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Green's function for the deflection of non-prismatic simply supported beams by an analytical approach

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Abstract. The influence line for deflection of non-prismatic simply supported beams has been developed. The methodology primarily comprises the determination of the Green's function of the governing differential equation and extension of the results to typical problems. The method of inverse operator along with the orthogonal eigenfunction expansion was employed and the final form of the solution is presented in an integral form, which can be solved by either direct methods or numerical techniques. An example problem and the closed-form solution for a particular class of non-prismatic beams, very often applied in practice, have been presented.

Key words: non-prismatic beam, influence line, deflection, Green's function, analytic solution.

1. INTRODUCTION

In civil engineering, there are two design criteria for structural members like columns and beams: the strength, and the serviceability, which require the internal shear forces, moments and deflections to be understood. The influence line for a beam is a graph or a curve, representing the variation of the shear force, bending moment, deflection or other parameters due to a unit load traversing along the beam axis [1–4]. The governing equation of the influence line can be found by applying a unit load at an arbitrary stationary location on the beam, such as x_0, and determination of the desired quantity, e.g., deflection, bending moment or shear force as a function of the position x. Also the influence line can be used to find the most critical position for a system of loads imposed on the beam, generating the extreme value for the quantity of interest. Alternatively, it shows where the critical value of the quantity of interest would occur if a unit load (or a system of loads) is applied to a beam. In civil engineering, an economic design often requires a variable cross-section to be utilized. Non-prismatic or non-uniform simply supported beams have wide applications in practice, e.g. in long-span bridges and industrial structures with plate-girders and hinge connections to support heavy loads. There are several methods in tackling with the problem of the deflection of non-prismatic beams [5]. For statically determinate prismatic beams, except in particular cases, a closed form solution is often available for influence lines [3,4]. However, a closed form solution of the influence line for the deflection of non-prismatic beams cannot be always obtained. Several methods have been developed to find the influence line or, in general, the deflection of non-prismatic beams. For example, the use of the energy method or Castigliano's theorem for the deflection of structural members can be mentioned [1,3,4].

Over the past decades, many researchers have investigated the problem of deflection of non-prismatic beams subjected to different boundary conditions. A wide range of methods, including closed-form solutions and numerical techniques has been developed: Karabalis and Beskas [6] developed a method based on exact stiffness and mass matrices for constant width linear height tapers. Eisenberger [7] developed an exact stiffness matrix for some particular cases of non-prismatic beam deflection analysis. Ganga Rao and Spyrakos [8] proposed a series solution for differential equations with variable properties over the domain. They handled the problem by assuming the solution to be expandable in terms of the generalized Fourier series and found the undetermined coefficients in a system of equations, leading to some *mode shape* functions. Such a technique can be extended to the problem of the deflection of non-prismatic beams. Static and vibrating behaviour of non-prismatic beams was studied by Eisenberger and Reich [9]. They approximated the moment of inertia and cross-sectional area of non-prismatic beams with some power series and presented their solution in terms of matrix equations. In [10,11] the deflection of non-prismatic beams under static loads, based on fundamental solutions of the governing differential equation (the fourth-order ODE) under rather general boundary conditions was investigated. A closed-form solution for beams with variations in both stiffness and moment of inertia was obtained. Al-Gahtani and Khan [12] used the boundary integral method (BIM) to find the deflection, shear force and moment of non-prismatic beams with general boundary conditions at both ends. There are also several analytical and numerical attempts for both static and dynamic responses of non-prismatic beams [5,13–15].

Applications of mathematical techniques to solve particular differential equations have been widely employed in the field of engineering. A very useful and versatile concept in mathematics, the Green's function, has been given a considerable attention in the past decades for a wide range of boundary value problems. Once the Green's function is found, the deflection or internal forces can be easily computed. As an example, Mehri et al. [16] derived the Green's function for dynamic analysis of beams under moving loads. Deflection of non-prismatic beams can be analysed by different methods and the associated Green's function can be obtained by several techniques. In this paper, the method of operator in conjunction with the Fourier series expansion is employed to find the Green's function of the governing differential equation for simply supported non-prismatic beams. The method is first described in few details and then applied to the problem under study.

2. THE METHOD OF OPERATOR

As stated before, the method of operator is a classical and powerful technique in dealing with differential equations of all types (ordinary or partial). This method is applicable to linear operators. The most useful application of this method is to non-homogeneous problems. Once the governing differential equation has been constituted, the method of operator seeks to find the inverse operator of the governing differential equation [17]. It is worth mentioning that while this method is best fitted to partial differential equations, this research uses the method of operator for ordinary differential equations. Given below is a short review of the essence of the method of operator.

Let's suppose that D is a linear differential operator acting upon an arbitrary function, u, defined in some functional space, resulting in a non-homogeneous function, $f(x)$, as follows:

$$Du = f(x). \tag{1}$$

Depending on the physics of the problem and on the degree of the differential operator, the governing differential equation may involve some boundary conditions. The main strategy of the method of operator is to find the inverse operator of the primary differential problem, i.e. D^{-1}. Once D^{-1} has been found, it can be applied to both sides of the problem. Since the original operator, D, is a differential operator, the inverse operator, D^{-1}, should be an integral operator. Therefore, the inverse operator is expected to take the following form [17]:

$$K = D^{-1} = \int g(x, x_0)\mathrm{d}x_0, \tag{2}$$

where $g(x, x_0)$ is the kernel of the integral transform, K. The inverse operator is assumed to possess the property $D^{-1}D = DD^{-1} = I$, where I is the identity operator. The main differential equation can be operated upon by this latter transformation. If it is applied to both sides of the governing differential equation, it yields

$$KDu = Kf(x) \Rightarrow D^{-1}Du = \int g(x, x_0) f(x_0)\mathrm{d}x_0. \tag{3}$$

It can be further simplified to result in the following equation:

$$u(x) = \int g(x, x_0)(Du)\mathrm{d}x_0 = \int g(x, x_0) Du \mathrm{d}x_0. \tag{4}$$

Note that the *associative law* has been applied to the last expression. For commutable operators, the order of the differential and integral operators is interchangeable and hence:

$$u(x) = D\int g(x, x_0)u\mathrm{d}x_0. \tag{5}$$

This expression is traditionally recast in the following form and $g(x, x_0)$ would be the associated Green's function

$$u(x) = \int G(x, x_0)u(x_0)\mathrm{d}x_0. \tag{6}$$

In this equation, $G(x, x_0)$ is $Dg(x, x_0)$.

Of particular importance is the inverse operator for a general differential operator. If the last expression is to be used for an arbitrarily chosen function, $u(x)$, the right integral must possess the property that it becomes zero when x is not equal to x_0 and it becomes $u(x)$ when $x = x_0$. The celebrated contribution of Dirac [18] ensures that this is always the case. It has been proved that the inverse operator is the solution of the related equation of the following form, in which the non-homogeneous term has been substituted by the Dirac's delta function [17]:

$$G(x, x_0) = Dg(x, x_0) = \delta(x - x_0), \tag{7}$$

where

$$\delta(x - x_0) = \begin{cases} \infty, & x = x_0, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

It should be noted that the Dirac's delta function, because of its rather complex nature, is formally described in terms of the derivative of the Heaviside unit step function [19]:

$$\delta(x - x_0) = \frac{\mathrm{d}H(x - x_0)}{\mathrm{d}x_0}, \tag{9}$$

where the Heaviside unit step function, $H(x - x_0)$, is defined as:

$$H(x - x_0) = \begin{cases} 1, & x > x_0, \\ 0, & x < x_0. \end{cases} \tag{10}$$

Diagrams of the Dirac's delta function and Heaviside unit step function are schematically shown in Fig. 1.

Therefore, once the Green's function has been found, solution of the differential equation subject to an arbitrary non-homogeneous term, $f(x)$, can be found as follows:

$$u(x) = \int g(x, x_0)(Du)\mathrm{d}x_0 = \int g(x, x_0)f(x_0)\mathrm{d}x_0. \tag{11}$$

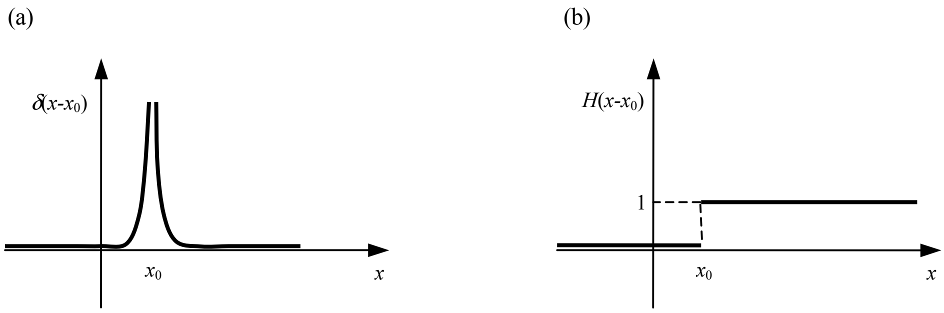

Fig. 1. Graphical representation of the Dirac's delta function (a) and Heaviside unit step function (b).

Thus, the problem of finding a particular solution for a given boundary value problem, expressed as an ordinary differential equation, will be reduced to finding the associated Green's function. The remaining part of this study is devoted to the development of a procedure based on the method of operator to find the inverse operator for the governing differential equation of the deflection of non-prismatic beams.

3. GOVERNING DIFFERENTIAL EQUATION AND SOLUTION TECHNIQUE

The method of operator is employed here to find the inverse operator of the problem under study. The governing differential equation, i.e. the deflection of a non-prismatic Bernoulli–Euler's beam of finite length has the following general form [2,10,11]:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI(x)\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2}\right)=f(x), \tag{12}$$

where $u(x)$ is the deflection (out of plane displacement) of the beam, $f(x)$ is the applied distributed or concentrated loads on the beam, E is the elasticity coefficient of the material and $I(x)$ is the moment of inertia, variable along the longitudinal axis of the beam.

It should be noted that, in general, E can also be variable along the beam [10]. But, for the sake of convenience, we drop this assumption since it would have no effect on the developed solution in its general form and it is not the case in most of practical problems. This equation along with certain boundary conditions can describe the deflection of all types of finite beams. In particular, in many civil engineering structures like bridge elements, the simply-supported beams are widely confronted and hence, they are in the focus of the current study. This assumption can be dropped and/or replaced with more general boundary conditions for all types of beams (for example, as the one presented in [12]. Assuming such special boundary conditions enables this method to be established based on well-known elementary functions. Although for general boundary conditions the presented approach holds, it requires further study to find thus obtained generalized Fourier series. These functions may be also very difficult to be expressed in terms of elementary functions. The corresponding boundary conditions require no deflection and no moment in either end of a beam of length L, i.e.

$$M(0) = M(L) = u(0) = u(L) = 0. \tag{13}$$

Once the deflection profile was found, it is an easy task to differentiate the obtained function to find the bending moment and shear force diagrams [3]. For instance, the moment distribution, $M(x)$, can be expressed by the following equation, which is required for further development of this approach:

$$EI(x)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = M(x). \tag{14}$$

The governing differential equation can be rewritten in the following form, while the operator D represents the twice derivative with respect to x:

$$D(EI(x)Du) = f(x). \tag{15}$$

Now, the problem is reduced to find the inverse operator D^{-1}. Following the method described in [17,20], it is required to solve the following differential equation, by making use of the Dirac's delta function:

$$D(EI(x)Du) = \delta(x - x_0). \tag{16}$$

The boundary conditions and the reduced problem to find D^{-1} are shown in Fig. 2.

This reduced problem still requires some further mathematical manipulations. Since the inner parenthesis represents the bending moment distribution along the beam, the corresponding differential equation can be further reduced as follows:

$$D(M(x)) = \delta(x - x_0). \tag{17}$$

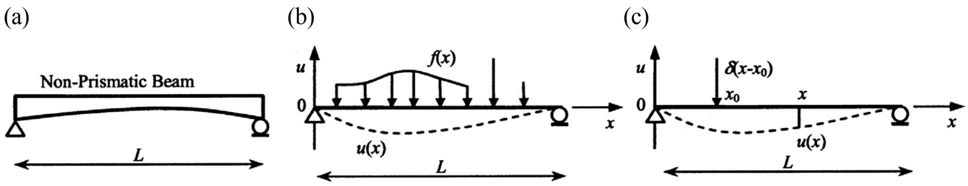

Fig. 2. The problem under study (a, b) and the reduced general problem (c).

Therefore

$$\frac{\mathrm{d}^2 M(x)}{\mathrm{d}x^2} = \delta(x - x_0). \tag{18}$$

Now comes the time to deal with the main problem, i.e. to invert the differential operator. This study intends to provide a more simple representation for the general solution. Before doing so, some preliminary manipulations are performed to simplify the problem. According to the bending moment diagram, this equation represents an ordinary differential equation subject to boundary conditions $M(0) = M(L) = 0.$ On the other hand, the physics of the problem requires the bending moment to be zero at both ends. Such conditions suggest using a half-range eigenfunction expansion of the bending moment in terms of the Fourier sine series as follows:

$$M(x) = \sum_{m=1}^{\infty} a_m \sin\frac{m\pi x}{L}, \tag{19}$$

with the coefficients a_m undetermined. This particular form guarantees the moment diagram to become zero at the boundaries. If this eigenfunction representation of $M(x)$ is operated upon by $D,$ the unknown coefficients can be found easily from the equation itself:

$$\frac{\mathrm{d}^2 M(x)}{\mathrm{d}x^2} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\sum_{m=1}^{\infty} a_m \sin\frac{m\pi x}{L} = \sum_{m=1}^{\infty} a_m \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\sin\frac{m\pi x}{L}\right) = \sum_{m=1}^{\infty} -a_m \left(\frac{m\pi}{L}\right)^2 \sin\frac{m\pi x}{L}$$

$$\Rightarrow DM(x) = \sum_{m=1}^{\infty} -a_m \left(\frac{m\pi}{L}\right)^2 \sin\frac{m\pi x}{L} = \delta(x - x_0). \tag{20}$$

It requires the Dirac's delta function to be expanded in the same way, i.e., in a half-range Fourier sine series expansion, which yields the following expressions:

$$\delta(x - x_0) = \sum_{m=1}^{\infty} A_m \sin\frac{m\pi x}{L}, \tag{21}$$

$$\sum_{m=1}^{\infty} -a_m \left(\frac{m\pi}{L}\right)^2 \sin\frac{m\pi x}{L} = \sum_{m=1}^{\infty} A_m \sin\frac{m\pi x}{L}. \tag{22}$$

The unknown coefficients, $A_m,$ however, can be found easily recalling the *sifting* property of the Dirac's delta function:

$$A_m = \frac{2}{L}\int_0^L \delta(z - x_0)\sin\frac{m\pi z}{L}\mathrm{d}z = \frac{2}{L}\sin\frac{m\pi x_0}{L}, \tag{23}$$

where z is the dummy integration variable. The unknown coefficients, a_m, can be found as follows:

$$a_m = -\frac{2L}{m^2\pi^2}\sin\frac{m\pi x_0}{L}. \tag{24}$$

Therefore, the function $M(x)$ is determined. It is now possible to rewrite the equation for the bending moment of the beam as a function of the deflection, u:

$$Du(x) = \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \frac{M(x)}{EI(x)} = \sum_{m=1}^{\infty} -\frac{2L}{EI(x)m^2\pi^2}\sin\frac{m\pi x_0}{L}\sin\frac{m\pi x}{L}. \tag{25}$$

This equation can be simplified as follows:

$$Du(x) = \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \sum_{m=1}^{\infty} f_m(x), \tag{26}$$

where

$$f_m(x) = -\frac{2L}{EI(x)m^2\pi^2}\sin\frac{m\pi x_0}{L}\sin\frac{m\pi x}{L}. \tag{27}$$

This equation has a solution in terms of a homogeneous solution and a particular integral as follows:

$$u = u_h + u_p = u_h + \sum_{m=1}^{\infty} u_{p(m)}. \tag{28}$$

However, the boundary conditions require the homogeneous solution to be identically zero, i.e. $u_h = 0$, and hence, only the particular integral remains to be found. To find a generally applicable particular solution for the last differential equation, it is possible to find the Green's function of the differential operator D. This can be done by the standard procedure described earlier, i.e. to solve the following differential equation:

$$Du(x,\xi) = Dg(x,\xi) = \delta(x-\xi), \tag{29}$$

subject to homogeneous boundary conditions. By making use of the Heaviside unit step function and integrating the preceding equation two times with respect to x and using properties of the Dirac's delta function, the Green's function will be found as

$$g(x,\xi) = \int H(x-\xi)\,\mathrm{d}x + x\alpha(\xi) + \beta(\xi), \tag{30}$$

where $\alpha(\xi)$ and $\beta(\xi)$ are to be determined. Applying the boundary conditions yields the following expressions for these two functions:

$$\alpha(\xi) = \begin{cases} -1 + \dfrac{\xi}{L} H(\xi), & \xi < L, \\ 0, & \xi > L, \end{cases} \tag{31a}$$

$$\beta(\xi) = \xi H(-\xi). \tag{31b}$$

Therefore, the Green's function for the differential operator, D, is fully determined. After making some manipulations, the deflection of the beam, u, will take the following form:

$$u(x, x_0) = \sum_{m=1}^{\infty} \int_0^L (x-\xi) H(x-\xi) f_m(\xi, x_0) \mathrm{d}\xi + x \int_0^L \frac{\xi - L}{L} f_m(\xi, x_0) \mathrm{d}\xi. \tag{32}$$

This last equation can be considered as the influence line for the deflection of non-prismatic beams. Knowing the functions f_m, it can be integrated to find the closed-form solution, or, if necessary, integrated numerically. Computations showed that numerical integration of the few first terms of f_m provides sufficiently accurate estimate of the deflection of both prismatic and non-prismatic beams. Following examples show the ability of the presented approach. It is remarkable that, in cases with closed-form solution, the location of the maximum deflection, bending moment or shear force can be found by differentiating the equation of the deflection curve. Moreover, in the range of elastic behaviour, the superposition assumption holds and once the closed-form or numerical integration of the derived equation is found, it can be easily extended to any arbitrary loading pattern.

4. EXAMPLES

Here, some simple examples are presented utilizing the presented method in computing the deflection of both prismatic (with known closed-form solution) and non-prismatic beams. It should be remarked that there are other methods like the energy method, which can be applied to such problems. Meanwhile, the proposed approach seems to be general, easily programmable and applicable for any pattern of applied load.

4.1. Deflection of prismatic beams

Let a simply supported prismatic beam of unit length and of unit stiffness $EI = 1$ carry a unit point load, P, at its half length, $L/2$. From mechanics of materials, the corresponding deflection is known to be $PL^3/48EI$, or, $1/48$ in this problem. Application of the presented method with the aid of different number of contributing terms is presented in Table 1. It is obvious that the solution converges with 0.01% error when only 3 terms are used. It is worth noting that the presented method for this particular case can be directly integrated to find the closed-form solution as well.

Table 1. Deflection of the middle of prismatic beams

Number of contributing terms	Current study	Closed-form solution	Error
1	0.020532	0.020833	0.000301
3	0.020786	0.020833	0.000048
5	0.020818	0.020833	0.000015
10	0.020830	0.020833	0.000003
20	0.020833	0.020833	0.000000

4.2. Deflection of non-prismatic beams

Variations in the cross-section of non-prismatic beams are practically very different. Moreover, integration of the last equation, except in a few cases, may result in very long and tedious equations. Therefore, the closed-form solution was obtained only for a very popular class of beams used in structural engineering. Such class of non-prismatic beams, confronted very often in practice, is shown in Fig. 3. For these particular beams, the closed form solution of the equation has been obtained by direct integration of the last expression, derived for non-prismatic beams with parameters defined in Fig. 3. Closed-form solution for this particular problem can be found by substitution of appropriate terms given in the Appendix. A specific problem, outlined below, has been solved using the numerical integration of the last equation beside the direct use of the closed-form solution.

A non-prismatic beam with geometry given in Fig. 4, has been solved with the presented method. The solution for the deflection at the point A (or B), has been found and compared to that provided by the energy method [4]. The influence line for the deflection of the points A and B are plotted in non-dimensional form in Fig. 5. The solution [4], based on the energy method, is (approximately) $0.029PL^3/(EI_0)$, where I_0 is the moment of inertia in the prismatic part of the beam (corresponding to h_0). Solutions have been computed, based on both closed-form equation and numerical integration of the last equation. Table 2 represents the results of the analysed cases employing different number of contributing terms for the deflection of point A (or B). It is noticeable that a

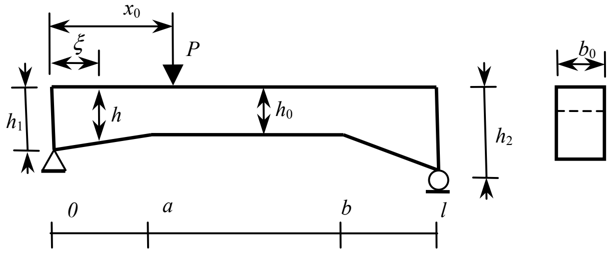

Fig. 3. A general non-prismatic beam with linear variations in the cross-section at both ends.

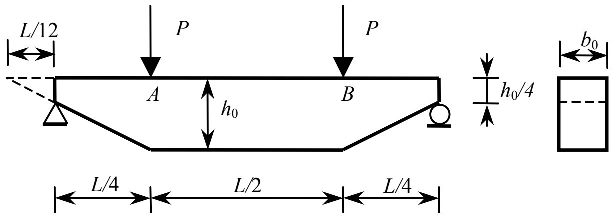

Fig. 4. Non-prismatic beam with variable height, carrying concentrated loads.

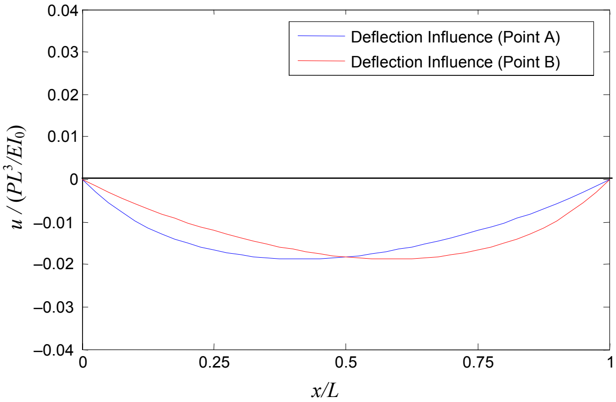

Fig. 5. Influence lines of deflection for the cases $x_0 = L/4$ (point A) and $3L/4$ (point B).

Table 2. Normalized deflection $u/(PL^3/EI_0)$ of a non-prismatic beam at points A and B

Number of contributing terms	Current study	Solution by energy method [4]	Error
1	0.0275	0.029	0.0015
3	0.0296	0.029	−0.0006
5	0.0289	0.029	0.0001

numerical integration has been used to find the solution. Again, it is evident that the solution converges very fast to that obtained by a closed form solution (or the value given by Popov [4]). The maximum deflection occurs somewhere close to $x = L/3$.

5. CONCLUSIONS

The influence line for deflection of simply-supported non-prismatic beams has been developed. The procedure mainly consists of the derivation of the corresponding Green's function for the governing differential equation. The method of inverse operator has been applied and the inverse differential operator was found as a basis for the Green's function of the main problem. The developed method showed a reasonably fast convergence by increasing the number of contributing terms. In elastic limits, the superposition assumption is applicable and, hence, any arbitrary loading pattern can be dealt with by the derived equation. Although the developed method is restricted only to simply supported beams, it has some advantages such as involvement of single integrals, which can be more easily evaluated or computed. Finally, the closed form solution for a class of non-prismatic beams, very often used in practice, has been derived.

APPENDIX

CLOSED-FORM SOLUTION FOR A SPECIFIC CASE OF NON-PRISMATIC BEAMS

The closed form solution of Eq. (32) for the case of a non-prismatic beam given in Fig. 3, is as follows:

$$y=\frac{a_m}{EI_0}\left[\int_0^a \frac{(x-\xi)H(x-\xi)\sin\frac{m\pi\xi}{L}}{\left[1+\frac{h_1-h_0}{h_0}\left(1-\frac{\xi}{a}\right)\right]^3}\mathrm{d}\xi+\int_a^b (x-\xi)H(x-\xi)\sin\frac{m\pi\xi}{L}\mathrm{d}\xi\right.$$

$$+\int_b^L \frac{(x-\xi)H(x-\xi)\sin\frac{m\pi\xi}{L}}{\left[1+\frac{h_2-h_0}{h_0}\left(1-\frac{L-\xi}{L-b}\right)\right]^3}\mathrm{d}\xi+\frac{x}{L}\int_0^a \frac{(\xi-L)\sin\frac{m\pi\xi}{L}}{\left[1+\frac{h_1-h_0}{h_0}\left(1-\frac{\xi}{a}\right)\right]^3}\mathrm{d}\xi$$

$$\left.+\frac{x}{L}\int_a^b (\xi-L)\sin\frac{m\pi\xi}{L}\mathrm{d}\xi+\frac{x}{L}\int_b^L \frac{(\xi-L)\sin\frac{m\pi\xi}{L}}{\left[1+\frac{h_2-h_0}{h_0}\left(1-\frac{L-\xi}{L-b}\right)\right]^3}\mathrm{d}\xi\right],$$

which can be evaluated by taking the following partial integrals, which appear in the main equation. The Heaviside's unit step function, H, has been replaced by 0 or 1, depending on the integration limits.

The 1st partial integral (for $x < \xi$):

$$\int_0^a \frac{(x-\xi)\sin\frac{m\pi\xi}{L}}{\left[1+\frac{h_1-h_0}{h_0}\left(1-\frac{\xi}{a}\right)\right]^3}\mathrm{d}\xi = \left(\frac{ah_0}{(a-1)h_1+h_0}\right)^3 \int_0^a \frac{(x-\xi)\sin\frac{m\pi\xi}{L}}{\xi^3}\xi = \left(\frac{ah_0}{(a-1)h_1+h_0}\right)^3$$

$$\times\left[\frac{-m\pi ci\left(\frac{m\pi\xi}{L}\right)}{L} - \frac{m^2\pi^2 xsi\left(\frac{m\pi\xi}{L}\right)}{2L^2} - \frac{x\sin\left(\frac{m\pi\xi}{L}\right)}{2\xi^2} - \frac{m\pi x\cos\left(\frac{m\pi\xi}{L}\right)}{2\xi L} + \frac{\sin\left(\frac{m\pi\xi}{L}\right)}{\xi}\right]_0^a$$

The 2nd partial integral (for $x < \xi$):

$$\int_a^b (x-\xi)\sin\frac{m\pi\xi}{L}\mathrm{d}\xi = \left[-\frac{L^2\sin\left(\frac{m\pi\xi}{L}\right)}{m^2\pi^2} - \frac{Lx\cos\left(\frac{m\pi\xi}{L}\right)}{m\pi} + \frac{L\xi\cos\left(\frac{m\pi\xi}{L}\right)}{m\pi}\right]_a^b$$

In these equations, si and ci functions are known tabulated ***sine*-integral** and ***cosine*-integral functions**, respectively, defined as [21]:

$$si(t) = \int_0^t \frac{\sin z}{z}\mathrm{d}z, \quad ci(t) = \int_0^t \frac{\cos z}{z}\mathrm{d}z.$$

The 3rd partial integral (for $x < \xi$):

First, two new variables are defined as follows:

$$A = h_0 L - h_2 b,$$
$$B = h_2 - h_0.$$

Integration gives

$$\int_b^L \frac{(x-\xi)\sin\frac{m\pi\xi}{L}}{\left[1+\frac{h_2-h_0}{h_0}\left(1-\frac{L-\xi}{L-b}\right)\right]^3}\mathrm{d}\xi = [h_0(L-b)]^3 \int_b^L \frac{(x-\xi)\sin\frac{m\pi\xi}{L}}{(A+B\xi)^3}\mathrm{d}\xi$$

$$=[h_0(L-b)]^3 \frac{1}{2A^4}\left[\frac{A^2(-Ax+2A\xi+B)\sin\frac{m\pi\xi}{L}}{(A\xi+B)^2} + \frac{m\pi}{L^2}\left[ci\left[\frac{m\pi(A\xi+B)}{AL}\right]\right.\right.$$

$$\times\left[m\pi(Ax+B)\sin\left(\frac{m\pi B}{AL}\right)-2AL\cos\left(\frac{m\pi B}{AL}\right)\right]$$

$$-si\left[\frac{m\pi(A\xi+B)}{AL}\right]\left[m\pi(Ax+B)\cos\left(\frac{m\pi B}{AL}\right)+2AL\sin\left(\frac{m\pi B}{AL}\right)\right]\Bigg]$$

$$\left.-\frac{m\pi A(Ax+B)\cos\frac{m\pi\xi}{L}}{L(A\xi+B)}\right]_b^L$$

The 4th partial integral:

$$\frac{x}{L}\int_0^a \frac{\left[(\xi-L)\sin\frac{m\pi\xi}{L}\right]}{\left[1+\frac{h_1-h_0}{h_0}\left(1-\frac{\xi}{a}\right)\right]^3}\mathrm{d}\xi=\frac{x}{L}\left[\frac{ah_0}{(a-1)h_1+h_0}\right]^3$$

$$\times\left[\frac{m\pi ci\left(\frac{m\pi\xi}{L}\right)}{L}+\frac{m^2\pi^2 si\left(\frac{m\pi\xi}{L}\right)}{2L}+\frac{L\sin\left(\frac{m\pi\xi}{L}\right)}{2\xi^2}-\frac{\sin\left(\frac{m\pi\xi}{L}\right)}{\xi}+\frac{m\pi\cos\left(\frac{m\pi\xi}{L}\right)}{2\xi}\right]_0^a$$

The 5th partial integral:

$$\frac{x}{L}\int_a^b(\xi-L)\sin\frac{m\pi\xi}{L}\mathrm{d}\xi=\frac{x}{L}\left[\frac{L^2\sin\left(\frac{m\pi\xi}{L}\right)}{m^2\pi^2}-\frac{L^2\cos\left(\frac{m\pi\xi}{L}\right)}{m\pi}-\frac{L\xi\cos\left(\frac{m\pi\xi}{L}\right)}{m\pi}\right]_a^b$$

The 6th partial integral:

First, two new variables are defined as follows:

$$A=h_0L-h_2b,$$
$$B=h_2-h_0.$$

Integration gives

$$\frac{x}{L}\int_b^L \frac{(\xi - L)\sin\frac{m\pi\xi}{L}}{\left[1+\frac{h_2-h_0}{h_0}\left(1-\frac{L-\xi}{L-b}\right)\right]^3}\mathrm{d}\xi = \frac{x}{L}[h_0(L-b)]^3\int_b^L \frac{(\xi - L)\sin\frac{m\pi\xi}{L}}{(A+B\xi)^3}\mathrm{d}\xi$$

$$= \frac{x[h_0(L-b)]^3}{2B^4L}\left[\frac{-B^2(A-Bl+2B\xi)\sin\left(\frac{m\pi\xi}{L}\right)}{(A+B\xi)^2} + \frac{m\pi}{L^2}\left[si\left[\frac{m\pi(A+B\xi)}{BL}\right]\right.\right.$$

$$\times\left[m\pi(A+BL)\cos\left(\frac{m\pi A}{BL}\right)+2BL\sin\left(\frac{m\pi A}{BL}\right)\right]$$

$$-ci\left[\frac{m\pi(A+B\xi)}{BL}\right]\left[m\pi(A+BL)\sin\left(\frac{m\pi A}{BL}\right)-2BL\cos\left(\frac{m\pi A}{BL}\right)\right]\Bigg]$$

$$\left.+\frac{m\pi B(A+BL)\cos\left(\frac{m\pi\xi}{L}\right)}{L(A+B\xi)}\right]_b^L$$

Solved example (specific case of the general solution):

Deflection at point A, where P is applied on A:

$$a_m = \frac{-2L}{m^2\pi^2}\sin\frac{m\pi}{4},\quad x=\frac{L}{4},\quad x_0=\frac{L}{4},\quad a=\frac{L}{4},\quad b=\frac{3L}{4}.$$

Deflection at point A, where P is applied on B:

$$a_m = \frac{-2L}{m^2\pi^2}\sin\frac{3m\pi}{4},\quad x=\frac{L}{4},\quad x_0=\frac{3L}{4},\quad a=\frac{L}{4},\quad b=\frac{3L}{4}.$$

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Greeni funktsiooni kasutamine mitteprismaatiliste lihttalade läbipainde uurimiseks analüütilisel meetodil

Mehdi Veiskarami ja Solmaz Pourzeynali

On esitatud mitteprismaatiliste lihttalade läbipainde mõjujoone koostamise arendatud metodoloogia, mis seisneb eelkõige Greeni funktsiooni leidmises läbipainde diferentsiaalvõrrandile ja tulemuste laiendamises tüüpülesannetele. On kasutatud pöördoperaatori meetodit koos ortogonaalse omafunktsiooni arendusega. Lõpplahend (mõjujoone võrrand) on esitatud integraalkujul, mille integraale saab arvutada kas analüütiliselt või numbriliselt. On esitatud näiteülesanne ja analüütiline lahend tegelikkuses sageli esinevatele mitteprismaatilistele lihttaladele.