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MATHEMATICS

# The life and work of Olof Thorin (1912–2004)

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**Abstract.** This paper reviews Olof Thorin's contributions to mathematical analysis, actuarial mathematics, and probability theory, though in reversed order. In probability theory he is known for his path-breaking work on infinite divisibility. In actuarial mathematics he contributed significantly to the ruin problem. However, his international fame very much relies on his work in mathematical analysis and his share in the Riesz–Thorin theorem. Data about his life and some personal recollections are also given.

**Key words:** actuarial mathematics, infinite divisibility, interpolation spaces, Riesz–Thorin theorem, ruin problem.

## 1. INTRODUCTION

Olof Thorin
(1912–2004)

Olof Thorin was never in the service of a university and worked as an actuarial mathematician most of his life. Still, the future will perhaps show that he was one of the best-known Swedish *pure* mathematicians in the second half of the 20th century. His international fame mainly relies on his share in the so-called Riesz–Thorin theorem.

Searching in MathSciNet for "Thorin", we got 13 hits, all concerning his stochastic activities only, which shows his influence also in this area. Searching for "Anywhere Thorin", one gets the score 173, with "Anywhere Riesz–Thorin", 113. Not a bad outcome!

In Sections 2 and 3, Thorin's achievements in the stochastic theatre are set forth by Bondesson and Grandell. In Section 4, Peetre reviews Thorin's contributions to mathematical analysis, giving first also some additional information about his life and death.

## 2. OLOF THORIN'S GENERALIZED Γ-CONVOLUTIONS

(Lennart Bondesson)

At the end of the 1970s Olof Thorin published four papers on infinite divisibility of probability distributions. He introduced and developed a new technique that turned out to be very powerful. The background, Thorin's contributions, and later developments are presented here.

### 2.1. Background

Infinite divisibility of probability distributions was introduced by de Finetti in 1929. A distribution with probability density $f$ is infinitely divisible if, for any $n \geq 1$, it can be written as a convolution $f = f_n * f_n * \cdots * f_n$ with $n$ components. A random variable $X$ with an infinitely divisible distribution can be represented as a sum of any number of independent and identically distributed variables $X_{n1}, X_{n2}, \ldots, X_{nn}$. The theory of infinite divisibility was developed in the 1930s by celebrities like Kolmogorov and Lévy in their study

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of stochastic processes. A general formula for the characteristic function (i.e. the Fourier transform) of an infinitely divisible distribution was derived early by Lévy (see, e.g., Feller [1], Chapter 17). However, the formula is not very useful in deciding whether or not a given probability density is infinitely divisible. In the late 1960s and early 1970s some criteria that were sufficient for a density to be infinitely divisible were developed. In particular, a density on $(0,\infty)$ is infinitely divisible if it is completely monotone, i.e. the signs of the derivatives $f^{(n)}, n=0,1,2,\ldots$, alternate (Goldie–Steutel theorem). But the class of such densities is not sufficiently rich to cover many probability distributions that appear in practice. In particular, the lognormal distribution is not in this class. In a survey Steutel [2] mentioned the infinite divisibility of the lognormal distribution as an open problem.

### 2.2. Thorin's contributions

Since the lognormal distribution with probability density

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma x}\exp\left(-\frac{1}{2}\left(\frac{\log x-\mu}{\sigma}\right)^2\right),$$

$$x>0,\ (\mu\in\mathbb{R},\sigma>0)$$

has been used to model annual claims on insurance companies, it was natural for the actuary Thorin to try to verify its infinite divisibility. The annual claims can be seen as sums of many independent partial claims. He succeeded to show that something much stronger than infinite divisibility holds for the lognormal distribution [3]. A gamma distribution with density $g(x)=(\Gamma(\beta))^{-1}\lambda^\beta x^{\beta-1}e^{-\lambda x}$, $x>0$ $(\beta,\lambda>0)$, has the *moment generating function* (mgf) $\phi(s)=\int_0^\infty e^{sx}g(x)\mathrm{d}x=(\lambda/(\lambda-s))^\beta$. The distribution is trivially infinitely divisible. If we convolve different gamma distributions and take limits, as Thorin did, we get a class of infinitely divisible distributions with mgf of the form (with $t$ instead of $\lambda$)

$$\phi(s)=\exp\left(as+\int_0^\infty\log\left(\frac{t}{t-s}\right)U(\mathrm{d}t)\right),\quad \Re(s)\le 0,$$

where $U(\mathrm{d}t)$ is a nonnegative measure and $a\ge 0$. Equivalently,

$$\phi'(s)/\phi(s)=a+\int_0^\infty\frac{1}{t-s}U(\mathrm{d}t). \tag{2.1}$$

Except for the minus sign in front of $s$, the right-hand side is the Stieltjes transform of a nonnegative measure. For a gamma distribution, the measure $U(\mathrm{d}t)$ has all its mass at one single point. These limit distributions were called generalized $\Gamma$-convolutions by Thorin. We also use the abbreviation GGC for them.

For the lognormal distribution, the mgf $\phi(s)=\int_0^\infty e^{sx}f(x)\mathrm{d}x$ is not explicit. But via Cauchy's integral formula for analytic functions, Thorin realized that to prove that the distribution is a GGC it suffices to verify that $\phi(s)$ can be analytically continued to a function that is defined in the cut complex plane $\mathbb{C}\setminus[0,\infty)$ and has no zeros there and is such that $\arg(\phi(s))$ is nondecreasing as $s$ increases along the upper side of the positive real line. Then

$$U(t)=\int_0^t U(\mathrm{d}t')=\pi^{-1}\arg(\phi(t)).$$

He was successful, but there were many obstacles for him to pass or often pass around with technical skill. His proof is very long and only a little about all the details can be mentioned here. One difficulty is that $U(\infty)=\infty$, which led him to first approximate the lognormal distribution with a distribution that corresponds to a product of a lognormal random variable and an independent gamma-distributed variable with integer parameter $\beta$. To show that $\arg(\phi(s)), s>0$, increases, he used and, ultimately, verified a very special condition, namely that, for each $c>0$, the function

$$h(x)=\exp\left(-c\,(\log(x+1+\sqrt{x^2+2x}))^2\right),\ x>0,$$

is completely monotone. This condition did not appear to be very central. According to a well-known theorem by Bernstein, a function is completely monotone if and only if it is the Laplace transform of a nonnegative measure (Feller [1], Chapter 13).

The lognormal paper was a very strong piece of research. Thorin, who was persistent, got fascinated by a difficult problem, which he could not leave until he had solved it. The paper was not his first one in the field. Somewhat earlier he had written a simpler one [4] about the Pareto distribution with density $f(x)=C(1+cx)^{-\gamma}, x>0$, where $C$ is a normalizing constant. Also this distribution was verified to be a GGC. Thorin's lognormal result meant that an important step forward was taken in the theory of infinite divisibility. It turned out that his technique could be generalized to give much more general results. In [5] he generalized the result to cover distributions corresponding to powers $X^q$, $|q|\ge 1$, of gamma variables $X$. Among other things, he had to use and prove that the function

$$\begin{aligned}h(x)=&\exp\Big(-c\,((x+1+\sqrt{x^2+2x})^\alpha\\&+(x+1+\sqrt{x^2+2x})^{-\alpha})\Big),\ x\ge 0,\end{aligned}$$

is completely monotone for all $c>0$ and $\alpha$ $(\alpha=1/q)$ such that $0<|\alpha|\le 1$.

In [6] Thorin also extended the GGC class to include distributions on the whole real line $\mathbb{R}$. However, there is no doubt that the lognormal paper was the most significant one of the four papers. In the literature the GGC class is also called the *Thorin class* of distributions and the measure $U$ is called the *Thorin measure*.

### 2.3. Later developments

Thorin's original theory was developed considerably during the 1980s. The booklet by Bondesson [7] and the recent book by Steutel and van Harn [8] describe much of what happened. Developments somewhat related to financial mathematics are described in, e.g., Barndorff-Nielsen et al. [9].

It turned out that Thorin's complete monotonicity condition, which for him was one of several conditions to check, was the only one that had to be verified. This became clear at the end of the 1980s. The following definition is central.

**Definition 1.** *A positive function* $f$ *on* $(0,\infty)$ *such that, for each* $u>0$, $f(uv)f(u/v)$ *is completely monotone as a function of* $w=v+v^{-1}$, *is called hyperbolically completely monotone* (*HCM*).

Actually, the condition is the Thorin condition for both the lognormal distribution and for the distribution of powers of gamma variables, as can be seen by putting $w=2(x+1)$. It is also easy to see that the functions

$$x^{\beta-1}\ (\beta\in\mathbb{R}),\ e^{-cx}\ (c>0),$$

and

$$(1+cx)^{-\gamma}\ (c>0,\gamma>0)$$

are HCM. Since complete monotonicity is preserved under multiplication and pointwise limits, it follows that also the HCM-class is closed with respect to multiplication and limits. In particular it follows that all functions of the form

$$f(x)=Cx^{\beta-1}\prod_{i=1}^{N}(1+c_i x)^{-\gamma_i},\quad x>0, \tag{2.2}$$

and limits thereof are HCM. It can be verified that the lognormal density is such a limit. It is also possible to verify, though it is more difficult, that all HCM-functions are limits of functions of the type (2.2).

A main theorem is the following result.

**Theorem 1.** *A probability density* $f$ *that is HCM is a GGC and hence infinitely divisible.*

This theorem is proved in [7]. It is accompanied there by a heuristic proof that can briefly be described as follows. Put for $s\in\mathbb{C}$, $\Re(s)\le 0$,

$$J(s)=\phi'(s)\overline{\phi(s)}=\int_0^\infty\int_0^\infty xe^{sx+\bar{s}y}f(x)f(y)\mathrm{d}x\mathrm{d}y.$$

Via the hyperbolic substitution $x=uv, y=u/v$ and a Bernstein representation $f(uv)f(u/v)=\int_{[0,\infty)}\exp(-u\lambda w)K(\mathrm{d}\lambda;u)$, where $K(\mathrm{d}\lambda;u)$ is a non-negative measure for each $u>0$, some manipulations, and the very formal substitution $v=u(\lambda-\bar{s})\rho$ (with $\rho>0$), one can see that

$$J(s)=\int_0^\infty 2u^2\int_0^\infty\left(\int_0^\infty u(\lambda-\bar{s})\right.\\ \left.\times\exp\left(-u^2|\lambda-s|^2\rho-\frac{1}{\rho}\right)\mathrm{d}\rho\right)K(\mathrm{d}\lambda;u)\mathrm{d}u.$$

The imaginary part of the integrand is positive in the upper half-plane. Neglecting the earlier restriction $\Re(s)\le 0$, we then have that $\phi'(s)/\phi(s)$ is a function with positive imaginary part in the upper half-plane. By a theorem of Pick and Nevanlinna from complex analysis, it is then representable in the form (2.1). QED

Now let $X$ be a random variable with density $f$. We write $X\sim$ HCM if $f$ is HCM. A remarkable result is that the HCM-class is closed with respect to multiplication and division of random variables.

**Theorem 2.** *If* $X\sim$ *HCM and* $Y\sim$ *HCM are independent random variables, then* $XY\sim$ *HCM,* $X/Y\sim$ *HCM, and* $X^q\sim$ *HCM for* $|q|\ge 1$.

By multiplying many independent gamma-distributed random variables and taking limits, we can get the lognormal distribution. Since the gamma densities are HCM, Thorin's GGC-result for the log-normal distribution is also obtained via Theorems 1 and 2.

We now look at the functions in (2.2). If the factor $Cx^{\beta-1}$ is neglected, the functions are indeed Laplace transforms of GGCs. There is also the following stronger result.

**Theorem 3.** *A function* $f$ *is HCM with* $f(0)=1$ *if and only if* $f$ *is the Laplace transform of a GGC, i.e. if and only if* $f(-s)$ *is the mgf of a GGC.*

The result provides a possibility for deciding whether a distribution is a GGC with *real* methodology. For instance, a stable distribution with Laplace transform $f(s)=\exp(-cs^\alpha)$, $0<\alpha\le 1$, is a GGC since $f$ is HCM. It can also be shown via Theorem 3 that if $X\sim$ GGC and $Y\sim$ HCM are independent random variables, then $XY\sim$ GGC. A difficult open problem is whether also the GGC-class is closed with respect to multiplication of independent random variables. Another open problem is whether, for $X\sim$ GGC, each power $X^q$, $q\ge 1$, also has its distribution in the GGC class.

Thorin's remarkable complete monotonicity condition started all this.

### 2.4. Final comments

Olof Thorin and I corresponded from 1976 onwards. He wrote very kind and somewhat ceremonious letters. When he retired in 1977 he paid attention to, among other things, Riemann's hypothesis about the location of the zeros of the $\zeta$-function. At the end of the 1980s he gave a GGC-formulation of the hypothesis that is reproduced in an HCM-version in [7], p. 93. The only thing one has to do to prove the hypothesis is to check that the real function

$$f(s) = \frac{\xi(\frac{1}{2})}{\xi(\frac{1}{2}+\sqrt{s})}$$

is HCM, where

$$\xi(z) = \frac{1}{2}z(z-1)\Gamma\left(\frac{z}{2}\right)\pi^{-z/2}\zeta(z)$$

is Riemann's $\xi$-function. However, that is not so simple to do. In his last letter to me Olof wrote that he then mainly spent his time solving crosswords. In 2004, the year when Olof died, the big book by Steutel and van Harn [8] appeared. It devotes much space to generalized $\Gamma$-convolutions and HCM-densities. But Olof never saw that book.

## 3. RUIN THEORY

(Jan Grandell)

In risk theory, or more precisely, collective risk theory, one considers a model for the development of risk business of an insurance company. The first attempt goes back to Lundberg [10], while the works of Lundberg [11] and Cramér [12] can be seen as introductions to today's theory. These papers appeared before the theory of stochastic processes was developed. They must therefore be viewed as pioneering works, not only in risk theory, but also in the general theory of stochastic processes. In Cramér [13] a stringent treatment is given, based on Wiener–Hopf methods.

The foundations of the classical risk model are provided by the following independent quantities:

(i) a Poisson process $N = \{N(t);\ t \geq 0\}$ with intensity $\lambda$;
(ii) a sequence $\{Z_k\}_1^\infty$ of independent and identically distributed positive random variables with distribution function $F$ and mean value $\mu$.

The *risk process* $X(t)$ is defined by

$$X(t) = ct - \sum_{k=1}^{N(t)} Z_k,$$

where $c$ is a positive real constant. In the model $N(t)$ is interpreted as the number of damages in the time interval $(0,t]$. At each jump of $N(t)$ the company has to pay a random amount $Z_k$. In compensation the company obtains a premium $c > \mu\lambda$ per time unit.

The *ruin probability* $\Psi(u)$ of a company facing the risk process $X(t)$ and having initial capital $u$ is defined by

$$\Psi(u) = P\{u + X(t) < 0 \text{ for some } t > 0\}.$$

Let $h(r) = \int_0^\infty (e^{rz}-1)\,\mathrm{d}F(z)$, and assume that there exists $r_\infty$, $0 < r_\infty \leq \infty$, such that $h(r) \uparrow \infty$ when $r \uparrow r_\infty$. A classical result in the Poisson case, which goes back to Lundberg [11] and Cramér [12], is the *Cramér–Lundberg* approximation

$$\lim_{u\to\infty} e^{Ru}\Psi(u) = C, \tag{3.1}$$

where the *Lundberg exponent* $R$ is the positive solution $r$ of the equation $\lambda h(r) = cr$ and $C = (c-\lambda\mu)/(\lambda h'(R) - c)$.

### 3.1. Olof Thorin's papers in risk theory

At the end of the 1960s Olof got a position as a research actuary at the company Trygg-Hansa. His principal obligation was to follow research in domains of interest for insurance business. His first paper related to insurance was [14], which treated certain auxiliary functions introduced by Cramér.

However, Olof's principal research concerned a generalization of risk theory to the case where the damages occur according to a renewal process. This means that the times between the damages are independent and identically distributed but their distribution need not be exponential as in the Poisson process case. The first treatment of the ruin problem in the renewal process case is due to Sparre-Andersen [15].

Now let $N$ be a renewal process with inter-occurrence time distribution function $K$ and let $S_k$ denote the epoch of the $k$th claim. Then the variables $S_1, S_2 - S_1, S_3 - S_2, \ldots$ are independent and $S_2 - S_1, S_3 - S_2, \ldots$ have the distribution function $K$. The process $N$ is called an *ordinary* renewal process if $S_1$ has the distribution function $K$ as well. The process $N$ is called a *stationary* renewal process if $K$ has finite mean $1/\lambda$ and if the distribution function $K_0$ of $S_1$ is given by

$$K_0(t) = \lambda \int_0^t (1 - K(s))\,\mathrm{d}s.$$

Let $\hat{k}(v) = \int_0^\infty e^{-vs}\,\mathrm{d}K(s)$ be the Laplace–Stieltjes transform of $K$. In the renewal case the Lundberg exponent $R$ is the positive solution $r$ of

$$(h(r)+1)\hat{k}(cr) = 1.$$

Thorin succeeded to show that the Cramér–Lundberg approximation (3.1) holds, both in the ordinary and the stationary case, with the same $R$ but with different constants $C$. It may be noticed that the Poisson process is the only renewal process for which those two cases coincide. In no case the constant $C$ is as explicit as in the Poisson case.

From 1970 onwards he published a number of papers on this topic. I restrict myself to mentioning here his concluding survey [16]. Olof took a keen interest not only in theory, but wrote also papers on numerical computation of ruin probabilities – most of them in cooperation with Nils Wikstad (see, e.g., [17]).

### 3.2. Personal recollections of Olof Thorin

From the end of the 1960s until the late 1990s Olof and I met rather regularly when he visited seminars at Stockholm University or at The Royal Institute of Technology. Despite the difference in age we became very good friends. Often Olof is pictured as an "elegant" elderly gentleman. This may be true, but he was also a very warm and considerate person.

## 4. OLOF THORIN AS AN ANALYST

(Jaak Peetre)

### 4.1. Personalia

I recall some salient facts about Olof Thorin's life, basing myself on a letter that he wrote to me and which is reproduced *in extenso* in [18].

Olof Thorin was born in Halmstad on 23 February 1912. After graduating from the gymnasium in 1929, he began, in the autumn of the same year, to study at Lund University. In 1933 he got the degree of "Fil.kand." (Candidate of Philosophy; subjects: mathematics, mechanics, mathematical statistics). He continued his postgraduate studies under the colourful Hungarian Marcel Riesz (see [19]) as advisor. Riesz assigned Thorin the task of looking for various extensions of his celebrated "Convexity Theorem" (nowadays called an interpolation theorem; see below). This led first (1937) to the degree of "Fil.lic." (Licentiate of Philosophy) for which he had to write a thesis, [20], and later (1948) also to a PhD thesis, [21]. It is on the results of these two theses that Olof Thorin's renown resides. In 1937 Thorin got a job at an insurance company. In 1946 he married. When he retired, in 1977, he is rumoured to have said the words: "Finally I am free to devote myself to mathematics!"

I met the Thorins only once on the occasion of ICM 70, right after our arrival at Nice airport (apparently we had been on the same plane). To the extent that I recall, they were both rather tall persons. In my memory I placed Olof in the category of "elderly distinguished gentlemen". We had a short discussion. He asked me how his result stood after so many years; diffident as I am, I could not sufficiently emphasize that he was a celebrity. But I am proud that I mentioned him in my talk [22]. Around 1970 I began to assemble material for an historical essay on interpolation. I received then the letter from him that became the basis of my later paper [18]. I had also replies from Mischa Cotlar and Antoni Zygmund. However, the latter said that he had forgotten most of what I had asked him about but suggested that I also contact Alberto Calderón, a piece of advice that I regretfully did not follow – and now it is too late. (Moral: Old mathematicians should always be interviewed before it is too late.) I also interrupted my project, to be continued only in 2000. Then I wrote again to Thorin asking for his consent to reproduce his letter; this he approved, and he was also most helpful in my work. We then became on quite friendly terms, exchanging at least season's greetings at the end of each year.

In the autumn of 2002 the couple was exposed to a great misfortune. Olof's wife Ingegerd (b. 1 April 1916) was taken to a convalescent home. Not long after, he himself had a fracture of the neck of his femur.

My last letter to Olof Thorin was written on 11 March 2003. There I suggested that I might visit him when I came to Stockholm next time. Reconsidering, I realize that this was rather tactless as we were not so close to each other, and he was old and in a difficult situation. Quite right! In his reply of 7 April 2003 he declines my visit. Quote: "…taking account of the primitive conditions in which I now live, I suggest that we postpone this meeting …". Further: "Regretfully, there is no other relative who could visit her more regularly." All this points to a rather isolated elderly couple.

Olof Thorin died on 14 February 2004, at Danderyd Hospital after, I believe, a short illness. Ingegerd died a year later. The Thorins had no children, which always is seen as a great misfortune, so also in their case.

### 4.2. The Riesz–Thorin theorem

At last, let us pass to (pure!) mathematics. This theorem can be formulated as follows (here I am following [23], pp. 2–3, with minor variations in the notation).

**Theorem** (Riesz–Thorin theorem). *Let $U$ and $V$ be two measure spaces equipped with positive measures $\mu$ and $\nu$. Moreover, let $L^p(\mu)$ and $L^q(\nu)$ denote* complex *Lebesgue spaces with respect to suitable positive measures $\mu$ and $\nu$ on $U$ and $V$, respectively. Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and assume that $T$ is a linear operator, such that*

$$T : L^{p_0}(\mu) \to L^{q_0}(\nu)$$

*with norm $M_0$ and*

$$T : L^{p_1}(\mu) \to L^{q_1}(\nu)$$

*with norm $M_1$. Then*

$$T : L^{p}(\mu) \to L^{q}(\nu)$$

*with norm*

$$M \leq M_0^{1-\theta} M_1^{\theta}, \tag{4.1}$$

*where $0 < \theta < 1$ and*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{4.2}$$

Clearly (4.1) means that the norm of $T$ is *logarithmically convex.*

The theorem was proved by Riesz [24] in the *real case*, however, subject to the restrictions $p_0 \leq q_0$ and $p_1 \leq q_1$. (In that case the three points $(p_0^{-1}, q_0^{-1})$, $(p_1^{-1}, q_1^{-1})$, and $(p^{-1}, q^{-1})$ in Figure 1 would all have to be taken in the triangle below the diagonal from (0,0) to (1,1).) Indeed, Riesz formulated only a finite-dimensional version of it, probably because, being old and prudent, he avoided invoking function spaces.

Thus it was an extension to the *complex* case that was given by Thorin. In [18] it is alluded to that it was a remark by Otto Frostman after a seminar that put him on the right track. There is also mention of an extraordinary appraisal of Thorin's proof expressed by Littlewood [25], p. 20, who speaks of *the most impudent idea in mathematics*.

The proof roughly runs as follows.

*Sketch of proof of the Riesz–Thorin theorem.* Choose $f$ with $\|f|_p = 1$ and let $h = Tf$. We have to estimate the integral

$$\langle h, g\rangle = \int_V h(y)g(y)\mathrm{d}\nu.$$

It follows from Hölder's inequality that

$$\|h\|_q = \sup\{|\langle h, g\rangle| : \|g\|_{q'} = 1\},$$

where $q' = \frac{q}{q-1}$ (the *conjugate* index to $q$). Here, as in [23], we will only deal with the cases where $p_0 \neq p_1$ and $q_0 \neq q_1$. This ensures that $p < \infty$ and $q' < \infty$, which means that we can assume that $f$ and $g$ take only finitely many values and are supported on sets of finite measure. (In fact, when $p_0 = p_1$ and/or $q_0 = q_1$, the proof is no more difficult.)

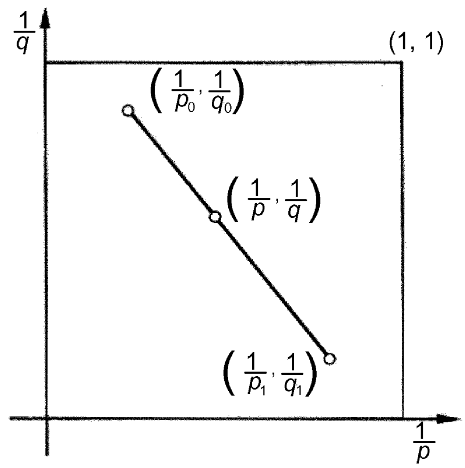


**Fig. 1.** A geometric illustration of formula (4.2).

The basic idea is now to perform a *deformation* of the points $f$ and $g$, imbedding them into suitable *complex curves*. The equations of these curves are determined by two analytic functions $\varphi(z)$ and $\psi(z)$, where $z$ is a complex variable restricted to the *closed* strip $S = \{0 \leq \Re z \leq 1\}$.

For $z \in S$ set

$$\frac{1}{p(z)} = \frac{1-z}{p_0} + \frac{z}{p_1}, \quad \frac{1}{q'(z)} = \frac{1-z}{q'_0} + \frac{z}{q'_1}.$$

Thorin's choices of $\varphi$ and $\psi$ are the following:

$$\varphi(z) = \varphi(x,z) = |f(x)|^{p/p(z)}\mathrm{sign}(f(x)), \quad x \in U;$$

$$\psi(z) = \psi(y,z) = |g(y)|^{q'/q'(z)}\mathrm{sign}(g(y)), \quad y \in V.$$

Further, we put

$$F(z) = \langle T\varphi(z), \psi(z)\rangle.$$

Clearly, $F(z)$ is continuous in $S$ and analytic in its interior with $F(\theta) = \langle Tf, g\rangle$. Moreover,

$$\|\varphi(it)\|_{p_0} = \|f\|_p^{p/p_0} = 1$$

and

$$\|\varphi(1+it)\|_{p_1} = \|f\|_p^{p/p_1} = 1 \ \forall t \in \mathbb{R},$$

and analogously for $\psi$. If we now apply the *Doetsch three line theorem* (a variation of Hadamard's better known *three circle theorem*), we find that $|\langle Tf, g\rangle| \leq M_0^{1-\theta}M_1^{\theta}$ and the proof is complete.

The Riesz–Thorin theorem has many important applications, in particular in harmonic analysis. For instance, it implies immediately the Hausdorff–Young theorem: if $\mathscr{F}$ is the Fourier transform on a locally compact Abelian group $G$ [1], then one has $\mathscr{F} : L^p(G) \to L^{p'}(\widehat{G})$, $1 \leq p \leq 2$, where $\widehat{G}$ is the dual group of $G$. This is a generalization of Plancherel's theorem ($p = 2$).

In parallel with Thorin's work, Jósef Marcinkiewicz[2], Zygmund's brilliant student, found another very important interpolation theorem. It was proved by completely different methods. Like Thorin, Marcinkiewicz published an announcement [26] of his result in 1939. These two theorems complement each other in several interesting and useful ways.

History took a new turn around 1960. Instead of only considering Lebesgue spaces in these kinds of contexts, mathematicians started "interpolating" between "abstract" spaces such as Banach spaces (Calderón, Kreĭn, Lions, etc.). In particular, Thorin's theorem was now incorporated in the so-called complex method (see [23], Chapter 5).

1 Examples are provided by the classical cases where $G$ is the real line $\mathbb{R}$ or the unit circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

2 Jósef Marcinkiewicz (1910–1941), as a reservist of the Polish Army probably became a victim of the Katyn massacre, when 4421 Polish officers were executed by the NKVD. His theorem is also applicable in *nonlinear* situations. Marcinkiewicz was probably the first to use the word "interpolation" in this context. Or might it have been Banach?

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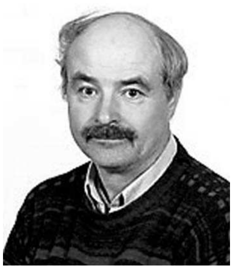

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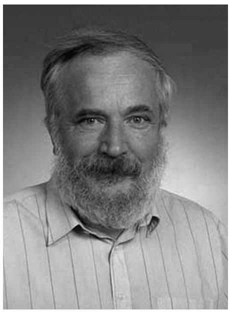

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## Olof Thorini elu ja töö

Lennart Bondesson, Jan Grandell ja Jaak Peetre

On antud ülevaade Olof Thorini (1912–2004) panusest matemaatika eri valdkondadesse: matemaatilisse analüüsi, tõenäosusteooriasse ja kindlustusmatemaatikasse. Tõenäosusteooria alal oli tema uurimisvaldkonnaks eeskätt lõpmatult jagunevate jaotuste teooria, kindlustusmatemaatikas arendas ta laostumisteooriat, leides hinnanguid kindlustusfirma laostumistõenäosusele. Tema tuntus matemaatikuna rajaneb aga eelkõige tulemustele matemaatilises analüüsis, kus olulisimad tulemused on saadud nn Rieszi-Thorini teoreemi näol ja sellega seotud teooria arendamisel. Artiklis on ära toodud ka Thorini eluloolised andmed ja autorite isiklikke mälestusi temast.