change in the total energy, in Eq. (17) the integration has to be performed from $0$ to $a$, whilst the ratio $a/b$, i.e. $c$, has to be kept constant (self-similar expansion):

$$\Delta\Pi = -\int_0^a \frac{M(\bar{a})}{\bar{a}}\mathrm{d}\bar{a}\Big|_{c=\text{const.}}. \quad (26)$$

Together with (25) this leads to

$$\Delta\Pi = -\frac{1}{2}M, \quad (27)$$

and on replacing $c$ by (21a) we arrive at

$$\Delta\Pi = -\frac{\pi a^2(\kappa+1)}{32G}\left[(\sigma_{\mathrm{I}}+\sigma_{\mathrm{II}})^2\left(1+\frac{b^2}{a^2}\right)-2(\sigma_{\mathrm{I}}^2-\sigma_{\mathrm{II}}^2)\left(1-\frac{b^2}{a^2}\right)+(\sigma_{\mathrm{I}}-\sigma_{\mathrm{II}})^2\left(1+\frac{b}{a}\right)^2\right]. \quad (1924) \qquad (28)$$

The result coincides with that given in [8]. As a special case we recover the “energy of the circular hole” (14) by setting $a=b=r_0$, and as a special case for the crack we find with $b=0$ the correct result for the “energy of the crack”:

$$\Delta\Pi = -\frac{\pi a^2\sigma_{\mathrm{II}}^2(\kappa+1)}{8G}, \quad (1924) \qquad (29)$$

i.e., $\Delta\Pi$ is not affected by the applied stress in line with the crack edges.

If the ellipsoidal cavity is subjected to simple shear $\tau$ at infinity, we receive the energy change from (28) by setting $\sigma_{\mathrm{I}}=+\tau$ and $\sigma_{\mathrm{II}}=-\tau$ with the result

$$\Delta\Pi = -\frac{\pi a^2\tau^2(\kappa+1)}{8G}\left(1+\frac{b}{a}\right)^2, \qquad (30)$$

which leads with $b=0$ to the correct expression for the “energy of the crack” subjected to far-field simple shear.

## 4. CONCLUSIONS

From the various possibilities of calculating the change in energy due to the formation of cavities in an infinite plate, the employment of the path-independent $M$ integral seems to be the most elegant one. After contributing some remarks to the old Griffith controversy, the energy of a circular hole can be calculated with a remarkably low expenditure of algebra. Even for the elliptical cavity the algebra is rather tractable. If the $M$ integral would have been at hand in 1920, the Griffith controversy would probably never had come into existence.

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## Elastsete plaatide potentsiaalenergia muutumine aukude ja pragude toimel

Reinhold Kienzler

Õõnsuste teke pingestatud elastsetes plaatides põhjustab nendes potentsiaalenergia muutusi. Eriti ulatuslikult on potentsiaalenergia muutumist viimastel kümnenditel uuritud pragude olemasolu korral. Artiklis on toodud esiteks mõned selgitused tuntud Griffithi vastuolu kohta ja siis arvutatud potentsiaalenergia muutused ringikujulise ning elliptilise voo ja piirväärtusena ka prao korral.