Nonessential functionals in multiobjective optimal control problems

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Abstract. We address the problem of obtaining well-defined criteria for multiple criteria optimal control problems. Necessary and sufficient conditions for an objective functional to be nonessential are proved. The results provide effective tools for determining nonessential objectives in multiobjective optimal control problems.

Key words: multiobjective optimal control problem, Pareto optimal solution, essential/ nonessential functionals.

1. INTRODUCTION

Multiobjective optimal control attracts more and more attention and is a topic of extensive current research (see, e.g., [1−3] and references therein). We consider multiobjective problems of optimal control governed by ordinary differential dynamical systems. This comprises an important class of problems which naturally appear in practical applications to economics [4], Ch. 8, and engineering modelling [5]. Our main goal is to extend and apply the results found in the literature on nonessential functions of mathematical static optimization programming [6,7] to functionals of optimal control theory.

It is well known that the concept of Pareto optimality (efficiency) plays a crucial role in optimal control [5,8]. The question of obtaining well-defined criteria for multiple criteria decision-making problems seems, however, to be considered in

the literature only for static multiobjective optimization problems (cf. [6,7] and references therein). In this work we investigate the problem of obtaining well-defined criteria for multiple criteria optimal control problems.

One of the approaches dealing with the problem of obtaining well-defined criteria for multiple criteria static decision-making problems is the concept of nonessential objective functions. A certain objective function is called nonessential if the set of efficient solutions is the same both with or without that objective function. Information about nonessential objectives helps a decision-maker to get insights and understand better a problem, and this might be a good starting point for further investigation or revision of the model. Dropping nonessential functions leads to a problem with a smaller number of objectives, which can be solved more easily. For this reason, the idea of nonessential objectives should be used as a regular feature in conditioning and analysis of multiple criteria programs [6,9,10]. To the best of the authors' knowledge, no study has been done in this field for optimal control problems. We are interested in generalizing the previous results on nonessential objectives found in the literature to cover multiobjective optimal control problems. More precisely, we generalize the concept of nonessential objective functions to functionals of optimal control systems and give the first steps on the corresponding theory. The main results provide methods for identifying nonessential objectives in multiobjective optimal control problems.

2. MULTIOBJECTIVE OPTIMAL CONTROL PROBLEMS

We consider a dynamical control system described by n *state variables* $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and r *control variables* $u = (u_1, \ldots, u_r) \in \mathbb{R}^r$, $r \leq n$. Both state and control variables vary with respect to the scalar variable $t \in \mathbb{R}$. Given a control vector function $u : [a, b] \to \mathbb{R}^r$, the state evolution over $[a, b]$, namely $x : [a, b] \to \mathbb{R}^n$, must satisfy the control system

$$\dot{x}(t) = h(t, x(t), u(t)), \tag{1}$$

the boundary conditions

$$x(a) = \alpha, \quad x(b) = \beta, \tag{2}$$

and m inequality constraints

$$g_i(t, x(t), u(t)) \leq 0, \quad i = 1, \ldots m. \tag{3}$$

We would like to find a piecewise-continuous control function $u(\cdot)$ and the corresponding state trajectory $x(\cdot)$, satisfying (1), (2), and (3), which minimizes a finite number N of (objective) functionals:

$$\min \int_a^b f(t, x(t), u(t)) \mathrm{d}t$$
$$= \min \left(\int_a^b f_1(t, x(t), u(t)) \mathrm{d}t, \cdots, \int_a^b f_N(t, x(t), u(t)) \mathrm{d}t \right).$$

All functions $f(t, x, u)$, $g(t, x, u)$, and $h(t, x, u)$ are assumed to be continuously differentiable with respect to t and x variables. To simplify notation, we write

$$I^N[x, u] = \int_a^b f(t, x(t), u(t))\mathrm{d}t$$

and

$$I_i[x, u] = \int_a^b f_i(t, x(t), u(t))\mathrm{d}t, \quad i = 1, \dots, N.$$

In general, there does not exist a pair of functions (x, u) that renders the minimum value to each functional I_i, $i = 1, \dots, N$, simultaneously, and one uses the concept of Pareto optimality. Let us denote by S the set of feasible solutions, i.e. the set of all admissible functions (x, u). The multiobjective optimal control problem consists of finding all feasible solutions that are efficient in the sense of Definition 2.1. This problem is denoted in the sequel by (P).

Definition 2.1. (Pareto optimality). *A pair of functions $(\tilde{x}, \tilde{u}) \in S$ is said to be an efficient (Pareto optimal)* solution of the problem (P) if, and only if, there exists no $(x, u) \in S$ such that $I^N[x, u] \leqq I^N[\tilde{x}, \tilde{u}]$, where

$$I^N[x, u] \leqq I^N[\tilde{x}, \tilde{u}]$$
$$\Leftrightarrow \forall i \in \{1, \dots, N\} : I_i[x, u] \leqslant I_i[\tilde{x}, \tilde{u}] \land \exists j \in \{1, \dots, N\} : I_j[x, u] < I_j[\tilde{x}, \tilde{u}].$$

The set of efficient solutions of (P) is denoted by S_E^N.

We remark that many practical applications that appear in engineering and economics can be written in the form of problem (P) [5].

The central result in optimal control theory is given by the celebrated Pontryagin maximum principle [11], which is a necessary optimality condition. A version of the Pontryagin maximum principle for Pareto optimal solutions of multiobjective optimal control problems was proved already in the 1960s [12]. Roughly speaking, one can say that the necessary and sufficient conditions for Pareto optimality are obtained by converting the multiobjective optimal control problem into a single optimal control problem or a family of single optimal control problems with an auxiliary scalar integral functional, possibly depending on some parameters [13,14]. For a gentle introduction to optimal control, including necessary and sufficient conditions and the issue of existence, we refer the reader to [15,16] (scalar case) and [5,8] (Pareto optimal solutions). Here we just recall three basic lemmas (cf. [8], Ch. 17) that relate the Pareto optimal solution of a multiobjective optimal control problem to the solutions of an appropriate scalar optimal control problem.

Lemma 2.2. ([8], §17.4) *If the feasible pair $(\tilde{x}, \tilde{u}) \in S$ is efficient for (P), then it minimizes each one of the scalar integral functionals*

$$I_i[x, u], \quad i \in \{1, \dots, N\},$$

subject to the constraints $(x, u) \in S$ *and*

$$I_j[x, u] - I_j[\tilde{x}, \tilde{u}] \leq 0, \quad j = 1, \ldots, N \text{ (and } j \neq i) .$$

Lemma 2.2 is very useful because it implies that the necessary conditions for optimal control subject to isoperimetric constraints [11,17] are also necessary for Pareto optimality in the multiobjective optimal control problem. As with the necessary conditions, the next two lemmas reduce the sufficient conditions for Pareto optimality to sufficient conditions for optimal control in the scalar optimal control problem.

Lemma 2.3. ([8], §17.5) *A feasible pair* $(\tilde{x}, \tilde{u}) \in S$ *is efficient for* (P) *if there exists a constant* $\gamma \in \mathbb{R}^N$, *with* $\gamma_i > 0$ *for* $i = 1, \ldots, N$ *and* $\sum_{i=1}^{N} \gamma_i = 1$, *such that*

$$\sum_{i=1}^{N} \gamma_i I_i[x, u] \geq \sum_{i=1}^{N} \gamma_i I_i[\tilde{x}, \tilde{u}]$$

for every $(x, u) \in S$.

Remark 2.4. Proof of Lemma 2.3 is very simple. Moreover, Lemma 2.3 is easily generalized. Let f be a strongly increasing function if $y \leqq \tilde{y} \Rightarrow f(y) < f(\tilde{y})$. Then $(\tilde{x}, \tilde{u}) \in S$ is efficient for (P) if there exists a strongly increasing function f and $f\left(I^N[x, u]\right) \geq f\left(I^N[\tilde{x}, \tilde{u}]\right) \forall (x, u) \in S$.

Unlike Lemma 2.3, not all the components of γ in the next Lemma 2.5 need to be nonzero. However, in Lemma 2.5 the minimum of $\sum_{i=1}^{N} \gamma_i I_i[x, u]$ must be achieved by a unique $(\tilde{x}, \tilde{u}) \in S$.

Lemma 2.5. ([8], §17.5) *A feasible pair* $(\tilde{x}, \tilde{u}) \in S$ *is efficient for* (P) *if there exists a constant* $\gamma \in \mathbb{R}^N$, *with* $\gamma_i \geq 0$ *for* $i = 1, \ldots, N$ *and* $\sum_{i=1}^{N} \gamma_i = 1$, *such that*

$$\sum_{i=1}^{N} \gamma_i I_i[x, u] > \sum_{i=1}^{N} \gamma_i I_i[\tilde{x}, \tilde{u}]$$

for every $(x, u) \in S$, $(x, u) \neq (\tilde{x}, \tilde{u})$.

Together with the Pontryagin maximum principle [11,17], Lemmas 2.2, 2.3, and 2.5 provide expedient tools to study concrete multiobjective problems of optimal control (cf. Section 4).

3. NONESSENTIAL FUNCTIONALS: MAIN RESULTS

We form a new multiobjective optimal control problem $(\tilde{P})$ from (P) by adding a new functional $I_{N+1}[x, u] = \int_a^b f_{N+1}(t, x(t), u(t)) \mathrm{d}t$ to problem (P). Let S_E^{N+1}

denote the set of efficient solutions of the problem $(\tilde{P})$. With this notation we introduce the definition of a nonessential functional.

Definition 3.1. *The functional I_{N+1} is said to be* nonessential *in $(\tilde{P})$ if, and only if, $S_E^N = S_E^{N+1}$. A functional which is not nonessential will be called* essential.

We are interested in characterizing integral functionals which do not change the set of efficient solutions (nonessential objective functionals). Throughout the text we denote by $S_i, i = 1, 2, \ldots, N, N + 1$, the set of optimal solutions of the scalar optimal control problem

$$\min I_i[x, u]$$

subject to S. We start with a simple example taken from [18].

Example 3.2. Consider a system characterized by a single state and control variable $(n = r = 1)$ that evolves according to the state equation

$$\dot{x}(t) = u(t)$$

with the control constraint set

$$U = \{u : [a, b] \rightarrow \mathbb{R} : |u(t)| \leq 1\}.$$

The system is to be transferred from a given initial state $x(0) = \xi \neq 0$ to a given terminal state $x(T) = 0$ within an unspecified bounded interval $[0, T]$. Functionals to be minimized are

$$I_1 = \int_0^T \mathrm{d}t, \quad I_2 = \int_0^T |u(t)|\mathrm{d}t.$$

Applying the Pontryagin maximum principle [11], one obtains:

$$S_1 = \{(x(t), u(t)) : u(t) = -\mathrm{sgn}\{\xi\}\}, \ \min \int_0^T \mathrm{d}t = |\xi|$$

and

$$S_2 = \{(x(t), u(t)) : u(t) = -\mathrm{sgn}\{\xi\}v(t)\},$$

where

$$v(t) \in V = \{v(t) : 0 \leq v(t) \leq 1, t \in [0, T], v(t) \not\equiv 0\},$$

$$\min \int_0^T |-\mathrm{sgn}\{\xi\}v(t)|\mathrm{d}t = |\xi|.$$

Details can be found in [18]. It is easy to see that $S_1 \cap S_2 = S_1$ (we can take $v(t) = 1, t \in [0, T]$). In this problem we have: $S_1 = S_E^1 = S_E^2 \subset S_2$. Hence I_2 is nonessential, but I_1 is essential (in order to see this, we need only to interchange indices).

340

Lemma 3.3. *We have $S_E^N \subset S_E^{N+1}$ if and only if for every $(x, u) \in S_E^N$ the following condition holds*:

$$\exists (x', u') \in S : I^N[x', u'] = I^N[x, u] \Rightarrow I_{N+1}[x', u'] = I_{N+1}[x, u].$$

Proof. Let $S_E^N \subset S_E^{N+1}$ and assume, on the contrary, that there exists $(\tilde{x}, \tilde{u}) \in S_E^N$ such that

$$\exists (x', u') \in S : I^N[x', u'] = I^N[\tilde{x}, \tilde{u}] \tag{4}$$

and

$$I_{N+1}[x', u'] \neq I_{N+1}[\tilde{x}, \tilde{u}]. \tag{5}$$

We conclude from (4) that $(x', u') \in S_E^N$. Therefore (x', u') is not in S_E^{N+1} or $(\tilde{x}, \tilde{u})$ is not in S_E^{N+1} by (5). This contradicts the fact that $S_E^N \subset S_E^{N+1}$. Let us now prove the second implication. If $S_E^N = \emptyset$, then $S_E^N \subset S_E^{N+1}$. Let $S_E^N \neq \emptyset$. Suppose that for every $(x, u) \in S_E^N$ there holds:

$$\exists (x', u') \in S : I^N[x', u'] = I^N[x, u] \Rightarrow I_{N+1}[x', u'] = I_{N+1}[x, u] \tag{6}$$

and S_E^N is not contained in S_E^{N+1}. In this case there exists $(\tilde{x}, \tilde{u})$ in S_E^N which is not in S_E^{N+1}. Hence

$$\exists (\hat{x}, \hat{u}) \in S : I^{N+1}[\hat{x}, \hat{u}] \leqq I^{N+1}[\tilde{x}, \tilde{u}]. \tag{7}$$

This gives $I^N[\hat{x}, \hat{u}] = I^N[\tilde{x}, \tilde{u}]$ and from (6) we have $I_{N+1}[\hat{x}, \hat{u}] = I_{N+1}[\tilde{x}, \tilde{u}]$. Consequently, $I^{N+1}[\hat{x}, \hat{u}] = I^{N+1}[\tilde{x}, \tilde{u}]$, contrary to (7). $\square$

Remark 3.4. Notice that in Example 3.2 the scalar optimal control problem

$$\min I_1[x, u]$$

subject to S has a unique solution. Therefore, Lemma 3.3 holds true for the example.

Definition 3.5. *A function $f : \mathbb{R}^N \to \mathbb{R}$ is nondecreasing if for y^1 and $y^2 \in \mathbb{R}^N$ $y^1 \leqq y^2$ implies $f(y^1) \leq f(y^2)$.*

Theorem 3.6. *If $I_{N+1}[x, u] = f(I^N[x, u]) \, \forall (x, u) \in S$, where $f : \mathbb{R}^N \to \mathbb{R}$, then $S_E^N \subset S_E^{N+1}$. Furthermore, $S_E^N = S_E^{N+1}$ if the function f is nondecreasing on the set $I^N(S)$.*

Proof. Let $(x, u) \in S_E^N$. If there exists $(x', u') \in S$ such that $I^N[x', u'] = I^N[x, u]$, then $f(I^N[x', u']) = f(I^N[x, u])$ and so $I_{N+1}[x', u'] = I_{N+1}[x, u]$. Therefore $S_E^N \subset S_E^{N+1}$ by Lemma 3.3.

We will now show the inclusion $S_E^{N+1} \subset S_E^N$. Let $(x, u) \in S$, (x, u) being not an element of the set S_E^N. In this case there exists $(x', u') \in S$ such that $I^N[x', u'] \leqq I^N[x, u]$. If f is nondecreasing on $I^N(S)$, we know that $I_{N+1}[x', u'] = f(I^N[x', u']) \leq f(I^N[x, u]) = I_{N+1}[x, u]$. Hence (x, u) is not an element of the set S_E^{N+1}. $\qquad\square$

Remark 3.7. Example 3.2 shows that the sufficient condition for an objective functional to be nonessential, given by Theorem 3.6, is not necessary.

Theorem 3.8. *Let* $S_{N+1} = \{(x^0, u^0)\}$. *If the functional* I_{N+1} *is nonessential, then* $(x^0, u^0) \in S_E^N$.

Proof. Let $S_E^N = S_E^{N+1}$. If (x^0, u^0) is not an element of the set S_E^N, then (x^0, u^0) is also not an element of the set S_E^{N+1}. In this case, there exists $(x', u') \in S$ such that $I^{N+1}[x', u'] \leqq I^{N+1}[x^0, u^0]$. So $I_{N+1}[x', u'] \leq I_{N+1}[x^0, u^0]$. This is a contradiction to the assumption that $S_{N+1} = \{(x^0, u^0)\}$. $\qquad\square$

Theorem 3.9. *Let the set* S *be compact. If the functional* I_{N+1} *is nonessential, then* $S_{N+1} \cap S_E^N \neq \emptyset$.

Proof. Consider the problem

$$\min \int_a^b f(t, x(t), u(t)) \mathrm{d}t \tag{8}$$

subject to S_{N+1}. Let $\tilde{S}$ denote the set of efficient solutions of the problem (8). By the compactness of the set S, the set $\tilde{S}$ is nonempty. Let $(x^0, u^0) \in \tilde{S}$. If (x^0, u^0) is not an element of S_E^N, then by assumption (x^0, u^0) is not an element of S_E^{N+1}. In this case, there exists $(x', u') \in S$ such that $I^{N+1}[x', u'] \leqq I^{N+1}[x^0, u^0]$. Hence $(x', u') \in S_{N+1}$. This contradicts the fact that (x^0, u^0) is an efficient solution of the problem (8). $\qquad\square$

Remark 3.10. Notice that I_2 is nonessential in Example 3.2 and we have $S_E^1 \cap S_2 \neq \emptyset$.

The next section provides an example of application of the obtained results to check whether a functional is nonessential.

4. AN ILLUSTRATIVE EXAMPLE

We illustrate the obtained results with a multiobjective control problem borrowed from [5], §4.3, where $N = 3$, $n = 2$, $r = 1$, $m = 4$, $a = 0$, $b = T$, with T not fixed. We consider a mobile rocket car with mass one running in rails on a closed region $-3 \leq x_1 \leq 3$ (we denote the position of the centre of the car at time t by $x_1(t)$), whose movement we can control with its accelerator u, where

the maximum allowable acceleration is 1 and the maximum break power is -1, i.e., $-1 \le u \le 1$ (negative force means break, positive force means acceleration). The dynamics of the system is given by Newton's second law, force equals mass times acceleration, which in our setting reads as $u(t) = \ddot{x}_1(t)$. The problem is to move the car from a given location to a pre-assigned destination. If the car is at a position $x_1 = 1$ at time $t = 0$, with no velocity, that is $\dot{x}_1(0) = 0$, we want to find a piecewise constant function $u(t)$ that drives the car to $x_1(T) = 0$ at some instant $T > 0$. The state of the system is given by the position $x_1(t)$ and the velocity $x_2(t) = \dot{x}_1(t)$ (where we are and how fast we are going at each instant of time t). Different objective functionals can be considered, for example, minimizing the time T (functional I_1 below); maximizing the velocity at T (maximizing $x_2(T)$, which corresponds to functional I_2 below); and a linear combination I_3 of these functionals: minimize

$$I_1 = \int_0^T 1 \mathrm{d}t, \quad I_2 = \int_0^T -u(t)\mathrm{d}t, \quad I_3 = I_1 + I_2,$$

subject to the control system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = u; \end{cases} \tag{9}$$

to the boundary conditions

$$x_1(0) = 1, \quad x_1(T) = 0, \quad x_2(0) = 0; \tag{10}$$

and inequality constraints

$$|u| \le 1, \quad |x_1| \le 3. \tag{11}$$

Let us denote by S_i, $i = 1, 2, 3$, the solution set of the scalar optimal control problem $\min I_i[x, u]$ subject to (9)–(11). We have:[1] $S_1 = \{(x^1, u^1)\}$ with

$$u^1 = -1,$$

$$x_1^1 = -\frac{t^2}{2} + 1, \quad x_2^1 = -t,$$

$$0 \le t \le T = \sqrt{2};$$

$S_2 = \{(x^2, u^2)\}$ with

$$u^2 = \begin{cases} -1 & \text{if } 0 \le t \le 2, \\ +1 & \text{if } 2 \le t \le T = 4 + \sqrt{6}, \end{cases}$$

$$x_1^2 = \begin{cases} -\frac{t^2}{2} + 1 & \text{if } 0 \le t \le 2, \\ \frac{t^2}{2} - 4t + 5 & \text{if } 2 \le t \le 4 + \sqrt{6}, \end{cases} \qquad x_2^2 = \begin{cases} -t & \text{if } 0 \le t \le 2, \\ t - 4 & \text{if } 2 \le t \le 4 + \sqrt{6}; \end{cases}$$

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[1] The solutions to the scalar optimal control problems $I_i[x, u] \to \min$ are found by application of the Pontryagin maximum principle [11]. Details can be found in [5], §4.3.

$S_3 = \{(x^3, u^3)\}$ with

$$u^3 = \begin{cases} -1 & \text{if } 0 \le t \le 1, \\ +1 & \text{if } 1 \le t \le T = 2; \end{cases}$$

$$x_1^3 = \begin{cases} -\frac{t^2}{2} + 1 & \text{if } 0 \le t \le 1, \\ \frac{t^2}{2} - 2t + 2 & \text{if } 1 \le t \le 2, \end{cases} \qquad x_2^3 = \begin{cases} -t & \text{if } 0 \le t \le 1, \\ t - 2 & \text{if } 1 \le t \le 2. \end{cases}$$

Direct calculations show that

$$I^2[x^1, u^1] = [\sqrt{2}, \sqrt{2}]^T = A,$$
$$I^2[x^2, u^2] = [4 + \sqrt{6}, -\sqrt{6}]^T = B,$$
$$I^2[x^3, u^3] = [2, 0]^T = C.$$

Let ζ denote the set $I^2(S_E^2)$. It is the continuous, convex curve $\widehat{AB}$ (details can be found in [5], §4.3). As $C \in \zeta$, we have $S_E^2 \cap S_3 \ne \emptyset$. Moreover, let us notice that I_3 has the form $I_3[x, u] = f(I^2[x, u])$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is a nondecreasing function. Therefore, the functional I_3 is nonessential by Theorem 3.6.

Remark 4.1. If we change the functional I_3 into

$$I_3[x, u] = \gamma_1 I_1[x, u] + \gamma_2 I_2[x, u], \tag{12}$$

where $\gamma_i \in \mathbb{R}$ and $\gamma_i \ge 0, i = 1, 2$ or

$$I_3[x, u] = [(I_1[x, u] - \sqrt{2})^p + (I_2[x, u] + \sqrt{6})^p]^{1/p}, \tag{13}$$

where $p \in [1, \infty]$, then again I_3 will be nonessential by Theorem 3.6. It is worth noting that functionals (12) and (13) can be used in order to find efficient solutions of the problem $\min I^2[x, u]$ subject to (9)–(11). We mentioned this in Section 2, details can be found in [5] and [8].

5. CONCLUSIONS

The problem of optimization of a vector-valued criterion often arises in connection with the solution of problems in the areas of planning, organization of production, operational research, and dynamical control systems. Currently, the problem of optimizing vector-valued criteria is a central part of control theory and great attention is being given to it in the design and construction of modern automatic control systems, for example in specific applications of seismology, energetic chemistry, and metallurgy. In this work we use the notion of Pareto optimality in control theory to define and investigate nonessential objective functionals of optimal control. For multiple criteria optimal control problems this notion seems to be new and not used before. We believe that the concept of a nonessential objective functional is an important issue in optimal control and we trust it will have an important role in the study of vector optimization problems of control theory. In future, it would be interesting to study the consequences of dropping nonessential objectives in multiobjective optimal control problems.

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Mitteolulised funktsionaalid mitmekriteeriumilistes optimaaljuhtimise ülesannetes

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On uuritud täpselt defineeritud optimaalsuskriteeriumi leidmise probleemi mitmekriteeriumilistes optimaaljuhtimise ülesannetes. On tõestatud sihifunktsiooni mitteolulisuse vajalikud ja piisavad tingimused. Saadud tulemused võimaldavad efektiivselt määrata mitteolulisi kriteeriume mitmekriteeriumilistes optimaaljuhtimise ülesannetes.